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An Exactly Solvable Travelling Wave Equation in the Fisher–KPP Class

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Abstract For a simple one dimensional lattice version of a travelling wave equation, we obtain an exact relation between the initial condition and the position of the front at any later time. This exact relation takes the form of an inverse problem: given the times t_n at which the travelling wave reaches the positions n, one can deduce the initial profile. We show, by means of complex analysis, that a number of known properties of travelling wave equations in the Fisher–KPP class can be recovered, in particular Bramson's shifts of the positions. We also recover and generalize Ebert–van Saarloos' corrections depending on the initial condition.

Keywords Fisher–KPP · Front equation · Travelling wave

1 Introduction

The study of the solutions of partial differential equations describing a moving interface from a stable to an unstable medium is a classical subject [1–5] in mathematics, theoretical physics and biology [6–9]. The prototype of such equations is the Fisher–KPP equation (after Fisher [10] and Kolmogorov–Petrovskii–Piskunov [11])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1}$$

where the field u satisfies $0 \le u(x,t) \le 1$ and where $f(u) \ge 0$. The unstable medium corresponds to $u = 0$ (i.e. $f(0) = 0$ and $f'(0) > 0$) and the stable one to $u = 1$ (i.e. $f(1) = 0$ and $f'(1) < 0$).

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One can show that equations of type (1) exhibit a continuous family W_v of travelling wave solutions

$$u(x,t) = W_v(x - vt) \tag{2}$$

indexed by their velocities v. Explicit expressions of the travelling waves are in general not known except for particular velocities [12]. The best known example, due to Ablowitz and Zeppetella [13], is $u = \left[1 + c\exp[(x - vt)/\sqrt{6}]\right]^{-2}$ for the Fisher–KPP equation (1) with $f(u) = u - u^2$ and $v = 5/\sqrt{6}$.

Apart from describing the shapes of these travelling wave solutions (2), a central question is to understand how the long time behavior of the solutions of (1) depends on the initial condition $u(x,0)$. In general this asymptotic regime is controlled by the rate of the exponential decay of this initial condition. A brief review of the properties of the travelling wave solutions of (1) and on the way the position and the asymptotic velocity of the solution depend on the initial condition is given in Sect. 2.

In the present paper we study a simple one dimensional lattice version of a travelling wave equation. In this lattice version we associate to each lattice site $n \in \mathbb{Z}$ a positive number $h_n(t)$ which plays the role of the field $u(x,t)$ and these $h_n(t)$ evolve according to

$$\frac{\mathrm{d}h_n(t)}{\mathrm{d}t} = \begin{cases} a h_{n-1}(t) + h_n(t) & \text{if } 0 \le h_n(t) < 1, \\ 0 & \text{if } h_n(t) \ge 1. \end{cases} \tag{3}$$

We see that the evolution of $h_n(t)$ is linear except for the saturation at $h_n(t) = 1$ which is the only non-linearity in the problem. This saturation simply means that whenever $h_n(t)$ reaches the value 1, it keeps this value forever. The evolution (3) therefore combines linear growth, spreading (or diffusion) because of the coupling between neighboring sites, and saturation, very much like in Fisher–KPP equation (1).

The aim of this paper is to show that the evolution (3) leads to behaviors very similar to those expected for the usual Fisher–KPP equation (1). Moreover a number of properties of the solutions of (3) are easier to determine than for the original Fisher–KPP equation (1). Our approach is essentially based on the exact relation (32) derived in Sect. 3 which relates the times t_n at which $h_n(t)$ reaches 1 for the first time to the initial condition $h_n(0)$. We show in Sect. 4 that from (32) one can obtain a precise description of the shape of the travelling wave solutions, in particular explicit formulas for their asymptotic decay. We also show in Sect. 5 that (3) shares with the Fisher–KPP equation most of the properties expected for the dependence of the position of the front on the initial condition. Our results are summarized in Sect. 6.

2 Some Known Properties of the Fisher–KPP Class

In this section we briefly recall some properties of the Fisher–KPP equation.

2.1 The Travelling Waves

For the Fisher–KPP equation (1) the shape $W_v(x)$ of the travelling wave (2) satisfies an ordinary differential equation

$$W_v'' + vW_v' + f(W_v) = 0 \tag{4}$$

with the boundary conditions $W_v(-\infty) = 1$ and $W_v(+\infty) = 0$. By linearizing (4) for small W_v (when x is large),

$$W_v'' + vW_v' + f'(0)W_v = 0, \tag{5}$$

one can see that, generically, $W_v(x)$ vanishes exponentially as $x \to \infty$

$$W_v(x) \sim e^{-\gamma x}, \tag{6}$$

with γ related to the speed v of the travelling wave by

$$v(\gamma) = \gamma + \frac{f'(0)}{\gamma}. \tag{7}$$

This relation shows that depending on v, the rate γ of the exponential decay is either real or complex, and these two regimes are separated by a critical velocity v_c where $v(\gamma)$ is minimum

$$v_c = v(\gamma_c) \qquad \text{where} \qquad v'(\gamma_c) = 0. \tag{8}$$

With $v(\gamma)$ given by (7), one gets $v_c = 2\gamma_c$ and $\gamma_c = \sqrt{f'(0)}$. Under certain conditions on the function $f(u)$ (such as $0 \le f(u) \le uf'(0)$ for all u see [4,5,14] and references therein), it is known that:

- For $0 < v < v_c$, the solutions γ of the equation $v(\gamma) = v$ are complex. The corresponding travelling waves solutions of (4) oscillate around 0 while decaying as $x \to \infty$.
- For $v > v_c$, the travelling wave is monotonically decreasing and decays for large x as

$$W_v(x) \simeq A\, e^{-\gamma_1 x} \qquad \text{with } A > 0, \tag{9}$$

  where γ_1 is the smallest solution of $v(\gamma) = v$.
- For $v = v_c$, the equation $v(\gamma) = v_c$ has a double root γ_c and the travelling wave is monotonically decreasing and decays for large x as

$$W_{v_c}(x) \simeq A\, x\, e^{-\gamma_c x} \qquad \text{with } A > 0. \tag{10}$$

Remark 1 The facts (9) and (10) for $v \ge v_c$ are not obvious and cannot be understood from the linearized equation (5) only. These are properties of the full non-linear equation (4), which can be proved under known conditions on the non-linearity $f(u)$ (such as $0 \le f(u) \le uf'(0)$). Fronts which satisfy these properties are called *pulled fronts*.

For well tuned non-linearities (which fail to satisfy these conditions), travelling waves for $v \ge v_c$ might not be monotone and the asymptotics (9) and (10) might be modified; for instance in (9), depending on the value of v, one could have $A < 0$ or a decay in $\exp(-\gamma_2 x)$ where γ_2 is the largest solution of $v(\gamma) = v$. Rather than (10), one could have $A \exp(-\gamma_c x)$ without the x prefactor. In all these cases, the front equation is then said to be *pushed* [14,15].

2.2 The Selection of the Velocity

The travelling waves W_v solutions of (4) move at a constant speed with a time independent shape. For general initial conditions $u(x, 0)$, the shape of the solution is time-dependent and the question of the selection of the speed is to predict the asymptotic shape and velocity of the solution $u(x, t)$ in the long time limit. For initial profiles decreasing from $u(-\infty, 0) = 1$ to $u(+\infty, 0) = 0$ it is known since the works of Bramson [4,5,9,14,16] under which conditions the shape of the solution $u(x, t)$ converges to a travelling wave W_v solution of (4) in the sense that one can find a displacement X_t such that

$$u(X_t + x, t) \to W_v(x), \quad \text{with } \frac{X_t}{t} \to v. \tag{11}$$

In particular it is known that if the initial condition $u(x,0)$ satisfies for large x:

- $u(x,0) \sim e^{-\gamma x}$ with $0 < \gamma < \gamma_c$,
  then the asymptotic velocity is $v(\gamma)$, the asymptotic shape is $W_{v(\gamma)}$ and

$$X_t = v(\gamma)t + \text{Cst}. \tag{12}$$

- $u(x,0) \ll x^\alpha e^{-\gamma_c x}$ for some $\alpha < -2$ (in particular for step initial conditions),
  the asymptotic velocity is $v_c = v(\gamma_c)$, the asymptotic shape is W_{v_c} and

$$X_t = v_c t - \frac{3}{2\gamma_c} \ln t + \text{Cst}. \tag{13}$$

- $u(x,0) \sim x^\alpha e^{-\gamma_c x}$ with $\alpha \geq -2$,
  the asymptotic velocity v_c and shape W_{v_c} are the same as in the previous case but the logarithmic correction to the position X_t is modified:

$$X_t = v_c t - \frac{1-\alpha}{2\gamma_c} \ln t + \text{Cst} \qquad \text{for } \alpha > -2, \tag{14}$$

$$X_t = v_c t - \frac{3}{2\gamma_c} \ln t + \frac{1}{\gamma_c} \ln\ln t + \text{Cst} \qquad \text{for } \alpha = -2. \tag{15}$$

(Initial conditions decaying too slowly would not lead to a travelling wave.)

