Probability distribution of the free energy of a directed polymer in a random medium

Éric Brunet* and Bernard Derrida†
Laboratoire de Physique Statistique, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cédex 05, France
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We calculate exactly the first cumulants of the free energy of a directed polymer in a random medium for the geometry of a cylinder. By using the fact that the nth moment $\langle Z^n\rangle$ of the partition function is given by the ground-state energy of a quantum problem of n interacting particles on a ring of length L, we write an integral equation allowing to expand these moments in powers of the strength of the disorder γ or in powers of n. For n small and $n\sim(L\gamma)^{-1/2}$, the moments $\langle Z^n\rangle$ take a scaling form which allows us to describe all the fluctuations of order $1/L$ of the free energy per unit length of the directed polymer. The distribution of these fluctuations is the same as the one found recently in the asymmetric exclusion process, indicating that it is characteristic of all the systems described by the Kardar-Parisi-Zhang equation in $1+1$ dimensions.

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I. INTRODUCTION

Directed polymers in a random medium is one of the simplest systems for which the effect of strong disorder can be studied [1–3]. At the mean-field level, it possesses a low-temperature phase, with a broken symmetry of replica [4,5] similar to mean-field spin glasses [6]. The problem is, however, much better understood than spin glasses; in particular, one can write [4,5] closed expressions of the mean-field free energy and one can predict the existence [7] of phase transitions in all dimensions $d+1>2+1$. It is also an interesting system from the point of view of nonequilibrium phenomena: through the Kardar-Parisi-Zhang (KPZ) equation [8,9], it is related to ballistic growth models and, in $1+1$ dimensions, to the asymmetric simple exclusion process (ASEP) [3,9].

In the theory of disordered systems, the replica approach plays a very special role. On the one hand, it is one of the most powerful theoretical tools and often the only possible approach to study some strongly disordered systems. On the other hand, it is difficult to tell in advance whether the predictions of the replica approach are correct or not. When it does not work, one can always try to break the symmetry of the replica [6]: this usually makes the calculations much more complicated without being certain that the results become correct. In the replica approach, the calculation usually starts with an integer number n of the replica. Then, as the limit of physical interest is the limit $n\to 0$, one has to extend to noninteger n results obtained for integer n. This is in fact the big difficulty of the replica approach, so it is useful to look at simple examples for which the n dependence can be studied in detail.

This is one of the motivations of the present work, where we show how to calculate integer and noninteger moments $\langle Z^n\rangle$ of the partition function Z of a directed polymer in $1+1$ dimensions. The geometry we consider is a cylinder infinite in the t direction and periodic, of size L, in the x direction (i.e., $x+L\equiv x$). The partition function $Z(x,t)$ of a directed polymer joining the points (0,0) and (x,t) on this cylinder is given by the path integral

$$Z(x,t)=\int_{(0,0)}^{(x,t)}\mathcal{D}y(s)\exp\left(-\int_0^t ds\left[\frac{1}{2}\left(\frac{dy(s)}{ds}\right)^2+\eta(y(s),s)\right]\right),\tag{1}$$

where the random medium is characterized by a Gaussian white noise $\eta(x,t)$,

$$\langle\eta(x,t)\eta(x',t')\rangle=\gamma\delta(x-x')\delta(t-t').\tag{2}$$

One of the main goals of the present work is to calculate the cumulants $\lim_{t\to\infty}\langle\ln^k Z(t)\rangle_c/t$ of the free energy per unit length of the directed polymer. These cumulants are the coefficients of the small-n expansion of $E(n,L,\gamma)$ defined as

$$E(n,L,\gamma)=-\lim_{t\to\infty}\frac{1}{t}\ln\left[\frac{\langle Z^n(x,t)\rangle}{\langle Z(x,t)\rangle^n}\right].\tag{3}$$

This $E(n,L,\gamma)$ was calculated exactly by Kardar [10] for integer n and $L=\infty$. His closed expression $E(n,\infty,\gamma)=-n(n^2-1)\gamma^2/24$ cannot, however, be continued to all values of n, in particular to negative n, as it would violate the fact that $\partial^2E(n,L,\gamma)/\partial n^2$ is negative. Therefore, one does not know the range of validity of this expression.