Notice that the solutions W_v of (4) can always be translated along the x axis, so the "Cst" in (12–15) depends on the particular solution of (4) that was chosen. It is often convenient to single out one particular solution W_v of (4): for example one may select the solution such that $W_v(0) = 1/2$ or such that $\int x W_v'(x)\,\mathrm{d}x = 0$. Once a particular prescription for W_v is chosen, the "Cst" in the equations above is well defined. It can be computed in some cases such as (12), but its analytic expression is not known in some other cases such as (13).

2.3 Vanishing Corrections

The convergence property (11) does not allow to define the displacement X_t to better than a constant: if X_t satisfies (11), then $X_t + o(1)$ also satisfies (11). It is however quite natural to choose a particular X_t, which one might call the position of the front. A possible choice could be

$$u(X_t, t) = c, \tag{16}$$

where $c \in (0,1)$ is a fixed given number. Another possible choice would be to interpret $-\partial u/\partial x$ as a probability density and pick X_t as its expectation:

$$X_t = -\int \mathrm{d}x\, x \frac{\partial u}{\partial x}. \tag{17}$$

Either definition (16) or (17) gives a position X_t which satisfies (11). With such a precise definition for X_t as (16) or (17), it makes sense to try to improve on (12–15) and determine higher order corrections. Ebert and van Saarloos [17, 18] have claimed that for steep enough initial conditions, the first correction to (13) is of order $t^{-1/2}$ and is universal: it depends neither on the initial condition, nor on the choice of (16) or (17), nor on the value c in (16), nor on the non-linearities. They found that

$$X_t = v_c t - \frac{3}{2\gamma_c} \ln t + \text{Cst} - 3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-1/2} + \cdots . \tag{18}$$

2.4 The Fisher–KPP Class

The main ingredients of the Fisher–KPP equation (1) which lead to travelling waves and fronts converging to those travelling waves are a diffusive term, a growth term and a saturation term. There exist many equations with the same ingredients which share the above properties (8–18) of the Fisher–KPP equation: the equation satisfied by the travelling waves (4), the dispersion relation (7) and the values of γ_c and v_c are modified, but everything else remains the same.

To give an example which appears in the problem of directed polymers on a tree [6], let us consider an evolution equation of the type

$$G(x, t+1) = \int G(x+\epsilon)^B \rho(\epsilon)\, \mathrm{d}\epsilon. \tag{19}$$

(In the directed polymers context, B is the branching ratio on the tree and $\rho(\epsilon)$ is the distribution of the random energies associated to edges of the tree). Then $u(x,t) = 1 - G(x,t)$ satisfies a discrete time evolution equation with an unstable uniform solution $u = 0$ and a stable one $u = 1$ as in (1). Even though (1) is continuous in time while (19) is discrete, they have similar properties: travelling waves for (19) are solutions of

$$W_v(x-v) = \int W_v(x+\epsilon)^B \rho(\epsilon)\, \mathrm{d}\epsilon \tag{20}$$

instead of (4). By linearizing the evolution of $G(x,t)$ around the unstable uniform solution $G = 1$ and by looking for travelling wave solutions of this linearized equation of the form $1 - G(x,t) \sim \exp[-\gamma(x - v(\gamma)t)]$, one gets a new dispersion equation which replaces (7):

$$v(\gamma) = \frac{1}{\gamma} \ln\left[B \int e^{\gamma\epsilon} \rho(\epsilon)\, \mathrm{d}\epsilon\right], \tag{21}$$

but all the above behaviors (8–18) remain valid with v_c and γ_c computed from (21) and (8). For example, for $B = 2$ and a uniform $\rho(\epsilon)$ on the unit interval (i.e. $\rho(\epsilon) = 1$ for $0 < \epsilon < 1$ and $\rho(\epsilon) = 0$ elsewhere) one gets, $v(\gamma) = \frac{1}{\gamma} \ln\left[\frac{2}{\gamma}(e^\gamma - 1)\right]$ which leads to $v_c \simeq 0.815172$ and $\gamma_c \simeq 5.26208$.

Example (19) is a front equation where time is discrete. One could also consider travelling wave equations where space is discrete, say $x \in \mathbb{Z}$. For instance, one could discretize the Laplacian in (1) or take (19) with a distribution $\rho(\epsilon)$ concentrated on integer values of ϵ. When space is discrete, special care should be taken: it is clear from (2) that while the front $u(x,t)$ lives on the lattice, the travelling wave $W_v(x)$ is defined for all real values x, and even when (2) holds, the shape of the front $W_v(x - vt)$ measured on the lattice evolves periodically in time with a period $1/v$. Furthermore, the convergence (11) no longer makes any sense. One can still try to define a specific position of X_t by something like the following generalization of (17):

$$X_t = \sum_{x \in \mathbb{Z}} x\big[u(x,t) - u(x+1,t)\big], \tag{22}$$

but with such a definition, even if the front is given by the travelling wave $W_v(x - vt)$, the difference $X_t - vt$ is no longer constant but becomes a periodic function in time because the shape of the front on the lattice evolves also periodically. Similarly, in the discrete space case, the Cst term in all the asymptotics (12–15) is in general replaced by a periodic function of time.

An alternative way to locate the front when time is continuous and space is discrete is to invert the roles of x and t: instead of defining X_t by $u(X_t, t) = c$ as in (16), one can define

t_x as the first time when the front at a given position x reaches a certain level c:

$$u(x, t_x) = c. \tag{23}$$

Note that when time and space are continuous, the functions X_t and t_x are reciprocal and one can write (12–15) as

$$t_x = \frac{x}{v(\gamma)} + \mathrm{Cst}', \qquad \text{for } u(x,0) \sim e^{-\gamma x} \text{ with } 0 < \gamma < \gamma_c, \tag{24}$$

$$t_x = \frac{x}{v_c} + \frac{3}{2\gamma_c v_c} \ln x + \mathrm{Cst}', \qquad \text{for } u(x,0) \ll x^\alpha e^{-\gamma_c x} \text{ for some } \alpha < -2, \tag{25}$$

$$t_x = \frac{x}{v_c} + \frac{1-\alpha}{2\gamma_c v_c} \ln x + \mathrm{Cst}', \qquad \text{for } u(x,0) \sim x^\alpha e^{-\gamma_c x} \text{ for } \alpha > -2, \tag{26}$$

$$t_x = \frac{x}{v_c} + \frac{3}{2\gamma_c v_c} \ln x - \frac{1}{\gamma_c v_c} \ln\ln x + \mathrm{Cst}', \qquad \text{for } u(x,0) \sim x^{-2} e^{-\gamma_c x}, \tag{27}$$

and, for steep enough initial conditions, one can write (18) as

$$t_x = \frac{x}{v_c} + \frac{1}{\gamma_c v_c}\left[\frac{3}{2}\ln x + \mathrm{Cst}' + 3\sqrt{\frac{2\pi v_c}{\gamma_c^3 v''(\gamma_c)}}\, x^{-1/2} + \cdots\right]. \tag{28}$$

The main advantage of (24–28) over (12–15,18) is that they still make sense when space is discrete (with a real constant Cst′, not a periodic function of time). We will see that they remain valid for our lattice model (3).

3 The Key Formula for the Position of the Front

In this section we consider the front $h_n(t)$ defined by (3) and we establish relation (32) between the initial condition $h_n(0)$ and the first times t_n at which $h_n(t)$ reaches the value 1. Here we limit our discussion to the case $a > 0$ and to initial conditions of the form

$$h_n(0) = \begin{cases} 1 & \text{for } n \le 0, \\ k_n & \text{for } n \ge 1, \end{cases} \tag{29}$$

where the k_n are non-negative, smaller than 1 and non-increasing, i.e.

$$1 > k_1 \ge k_2 \ge k_3 \ge \cdots \ge 0. \tag{30}$$

Clearly, as $a > 0$, for a monotonic initial condition (30), the solution $h_n(t)$ of (3) remains monotonic at any later time. One can define t_n as the time when $h_n(t)$ reaches 1 for the first time (i.e. $h_n(t) = 1$ for $t \ge t_n$ while $h_n(t) < 1$ for $t < t_n$). The monotonicity (30) of the initial condition implies the monotonicity of the times t_n

$$0 < t_1 < t_2 < \cdots < t_n < \cdots \tag{31}$$

Most of the properties of the solutions of (3) with the initial conditions (29) discussed in this paper will be based on the following exact formula

$$\sum_{n=1}^{\infty} k_n \lambda^n = -\frac{a\lambda}{1+a\lambda} + \frac{a+1}{1+a\lambda} \sum_{n=1}^{\infty} e^{-(1+a\lambda)t_n} \lambda^n, \tag{32}$$

which relates the generating function of the initial condition $\{k_n\}$ to the times $\{t_n\}$.

Formula (32) can be derived as follows. If one defines the generating functions

$$H_m(t) = \sum_{n \geq m} h_n(t) \lambda^{n-m+1}, \tag{33}$$

one can see that for $t_{m-1} \leq t \leq t_m$ (with the convention that $t_0 = 0$) the evolution of $H_m(t)$ is given by

$$\frac{\mathrm{d}H_m(t)}{\mathrm{d}t} = (1+a\lambda)H_m(t) + a\lambda. \tag{34}$$

This of course can be easily solved to give

$$H_m(t) = -\frac{a\lambda}{1+a\lambda} + \Phi_m e^{(1+a\lambda)t}, \tag{35}$$

where the Φ_m's are constants of integration. These Φ_m's can be determined by matching the solutions at times $0, t_1, t_2, \ldots$:

$$H_1(0) = \sum_{n=1}^{\infty} k_n \lambda^n, \qquad H_m(t_m) = \lambda\big(1 + H_{m+1}(t_m)\big), \tag{36}$$

and one gets that for $t_{m-1} \leq t \leq t_m$

$$H_m(t) = -\frac{a\lambda}{1+a\lambda} + \frac{a+1}{1+a\lambda} \sum_{n=m}^{\infty} e^{(1+a\lambda)(t-t_n)} \lambda^{n+1-m}. \tag{37}$$

Then (32) follows by taking $m = 1$ and $t = 0$ in (37).

Remark that formula (32) appears as the solution of a kind of inverse problem: given the times t_n, one can compute the initial profile k_n by expanding in powers of λ. This gives expressions of k_n in terms of the times t_m's for $m \leq n$. Alternatively one can determine the times t_n in terms of the initial profile k_m for $m \leq n$:

$$e^{-t_1} = \frac{a+k_1}{a+1}, \quad e^{-t_2} = \frac{ak_1+k_2}{a+1} + at_1 e^{-t_1}, \quad e^{-t_3} = \frac{ak_2+k_3}{a+1} + at_2 e^{-t_2} - \frac{(at_1)^2}{2} e^{-t_1}, \tag{38}$$

etc. Unfortunately these expressions become quickly too complicated to allow to determine how the times t_n depend asymptotically on the initial profile $\{k_n\}$ for large n. How these asymptotics can be understood from (32) will be discussed in Sect. 5.

4 Travelling Wave Solutions

4.1 The Exact Shape of the Travelling Waves

As usual, with travelling wave equations, the first solutions one can try to determine are travelling wave solutions moving at a certain velocity v. Because the $h_n(t)$ are defined on a lattice, a travelling wave solution moving at velocity v satisfies

$$h_n(t) = h_{n+1}\left(t + \frac{1}{v}\right). \tag{39}$$

Clearly this implies that the times t_n form an arithmetic progression, and by shifting the origin of time, one can choose

$$t_n = \frac{n}{v}. \tag{40}$$

This immediately gives, using (37), the generating function of the front shape at all times: for example for $0 \le t \le t_1 = 1/v$, one takes $m = 1$ in (37) and gets

$$\sum_{n\ge 1} h_n(t)\lambda^n = -\frac{a\lambda}{1+a\lambda} + \frac{a+1}{1+a\lambda} \times \frac{\lambda e^{(1+a\lambda)t}}{e^{(1+a\lambda)/v} - \lambda}. \tag{41}$$

Another way of determining the travelling wave solutions is to look directly for solutions of (3) of the form (39). One sees that W_v must satisfy

$$W_v(x) = 1 \quad \text{for } x \le 0, \qquad W_v(x) + aW_v(x-1) + vW_v'(x) = 0 \quad \text{for } x > 0. \tag{42}$$

These equations can be solved iteratively: for $x \le 0$, one already knows that $W_v(x) = 1$. For $x \in [0, 1]$ one has therefore $W_v + vW_v' + a = 0$, which implies for ($x \in [0, 1]$) that $W_v(x) = (a + 1)e^{-x/v} - a$ (the integration constant being fixed by continuity at $x = 0$). Knowing $W_v(x)$ for $x \in [0, 1]$, one can solve (42) for $x \in [1, 2]$ and so on.

$$W_v(x) = \begin{cases} 1 & \text{if } x \le 0, \\ (a+1)e^{-x/v} - a & \text{if } x \in [0,1], \\ \frac{a(1+a)}{v}(1-v-x)e^{(1-x)/v} + (1+a)e^{-x/v} + a^2 & \text{if } x \in [1,2], \\ \dots \end{cases} \tag{43}$$

In fact one can solve directly (42) by considering for $x \in [0, 1]$ the generating function $(\lambda, x) \mapsto \sum_n \lambda^n W_n(x + n)$. Then, as can be checked directly from (41) and (42), $W_v(x)$ and $h_n(t)$ are related for all $n \in \mathbb{Z}$ and $t \ge 0$ by

$$h_n(t) = W_v(n - vt). \tag{44}$$

4.2 The Decay of the Travelling Waves

The large n behavior of the travelling wave $h_n(t)$ (or equivalently the behavior of $W_v(x)$ for large x) can be understood by analyzing the singularities in λ of the right hand side of (41). These singularities are poles located at all the real or complex zeros of

$$e^{(1+a\lambda)/v} - \lambda = 0. \tag{45}$$

(one checks there is no pole at $\lambda = -1/a$) and each pole gives rise to an exponential decay in $h_n(t)$. Using $\lambda = \exp(\gamma)$, (45) can be rewritten into

$$v(\gamma) = \frac{1 + ae^{\gamma}}{\gamma}. \tag{46}$$

which is the dispersion relation for (3), similar to (7) or to (21). In fact, one can obtain (46) as in Sect. 2 by looking for the velocity v compatible in (42) with an exponentially decaying travelling wave of the form $W_v(x) \sim e^{-\gamma x}$.

One is then led to distinguish three cases depending on the number of real solutions of (46). There is a critical value v_c where (46) has a double zero on the real axis. This critical velocity v_c and the corresponding decay rate γ_c are the solution of

$$a(\gamma_c - 1)e^{\gamma_c} = 1, \qquad v_c = \frac{1}{\gamma_c - 1}. \tag{47}$$

1. For $v < v_c$, there is no real λ solution of (45), but there are complex roots. The large n behavior of $h_n(t)$ is then governed by the two roots λ_1 and λ_1^* of (45) closest to the origin

$$h_n(0) \simeq \frac{(a+1)v}{(1+a\lambda_1)(v-a\lambda_1)}\lambda_1^{-n} + \text{c.c.} \tag{48}$$

Because λ_1 and λ_1^* are complex, $h_n(t)$ changes its sign as n varies. So for $v < v_c$, as in the Fisher–KPP case, there are travelling wave solutions, but they fail to give positive profiles.

2. For $v = v_c$ given by (47), there is a double real root $\lambda_c = e^{\gamma_c}$ of (45). Then for large n the profile is of the form

$$h_n(0) \simeq \frac{2(1+a)}{1+v_c}\left[n + \frac{1+4v_c}{3(1+v_c)}\right]e^{-\gamma_c n}. \tag{49}$$

3. For $v > v_c$ there are two real roots $1 < \lambda_1 < \lambda_2$ of (45) and the large n behavior is controlled by the smallest root:

$$h_n(0) \simeq \frac{(a+1)v}{(1+a\lambda_1)(v-a\lambda_1)}\lambda_1^{-n}. \tag{50}$$

We see that for all velocities, we get *explicit* expressions of the prefactors of the exponential decay of the travelling waves. These prefactors are in general not known for more traditional travelling wave equations, such as the Fisher–KPP equation (1).

In each case, corrections to (48–50) can be obtained from the contributions of the other roots of (45). For instance, in the cases $v < v_c$ or $v > v_c$ one could write

$$h_n(0) \simeq \sum_r \frac{(a+1)v}{(1+a\lambda_r)(v-a\lambda_r)}\lambda_r^{-n}, \tag{51}$$

where the sum is over all the complex roots λ_r of (45). In Fig. 1, we compare the exact solution (43) of (42) with the asymptotic expansion (51) truncated to a finite number of roots of (45) closest to the origin and one can see that the truncation gives a very good fit of the actual solution.

We have seen that the travelling waves for $v < v_c$ were oscillatory. For $v \geq v_c$, they decrease monotonically towards 0; this can be seen directly from equation (42) verified by $W_v(x)$: write $W_v(x) = R(x)e^{-\gamma x}$ with γ a real positive number related to v through the

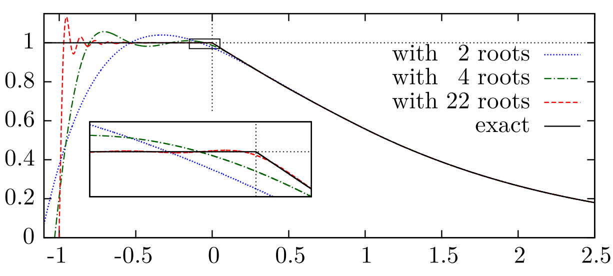

Fig. 1 The travelling wave $W_v(x)$ solution of (42) for $v = 4$ and $a = 1$ as a function of x. The *plain line* labeled "exact" is the exact small-x solution (43). The *dashed lines* are the sums (51) truncated to a given number of first terms: "with 2 roots" means only the two real roots λ_1 and λ_2, "with 4 roots" means the two real roots and the first pair of complex conjugate roots and "with 22 roots" means the two real roots and the ten pairs of complex conjugates roots closest to the origin. The *inset* is a zoom of the small rectangle around $x = 0$ and $W_v = 1$

dispersion relation (46) (Notice that choosing such a γ is impossible if $v < v_c$). Then (42) gives

$$R(x) = e^{\gamma x} \quad \text{for } x \le 0, \qquad ae^{\gamma}\big[R(x-1) - R(x)\big] + vR'(x) = 0 \quad \text{for } x > 0. \tag{52}$$

As $R(x)$ is strictly increasing for $x < 0$ it must be strictly increasing for all reals; otherwise, on the first local maximum x_m, one would have $R'(x_m) = 0$ and $R(x_m) > R(x_m - 1)$ which is incompatible with (52). Hence, $W_v(x)$ is positive and, from (42), strictly decreasing.

5 How the Initial Condition Determines the Asymptotic Regime

We now discuss how the position of the front at large times (or equivalently the large n asymptotics of the times t_n) depends on the initial condition.

First, by using mostly a comparison property, we will show that the final velocity of the front is determined by the large n decay of the initial condition k_n. Then, we will recover the logarithmic corrections (12–15) and sub-leading terms as in (18) by analyzing the key relation (32) between the initial profile k_n and the times t_n.

We write (32) as

$$(1 + a\lambda)K(\lambda) = -a\lambda + (a+1)T(\lambda), \tag{53}$$

where the two functions $K(\lambda)$ and $T(\lambda)$ are defined by

$$K(\lambda) = \sum_{n=1}^{\infty} k_n \lambda^n, \qquad T(\lambda) = \sum_{n=1}^{\infty} e^{-(1+a\lambda)t_n}\lambda^n. \tag{54}$$

The large n behavior of the k_n's and of the t_n's determines the domain of convergence of these two sums and one can try to use (53) to relate their singularities.

When $\lambda = e^{\gamma} > 1$, we will often use the following form of $T(\lambda)$ written in terms of the dispersion relation $v(\gamma)$:

$$T(e^{\gamma}) = \sum_{n=1}^{\infty} e^{\gamma[n - v(\gamma)t_n]}. \tag{55}$$

5.1 Selection of the Velocity

Let us first show that the final velocity of the front is determined by the large n behavior of the initial condition k_n in the same way as for other equations in the Fisher–KPP class. To do this, we use an obvious comparison property; considering two initial conditions $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with the corresponding times $\{t_n^{(1)}\}$ and $\{t_n^{(2)}\}$, one has

$$\text{if } 0 \le k_n^{(1)} \le k_n^{(2)} \text{ for all } n, \text{ then } t_n^{(1)} \ge t_n^{(2)} \text{ for all } n. \tag{56}$$

To keep the discussion simple, we focus only on initial conditions $\{k_n\}$ with $k_n \ge 0$ and the following simple asymptotics:

- If $k_n \sim n^{\alpha} e^{-\gamma n}$ with $0 < \gamma < \gamma_c$.
  Pick an $\epsilon > 0$ small enough so that $0 < \gamma - \epsilon$ and $\gamma + \epsilon < \gamma_c$, and consider the two travelling waves going at velocities $v(\gamma - \epsilon)$ and $v(\gamma + \epsilon)$ (they decay respectively like $e^{-(\gamma-\epsilon)n}$) and $e^{-(\gamma+\epsilon)n}$). It is clear that the initial condition $\{k_n\}$ can be sandwiched between these two travelling waves suitably shifted in space, so that, by using the comparison property one gets

$$\frac{1}{v(\gamma-\epsilon)} \le \liminf_{n\to\infty} \frac{t_n}{n} \le \limsup_{n\to\infty} \frac{t_n}{n} \le \frac{1}{v(\gamma+\epsilon)}. \tag{57}$$

Now take $\epsilon \to 0$ to get

$$\lim_{n\to\infty} \frac{t_n}{n} = \frac{1}{v(\gamma)}. \tag{58}$$

- If $k_n = 0$.
  It takes a time t_n to have $h_n(t) = 1$. But at time t_n, the $h_{n+m}(t)$ for $m > 0$ are positive so that, from the comparison property, one has

$$t_{n+m} \le t_n + t_m. \tag{59}$$

  The sequence $\{t_n\}$ is sub-additive and therefore t_n/n has a limit which we call $1/v$. By comparing the initial profile $k_n = 0$ to the travelling wave going at velocity v_c, one must have $1/v \ge 1/v_c$.
  We are now going to show that $1/v$ cannot be strictly greater than $1/v_c$. Indeed, if we had $1/v > 1/v_c$, the series (55) defining $T(\lambda)$ would be uniformly convergent on the whole positive real axis λ because $v(\gamma)t_n/n$ would eventually be larger than $1+\epsilon$ for some $\epsilon > 0$. One would then get

$$T'(\lambda) = \sum_{n\ge 1} \lambda^n e^{-(1+a\lambda)t_n} \left[\frac{n}{\lambda} - at_n\right] \qquad \text{for all } \lambda \ge 0 \tag{60}$$

  because the series (60) would also be uniformly convergent.
  However, for real and large enough λ (at least for $\lambda > \min_n n/(at_n)$), one would obtain $T'(\lambda) < 0$. But, with $k_n = 0$ one has $K(\lambda) = 0$ and from (53) $T'(\lambda) = a/(a+1)$, in contradiction with $T'(\lambda) < 0$.
  We conclude that one must have

$$\lim_{n\to\infty} \frac{t_n}{n} = \frac{1}{v_c} \quad \text{if} \quad k_n = 0 \quad \text{for} \quad n \ge 1. \tag{61}$$

- If $k_n \sim n^\alpha e^{-\gamma_c n}$ or if $k_n = o\big(e^{-\gamma_c n}\big)$.
  Again, by the comparison property, the initial condition can be sandwiched between $k_n = 0$ and, for any $\epsilon > 0$, the suitably shifted travelling wave going at velocity $v(\gamma_c - \epsilon)$. This leads to conclude that

$$\lim_{n\to\infty} \frac{t_n}{n} = \frac{1}{v_c}. \tag{62}$$

The velocity selection thus works as for other equations of the Fisher–KPP class.

5.2 Sub-leading Corrections

We limit the discussion to initial conditions similar to those discussed in the previous section which lead to a front with some asymptotic velocity V:

$$\lim_{n\to\infty} \frac{t_n}{n} = \frac{1}{V}. \tag{63}$$

We also assume that $k_n \ge 0$ which implies that $V \ge v_c$ as was shown in the previous section.

If $V > v_c$, write $V = v(\gamma_1) = v(\gamma_2)$ with $0 < \gamma_1 < \gamma_c < \gamma_2$. We have seen that this velocity is reached for initial conditions such as $k_n \sim n^\alpha e^{-\gamma_1 n}$. In (55), it is then clear that the series $T(\lambda)$ is divergent for $\lambda \in (e^{\gamma_1}, e^{\gamma_2})$ and convergent for $\lambda < e^{\gamma_1}$ or $\lambda > e^{\gamma_2}$. Furthermore, in (54), the radius of convergence of $K(\lambda)$ is e^{γ_1} and, as $k_n > 0$, the function $K(\lambda)$ must have a singularity at $\lambda = e^{\gamma_1}$. We thus see that both $T(\lambda)$ and $K(\lambda)$ become singular as λ approaches e^{γ_1} from below on the real axis. By matching the singularities of these two functions, we will obtain the sub-leading corrections to t_n for large n.

For $V = v_c$, if the initial condition is $k_n \sim n^\alpha e^{-\gamma_c n}$, the same argument applies: both $T(\lambda)$ and $K(\lambda)$ are singular when λ reaches e^{γ_c}, and one must match the singularities. But, with $V = v_c$, one could also have an initial condition which decays faster than $e^{-\gamma_c n}$ and for which the radius of convergence is larger than e^{γ_c} (even, possibly, infinite). Then, of course, $K(\lambda)$ would have no singularity at $\lambda = e^{\gamma_c}$, even though the convergence of $T(\lambda)$ would remain problematic when λ approaches e^{γ_c}. We will see that the large n behavior of t_n is tuned to "erase" the singularities in $T(\lambda)$ at e^{γ_c} to satisfy (53).

We attack the problem by assuming that the t_n are given and we try to obtain the asymptotics of the k_n. The starting point is thus to assume a velocity $V = v(\gamma_1)$ with $\gamma_1 \leq \gamma_c$, and study $T(\lambda)$ when λ gets close to e^{γ_1}. If one chooses, in all generality,

$$t_n = \frac{n}{V} + \frac{\delta_n}{\gamma_1 V}, \tag{64}$$

where $\delta_n/n \to 0$, one gets from (55)

$$T(e^\gamma) = \sum_{n=1}^{\infty} e^{\gamma\left[1-\frac{v(\gamma)}{V}\right]n - \frac{\gamma v(\gamma)}{\gamma_1 V}\delta_n}. \tag{65}$$

Now we want to take $\gamma = \gamma_1 - \epsilon$ and expand for small ϵ in order to extract the nature of the singularity. Two cases arise:

- If $V > v_c$ (which means $\gamma_1 < \gamma_c$), then $v'(\gamma_1) < 0$ and to leading order

$$T(e^{\gamma_1-\epsilon}) = \sum_{n=1}^{\infty} \exp\left[\left(\frac{\gamma_1 v'(\gamma_1)}{V}\epsilon + \cdots\right)n - (1 - \mu\epsilon + \cdots)\delta_n\right] \quad \text{for } V > v_c, \tag{66}$$

  with $\mu = 1/\gamma_1 + v'(\gamma_1)/V$.
- If $V = v_c$ (which means $\gamma_1 = \gamma_c$), then $v'(\gamma_c) = 0$ and one must push the expansion further:

$$T(e^{\gamma_c-\epsilon}) = \sum_{n=1}^{\infty} \exp\left[\left(-\frac{\gamma_c v''(\gamma_c)}{2v_c}\epsilon^2 + \cdots\right)n - \left(1 - \frac{1}{\gamma_c}\epsilon + \cdots\right)\delta_n\right] \quad \text{for} \quad V = v_c. \tag{67}$$

It is already clear that cases $V > v_c$ and $V = v_c$ need to be discussed separately. Equations (66) and (67) are the starting point of our analysis which is presented in detail in the following subsections.

We will make heavy use of the following formulas: for $\epsilon > 0$ small,

$$\sum_{n\geq 1} n^\alpha e^{-\epsilon n}\Big|_{\text{singular}} = \begin{cases} \frac{\Gamma(1+\alpha)}{\epsilon^{1+\alpha}} & \text{if } \alpha \text{ is not a negative integer,} \\ \dfrac{(-1)^\alpha \epsilon^{-\alpha-1} \ln \epsilon}{(-\alpha-1)!} & \text{if } \alpha \text{ is a negative integer.} \end{cases} \tag{68}$$

$$\sum_{n\geq 1} (\ln n) n^\alpha e^{-\epsilon n}\Big|_{\text{singular}} = \begin{cases} \frac{-\Gamma(1+\alpha)\ln\epsilon + \mathcal{O}(1)}{\epsilon^{1+\alpha}} & \text{if } \alpha \text{ is not a negative integer,} \\ \dfrac{(-\epsilon)^{-\alpha-1}}{(-\alpha-1)!}\left[\dfrac{\ln^2\epsilon}{2} + \mathcal{O}(\ln\epsilon)\right] & \text{if } \alpha \text{ is a negative integer,} \end{cases} \tag{69}$$

where the meaning of "singular" for a function $F(\epsilon)$ with a singularity at 0 is that the difference between $F(\epsilon)$ and $F(\epsilon)\big|_{\text{singular}}$ is a regular function of ϵ which can be expanded as a power series.

5.2.1 For $V > v_c$

As explained above we write $V = v(\gamma_1)$ with $\gamma_1 < \gamma_c$, and we choose t_n such that t_n/n that converges to $1/V$. If one chooses

$$t_n = \frac{n}{V} + \frac{B \ln n + C}{\gamma_1 V}, \tag{70}$$

by keeping the leading order in (66) and using (68) one gets for B not a positive integer

$$T(e^{\gamma_1 - \epsilon})\Big|_{\text{singular}} \simeq \Gamma(1-B)e^{-C}\left(\frac{V}{-v'(\gamma_1)\gamma_1\epsilon}\right)^{1-B}. \tag{71}$$

It is then easy to check that matching the singularities leads to an initial condition decaying as

$$k_n \simeq \frac{(1+a)e^{-C}}{-v'(\gamma_1)\gamma_1^2}\left[\frac{Vn}{-\gamma_1 v'(\gamma_1)}\right]^{-B} e^{-\gamma_1 n}. \tag{72}$$

Remarks: As can be easily checked, even though (71) is not valid if B is a positive integer, (72) is. One can also check that for $B = C = 0$ one recovers the asymptotics (50) of the travelling wave.

5.2.2 For $V = v_c$

if $V = v_c$, the main difference with the previous case is that $v(\gamma_c - \epsilon) - v(\gamma_c) \sim \epsilon^2$ as $\epsilon \to 0$ and one must use the expansion (67) instead of (66). As before, we choose a specific form for the times t_n which allow to easily make the comparison with the different cases (13–15) of the Fisher–KPP equation:

$$t_n = \frac{n}{v_c} + \frac{B \ln n + C}{\gamma_c v_c}. \tag{73}$$

With $\delta_n = B \ln + C$ into (67), one obtains generically (when $B \notin \{1, 3/2, 2, 5/2, 3, \ldots\}$, see discussion below)

$$T(e^{\gamma_c - \epsilon})\Big|_{\text{singular}} \simeq e^{-C}\Gamma(1-B)\left(\frac{\gamma_c v''(\gamma_c)}{2v_c}\right)^{B-1}\epsilon^{2B-2}. \tag{74}$$

Then, using (68) again and (53), one gets

$$k_n \simeq \frac{a+1}{\gamma_c v_c}e^{-C}\left(\frac{\gamma_c v''(\gamma_c)}{2v_c}\right)^{B-1}\frac{\Gamma(1-B)}{\Gamma(2-2B)}n^{1-2B}e^{-\gamma_c n}. \tag{75}$$

We see that the asymptotics of the initial condition (75) and of the times (73) for large n are related as in the Fisher–KPP case (26) and that the constant term in (26) can be determined. As in (70), one must be careful when B is a positive integer: (74) should be modified to include the logarithmic correction of (68), but (75) is not modified as can easily be checked (the ratio of the two Gamma functions has a limit).

There is another difficulty when $B \in \{3/2, 5/2, 7/2, \ldots\}$: for these values, the ratio of Gamma functions in (75) is zero. This means that an initial condition $\{k_n\}$ leading to (73) with $B = 3/2$ (for instance) must decrease faster than $n^{-2}e^{-\gamma_c n}$. For these special values of B, the right hand side of (74) is actually regular as ϵ^{2B-2} is a non-negative integer power of ϵ; any singular part of $T(e^{\gamma_c - \epsilon})$ must come from higher order terms.

We are now going to show that no non-negative initial condition $\{k_n\}$ can lead to a time sequence $\{t_n\}$ with an asymptotic expansion starting as in (73) with $B > 3/2$. To do so, we will show that the initial condition $k_n = 0$ leads to (73) with $B = 3/2$ (plus higher order corrections). As any non-negative initial condition must lead to times $\{t_n\}$ which are smaller than the times of the $k_n = 0$ initial condition, this will prove that B cannot be larger than 3/2.

Consider therefore the case $k_n = 0$; one has $K(\lambda) = 0$ and, from (53), one gets $T(\lambda) = a\lambda/(a+1)$. Obviously, $T(\lambda)$ has no singularity as λ approaches e^{γ_c}, so the right hand side of (74) must be regular, which implies that $B \in \{3/2, 5/2, 7/2, \ldots\}$. We now rule out any value other than $B = 3/2$ by looking at the term of order ϵ in the expansion of $T(e^{\gamma_c-\epsilon})$. One can check that the only terms of order ϵ come from the $\epsilon\delta_n/\gamma_c$ in (67) and from (74) if $B = 3/2$, so one has

$$T(e^{\gamma_c-\epsilon}) = T(e^{\gamma_c}) + \left[\frac{1}{\gamma_c}\sum_{n=1}^{\infty}\delta_n e^{-\delta_n} + e^{-C}\Gamma(-1/2)\left(\frac{\gamma_c v''(\gamma_c)}{2v_c}\right)^{1/2}\mathbb{1}_{B=3/2}\right]\epsilon + o(\epsilon). \tag{76}$$

Notice (64) that $\delta_n \geq 0$ for the $k_n = 0$ initial condition because it is below the travelling wave at velocity v_c for which $\delta_n = 0$. The first term of order ϵ in (76) is therefore positive; on the other hand, the second term (only if $B = 3/2$) is negative. But, from $T(\lambda) = a\lambda/(a+1)$ the term of order ϵ must be negative; therefore one must have $B = 3/2$ for the zero initial condition and, therefore, $B \leq 3/2$ for any non-negative initial condition.

To summarize, the relationship between the times (73) and the initial condition (75) we have established in this section is valid only for $B < 3/2$ because we only consider non-negative initial conditions. Furthermore, to have (73) with $B = 3/2$, one must have an initial condition decreasing faster than the $n^{-2}e^{-\gamma_c n}$ suggested by (75). No non-negative initial condition can lead to (73) with $B > 3/2$.

5.2.3 For $V = v_c$ and $B = 3/2$

The case $B = 3/2$ is of course the most delicate and it corresponds to (13, 15, 18) in the Fisher–KPP case. For the t_n given by (73) the leading singularity is not (74) but rather

$$T(e^{\gamma_c-\epsilon})\Big|_{\text{singular}} \simeq 3e^{-C}\sqrt{\frac{2\pi v''(\gamma_c)}{\gamma_c v_c}}\,\epsilon^2\ln\epsilon. \tag{77}$$

(This term comes from the first order expansion of the $\epsilon\delta_n/\gamma_c$ term in (67).) Relating this to the $\{k_n\}$ through (53), it leads through (68) to $k_n \sim n^{-3}e^{-\gamma_c n}$ with a negative prefactor. So there is no way for a non-negative initial condition to be compatible with exactly (73), without any extra term.

Therefore, we need to add some corrections to (73) when $B = 3/2$. Let us consider a correction of the form

$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + C + Dn^{-\xi}}{\gamma_c v_c} \tag{78}$$

for some $\xi > 0$. Plugging this correction into (67) one gets

$$T(e^{\gamma_c-\epsilon}) = \sum_{n=1}^{\infty} e^{-C-\frac{\gamma_c v''(\gamma_c)}{2v_c}\epsilon^2 n}\, n^{-\frac{3}{2}}\left[1 + \frac{3\epsilon}{2\gamma_c}\ln n - Dn^{-\xi} + \cdots\right], \tag{79}$$

where the "$\cdots$" contains smaller order terms of orders $n\epsilon^3, n^{-2\xi}, \epsilon n^{-\xi}, \epsilon^2\ln^2 n$, etc. Consider in turns the terms in the square bracket. The "1" leads to the right hand side of (74) with

$B = 3/2$, which is simply a regular term linear in ϵ. The term in $\epsilon \ln n$ gives the right hand side of (77) and the $-Dn^{-\xi}$ contribution can be computed from

$$\sum_{n=1}^{\infty} e^{-\frac{\gamma_c v''(\gamma_c)}{2v_c}\epsilon^2 n} n^{-\frac{3}{2}-\xi}\Big|_{\text{singular}} = \begin{cases} \Gamma(-\frac{1}{2}-\xi)\left(\frac{\gamma_c v''(\gamma_c)}{2v_c}\right)^{\frac{1}{2}+\xi}\epsilon^{1+2\xi} & \text{if } \xi \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}, \\ 2\frac{\gamma_c v''(\gamma_c)}{2v_c}\epsilon^2 \ln \epsilon & \text{if } \xi = \frac{1}{2}. \end{cases} \tag{80}$$

Several subcases must be considered

- If $\xi > 1/2$ this is smaller than $\epsilon^2 \ln \epsilon$; therefore the leading singularity is still given by (77) which is incompatible with a non-negative initial condition.
- If $0 < \xi < 1/2$ the leading singularity for $T(e^{\gamma_c - \epsilon})$ is $\epsilon^{1+2\xi}$ as given by (80). This leads to
$$k_n \simeq -De^{-C}\frac{1+a}{\gamma_c v_c}\frac{\Gamma(-\frac{1}{2}-\xi)}{\Gamma(-1-2\xi)}\left(\frac{\gamma_c v''(\gamma_c)}{2v_c}\right)^{\frac{1}{2}+\xi} n^{-2-2\xi}e^{-n\gamma_c}. \tag{81}$$
With $0 < \xi < \frac{1}{2}$, this is positive if $D > 0$.
- If $\xi = 1/2$ the corrections from (80) and from (77) are both of order $\epsilon^2 \ln \epsilon$. This leads to
$$k_n \simeq 2\frac{1+a}{\gamma_c v_c}e^{-C}\left[D\frac{\gamma_c v''(\gamma_c)}{v_c} - 3\sqrt{\frac{2\pi v''(\gamma_c)}{\gamma_c v_c}}\right]n^{-3}e^{-\gamma_c n}, \tag{82}$$
which is positive if D is large enough. Notice also that the square bracket in (82) vanishes for
$$D = 3\sqrt{\frac{2\pi v_c}{\gamma_c^3 v''(\gamma_c)}}. \tag{83}$$
This means that initial conditions decaying faster than $n^{-3}e^{-\gamma_c n}$ (including the zero initial condition) must lead to (78) with $\xi = 1/2$ and D given by (83). This is exactly the prediction (28).

To finish, notice that we found the first terms of the asymptotic expansion for the times t_n when the initial condition decays as $n^\alpha e^{-n\gamma_c}$ when $\alpha > -2$ (see (73) for $B < 3/2$) and when $\alpha < -2$ (it is of the form (78) with $\xi = -1 - \alpha/2$ for $-3 < \alpha < -2$ and $\xi = 1/2$ for $\alpha \le -3$), but we did not yet considered the case where $k_n \simeq n^{-2}e^{-n\gamma_c}$. One can check that by taking, as in (27),

$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n - \ln\ln n + C}{\gamma_c v_c}, \tag{84}$$

one obtains

$$T(e^{\gamma_c - \epsilon})\Big|_{\text{singular}} \simeq e^{-C}\sqrt{\frac{8\pi\gamma_c v''(\gamma_c)}{v_c}}\,\epsilon \ln \epsilon, \tag{85}$$

which leads to

$$k_n \simeq \frac{1+a}{\gamma_c v_c}e^{-C}\sqrt{\frac{8\pi\gamma_c v''(\gamma_c)}{v_c}}\,n^{-2}e^{-\gamma_c n}. \tag{86}$$

6 Summary

In the previous section, we have computed the initial conditions k_n as a function of the times t_n. Table 1 summarizes our results.

Table 1 Asymptotics of t_n as a function of the leading behavior of the initial condition k_n

1	$k_n \sim n^\alpha e^{-\gamma n}$ with $\gamma < \gamma_c$	$t_n \simeq \frac{n}{v(\gamma)} + \frac{1}{\gamma v(\gamma)}\Big[-\alpha \ln n + C\Big]$	see (70, 72)
2	$k_n \sim n^\alpha e^{-\gamma_c n}$ with $\alpha > -2$	$t_n \simeq \frac{n}{v_c} + \frac{1}{\gamma_c v_c}\Big[\frac{1-\alpha}{2}\ln n + C\Big]$	see (73, 75)
3	$k_n \sim n^{-2} e^{-\gamma_c n}$	$t_n \simeq \frac{n}{v_c} + \frac{1}{\gamma_c v_c}\Big[\frac{3}{2}\ln n - \ln\ln n + C\Big]$	see (84, 86)
4	$k_n \sim n^\alpha e^{-\gamma_c n}$ with $-3 \le \alpha < -2$	$t_n \simeq \frac{n}{v_c} + \frac{1}{\gamma_c v_c}\Big[\frac{3}{2}\ln n + C + D n^{1+\frac{\alpha}{2}}\Big]$	see (78, 81, 82)
5	$k_n \ll n^\alpha e^{-\gamma_c n}$ for some $\alpha < -3$	$t_n \simeq \frac{n}{v_c} + \frac{1}{\gamma_c v_c}\Big[\frac{3}{2}\ln n + C + 3\sqrt{\frac{2\pi v_c}{\gamma_c^3 v''(\gamma_c)}}\, n^{-\frac{1}{2}}\Big]$	see (78, 82, 83)

Table 2 Asymptotic expansion of X_t as a function of the leading behavior of the initial condition $u(x,0)$

1	$u(x,0) \sim x^\alpha e^{-\gamma x}$ with $\gamma < \gamma_c$	$X_t \simeq v(\gamma)t + \frac{\alpha}{\gamma}\ln t + C'$
2	$u(x,0) \sim x^\alpha e^{-\gamma_c x}$ with $\alpha > -2$	$X_t \simeq v_c t + \frac{\alpha - 1}{2\gamma_c}\ln t + C'$
3	$u(x,0) \sim x^{-2} e^{-\gamma_c x}$	$X_t \simeq v_c t - \frac{3}{2\gamma_c}\ln t + \frac{1}{\gamma_c}\ln\ln t + C'$
4	$u(x,0) \sim x^\alpha e^{-\gamma_c x}$ with $-3 \le \alpha < -2$	$X_t \simeq v_c t - \frac{3}{2\gamma_c}\ln t + C' - D' t^{1+\frac{\alpha}{2}}$
5	$u(x,0) \ll x^\alpha e^{-\gamma_c x}$ for some $\alpha < -3$	$X_t \simeq v_c t - \frac{3}{2\gamma_c}\ln t + C' - 3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-\frac{1}{2}}$

These asymptotics agree with all previously known results discussed in Sect. 2. Case 4 is a new prediction, and the domain of validity of Ebert–van Saarloos correction (18) from [17] is made precise (case 5).

The constant C can easily be computed in cases 1 to 3, but we did not manage to get a closed expression in cases 4 and 5. Similarly, we have no expression for D in case 4; in particular, for $\alpha = -3$, the D of case 4 is not given by the prefactor of $n^{-1/2}$ in case 5 because for $\alpha = -3$ the right hand side of (82) must not vanish.

The vanishing terms $n^{1+\alpha/2}$ and $n^{-1/2}$ in cases 4 and 5 depend only on the leading behavior of k_n for large n. One could compute higher order corrections in cases 1 to 3 using the same technique by looking at the next singularities in $T(e^{\gamma-\epsilon})$, but one would need to know a bit more about the asymptotic behavior of k_n: one would find that

$$\text{If } k_n = A n^\alpha e^{-\gamma n}\left(1 + o\Big(\frac{\ln n}{n}\Big)\right),$$

$$\text{then } t_n = \big[\text{as above}\big] + \begin{cases} D\dfrac{\ln n}{n} & \text{in case 1 for } \alpha \neq 0, \\ D\dfrac{\ln n}{\sqrt{n}} & \text{in case 2 for } \alpha \notin \{-1, 0, 1\}, \\ D\dfrac{1}{\sqrt{n}} & \text{in case 3,} \end{cases} \tag{87}$$

where the prefactor D could be computed in each case. These vanishing corrections in cases 1 to 3 are less universal than in case 4 to 5 as they do not depend *only* on the leading behavior of k_n for large n, but also on the fact that the sub-leading behavior of k_n decays fast enough compared to the leading behavior. For case 1 with $\alpha = 0$ and case 2 with $\alpha = 1$, the initial condition behaves asymptotically as the travelling wave eventually reached by the front, and vanishing corrections might depend on the initial condition in a more complicated way. Case 2 with $\alpha = -1$ or $\alpha = 0$ are border cases with slightly different corrections.

If we conjecture that the new results (cases 4 and 5) of Table 1 hold for the whole Fisher–KPP class one can obtain, by inverting the relations between t_n and n of Table 1, the asymptotics of the position X_t for initial conditions of the form $u(x, 0) \sim x^\alpha e^{-\gamma x}$. This is done in Table 2.

7 Conclusion

The main result of the present work is the exact relation (32) between the initial condition and the positions of the front at time t for the model (3). Relating the asymptotics of the t_n's to those of the k_n's, using the exact relation (32) is an interesting but not easy problem of complex analysis. It allows to obtain precise expressions of the shape of the travelling waves, including prefactors which are usually not known in the usual equations of the Fisher–KPP type. It also allows one to recover the known long time asymptotics of the front position, and to get previously unknown results; in particular, we have shown how fast an initial condition should decay to exhibit the Ebert–van Saarloos correction, and that there is a range of initial conditions which exhibit the $-3/2 \ln t$ Bramson logarithmic term but for which the Ebert–van Saarloos correction is modified (See cases 4 and 5 of Tables 1 and 2).

As shown here the analysis of the asymptotics (3), using complex analysis, is tedious but rather straightforward. Higher corrections to the asymptotics of the position could be determined. One could also try to study how, depending on the initial condition, the asymptotic shape is reached. Furthermore, it would be interesting to generalize (3) to evolutions involving more than two neighboring sites, or to a non-lattice version of the model. More challenging would be to attack the noisy version of the problem [19,20].

Appendix: An Heuristic Derivation of the Positions of the Front

In this appendix we show that several expressions of the position of the front for a Fisher–KPP front can be recovered by considering a simplified version of the Fisher–KPP equation (1) where the non-linear term is replaced by an absorbing boundary. Consider the following linearized Fisher–KPP equation with a given time-dependent boundary X_t with $X_0 = 0$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f'(0)u \quad \text{if } x > X_t, \\ u(X_t, t) = 0. \end{cases} \tag{88}$$

For a given $y > 0$, we look at the value $u(X_t + y, t)$ of the solution at a distance y from the boundary. Intuitively, if X_t increases too quickly with t, this quantity is pushed to zero. On the other hand, if X_t increases too slowly, it diverges with t. It is only for finely tuned choices of X_t that $u(X_t + y, t)$ remains of order 1.

Now we suppose that X_t is no longer given *a priori* but is instead determined by

$$u(X_t + 1, t) = 1. \tag{89}$$

It has been shown [21] that the solution of (88, 89) for compactly supported initial conditions leads to the same long time asymptotics for X_t as for the Fisher–KPP equation (see Sect. 2): one recovers the Bramson term (13) and the Ebert–van Saarloos correction (18).

For initial conditions decaying fast enough, one expects X_t to be asymptotically linear. If X_t were really linear (not only asymptotically but at all times), (88) would be very easy to solve. In this Appendix, we solve a simplified version of (88) where the boundary is replaced by a straight line. This allows us to recover the velocity and the logarithmic corrections (12–15) of the Fisher–KPP equation.

The version we actually consider is therefore the following: For each given time t, we replace the boundary by a linear boundary of slope X_t/t and solve

$$\begin{cases} \dfrac{\partial u}{\partial s} = \dfrac{\partial^2 u}{\partial x^2} + f'(0)u \quad \text{if } x > \frac{X_t}{t}s, \\ u\Big(\dfrac{X_t}{t}s, s\Big) = 0. \end{cases} \tag{90}$$

We then tune the value of X_t to satisfy (89) at time t.

For an initial condition $\delta(x-x_0)$ the solution to (90) is

$$g(x,s|x_0) = \frac{e^{f'(0)s}}{\sqrt{4\pi s}}\left[\exp\left(-\frac{(x-x_0)^2}{4s}\right) - \exp\left(\frac{X_t}{t}x_0 - \frac{(x+x_0)^2}{4s}\right)\right]. \tag{91}$$

Taking $s=t$ and writing $x = X_t + y$, one obtains

$$g(X_t+y,t|x_0) = \frac{1}{\sqrt{4\pi t}}\exp\left[f'(0)t - \frac{(X_t+y)^2 + x_0^2 - 2X_t x_0}{4t}\right] 2\sinh\left(\frac{yx_0}{2t}\right). \tag{92}$$

Given a general initial condition $u(x_0,0)$ for $x_0>0$ one has

$$u(X_t+y,t) = \int_0^\infty \mathrm{d}x_0\, g(X_t+y,t|x_0)u(x_0,0), \tag{93}$$

which, after writing $X_t = ct - \delta_t$ with $\delta_t \ll t$, leads to

$$\begin{aligned} &u(X_t+y,t) = \frac{1}{\sqrt{\pi t}}\exp\left[t\Big(f'(0)-\frac{c^2}{4}\Big) - \frac{c}{2}(y-\delta_t) - \frac{(y-\delta_t)^2}{4t}\right] \times I_t(y), \\ &\text{with} \quad I_t(y) = \int_0^\infty \mathrm{d}x_0\, u(x_0,0)\exp\left[\frac{cx_0}{2} - \frac{\delta_t x_0}{2t} - \frac{x_0^2}{4t}\right]\sinh\left(\frac{yx_0}{2t}\right). \end{aligned} \tag{94}$$

Depending on the initial condition $u(x_0,0)$, we can now determine for which values of c and δ_t the front $u(X_t+y,t)$ remains of order 1 for y of order 1 as t increases.

- For $u(x_0,0) \simeq Ae^{-\gamma x_0}$ with $\gamma < c/2$, one finds that the integral $I_t(y)$ is dominated by $x_0 \simeq (c-2\gamma)t$. One obtains

$$\begin{aligned} &I_t(y) \simeq A\sqrt{4\pi t}\,\sinh\Big(\frac{c-2\gamma}{2}y\Big)\exp\left[\Big(\frac{c^2}{4} - \gamma c + \gamma^2\Big)t - \frac{c-2\gamma}{2}\delta_t\right], \\ &\text{and } u(X_t+y,t) \simeq 2A\sinh\left[\frac{c-2\gamma}{2}y\right]\exp\left[\big(f'(0) - \gamma c + \gamma^2\big)t + \gamma\delta_t - \frac{c}{2}y\right]. \end{aligned} \tag{95}$$

Writing $u(X_t+y,t) \sim 1$ leads to $c = \gamma + f'(0)/\gamma = v(\gamma)$ and $\delta_t \simeq \text{Cst}$. The starting hypothesis $\gamma < c/2$ then translates into $\gamma < \gamma_c = \sqrt{f'(0)}$. We conclude that

$$\text{For } u(x_0,0)\sim e^{-\gamma x_0} \text{ with } \gamma<\gamma_c, \qquad X_t\simeq v(\gamma)t+C, \tag{96}$$

as in (12).

- For $u(x_0,0)\simeq Ax_0^\alpha e^{-\gamma x_0}$ with $\gamma<c/2$, the integral $I_t(y)$ is again dominated by $x_0\simeq(c-2\gamma)t$. The large t expression of $u(X_t+y,t)$ has an extra term $[(c-2\gamma)t]^\alpha$ which is canceled by taking now $\delta_t\simeq-\frac{\alpha}{\gamma}\ln t+\text{Cst}$. (The value of c remains the same.) We conclude that

$$\text{For } u(x_0,0)\sim x_0^\alpha e^{-\gamma x_0} \text{ with } \gamma<\gamma_c, \qquad X_t\simeq v(\gamma)t+\frac{\alpha}{\gamma}\ln t+C. \tag{97}$$

- For $u(x_0,0)\ll e^{-\gamma x_0}$ for some $\gamma>c/2$ (steep initial condition), the integral $I_t(y)$ is dominated by x_0 of order 1. This leads to

$$\begin{aligned} &I_t(y)\simeq\frac{y}{2t}\int_0^\infty \mathrm{d}x_0\,u(x_0,0)x_0\exp\Big[\frac{cx_0}{2}\Big],\\ &\text{and}\quad u(X_t+y,t)\sim\frac{y}{t^{3/2}}\exp\left[t\Big(f'(0)-\frac{c^2}{4}\Big)-\frac{c}{2}(y-\delta_t)\right]. \end{aligned} \tag{98}$$

One needs to take $c=2\sqrt{f'(0)}=v_c=2\gamma_c$ and $\delta_t=\frac{3}{2\gamma_c}\ln t+\text{Cst}$. The starting hypothesis $\gamma>c/2$ translates into $\gamma>\gamma_c$ and we conclude that

$$\text{For } u(x_0,0)\ll e^{-\gamma x_0} \text{ for some } \gamma>\gamma_c, \qquad X_t\simeq v_ct-\frac{3}{2\gamma_c}\ln t+C, \tag{99}$$

as in (13).

- For $u(x_0,0)\simeq Ax_0^\alpha e^{-\frac{c}{2}x_0}$, depending on the value of α, the integral $I_t(y)$ is dominated by values of x_0 of order 1 or of order $\sqrt{t}$. In any case, $x_0\ll t$ and one can simplify $I_t(y)$ into

$$I_t(y)\simeq\frac{y}{2t}\int_0^\infty \mathrm{d}x_0\,u(x_0,0)x_0\exp\left[\frac{c}{2}x_0-\frac{x_0^2}{4t}\right]. \tag{100}$$

When $\alpha<-2$, this integral is dominated by x_0 of order 1, the Gaussian term can be dropped and one recovers (98) and (99).
When $\alpha\geq-2$, the integral is dominated by x_0 of order $\sqrt{t}$. One gets

$$I_t(y)\simeq A\frac{y}{2t}\int_1^\infty \mathrm{d}x_0\,x_0^{\alpha+1}\exp\left[-\frac{x_0^2}{4t}\right]\simeq\begin{cases} Ay2^\alpha\Gamma\Big(1+\dfrac{\alpha}{2}\Big)t^{\frac{\alpha}{2}} & \text{if } \alpha>-2,\\ Ay\dfrac{\ln t}{4t} & \text{if } \alpha=-2.\end{cases} \tag{101}$$

Into (94) one must therefore take $c=2\sqrt{f'(0)}=v_c=2\gamma_c$ and $\delta_t=\frac{1-\alpha}{2\gamma_c}\ln t+\text{Cst}$ if $\alpha>-2$ or $\delta_t=\frac{3}{2\gamma_c}\ln t-\frac{1}{\gamma_c}\ln\ln t$ if $\alpha=-2$. We conclude that

$$\text{For } u(x_0,0)\sim x_0^\alpha e^{-\gamma_c x_0}, \quad X_t\simeq\begin{cases} v_ct-\dfrac{3}{2\gamma_c}\ln t+C & \text{if } \alpha<-2,\\ v_ct-\dfrac{3}{2\gamma_c}\ln t+\dfrac{\ln\ln t}{\gamma_c}+C & \text{if } \alpha=-2,\\ v_ct-\dfrac{1-\alpha}{2\gamma_c}\ln t+C & \text{if } \alpha>-2,\end{cases} \tag{102}$$

as in (13–15).

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