The second motivation of the present work is to test the universality class of the KPZ equation. The problem (1) of a directed polymer in a random medium is described by the KPZ equation as several other problems such as growing interfaces or exclusion processes [3]. For certain models of this class, the asymmetric exclusion processes, the distribution of the total current Y_t integrated over time t, has been calculated exactly [11–15] in the long-time limit. For large t, the generating function of this integrated current Y_t on a ring of L sites takes the form [11,12]

$$\ln\langle e^{\alpha Y_t}\rangle\sim\Lambda_{\max}(\alpha)t,\tag{4}$$

*Electronic address: Eric.Brunet@physique.ens.fr
†Electronic address: Bernard.Derrida@physique.ens.fr

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and it was shown [11–14], when L is large and when the parameter α in Eq. (4) is of order $L^{-3/2}$ that $\Lambda_{\max}(\alpha)$ takes the following scaling form:

$$\Lambda_{\max}(\alpha)-\alpha K_1=K_2G(\alpha K_3), \tag{5}$$

where K_1, K_2, and K_3 are three constants which depend on the system size L, the density of particles, and the asymmetry.

The interesting aspect of Eq. (5) is that the function $G(\beta)$ is universal [12,14,16] in the sense that it does not depend on any of the microscopic parameters which define the model. It is given (in a parametric form) by

$$\beta=-\sum_{p=1}^{+\infty}\frac{\epsilon^p}{p^{3/2}}, \tag{6}$$

$$G(\beta)=-\sum_{p=1}^{+\infty}\frac{\epsilon^p}{p^{5/2}}. \tag{7}$$

In the correspondence [3] between the directed polymer problem and the asymmetric exclusion process through the KPZ equation, the role played by $\ln(Z(t))$ is the ratio Y_t/L. Comparing $\langle\exp(\alpha Y_t)\rangle$ and $\langle Z^n(t)\rangle$ in Eqs. (3) and (4), we see that n corresponds to αL and $E(n,L,\gamma)$ to $\Lambda_{\max}(\alpha)$. If the function $G(\beta)$ is characteristic of systems described by the KPZ equation, we expect in the scaling regime (large L and $n\sim L^{-1/2}$) a relation similar to Eq. (5) between $E(n,L,\gamma)$ [defined by Eq. (3)] and n. This is indeed one of the main results of the present work: when L is large and $n\sim L^{-1/2}$, we find

$$E(n,L,\gamma)=\frac{n\gamma^2}{24}-\frac{\sqrt{\gamma}}{2\sqrt{2\pi}L^{3/2}}G(-n\sqrt{2\pi L\gamma}). \tag{8}$$

It is clear that in order to establish this relation we have to calculate noninteger moments of the partition function.

The paper is organized as follows. In Sec. II, we recall how the replica approach of Eq. (1) can be formulated as a quantum problem with n particles on a ring and how this problem can be solved by the Bethe ansatz when the noise is δ correlated as in Eq. (2). In Sec. III, we write an integral equation (26) which, together with some symmetry conditions (27) and (28), allows us to solve the Bethe equations of Sec. II. The main advantage of Eq. (26) is that the strength c of the disorder (where $c=\gamma L/2$) and the number of the replica appear as continuous parameters. We show how expansions in powers of c or in powers of the number n of replica can be obtained from this integral equation. In the expansion of the energy $E(n,L,\gamma)$ in powers of c, all the coefficients are polynomials in n. This allows us to define $E(n,L,\gamma)$ for a noninteger n at least perturbatively in c. At the end of Sec. III, we show how to generate a small-n expansion which solves the integral equation (26). We also give explicit expressions up to order n^3 and we notice that in this small-n expansion of the energy, we have to deal with coefficients that are functions of c with a zero radius of convergence. The content of Secs. II and III is essentially a recall of a method developed in our previous work [17]. In Sec. IV, we show that the recursion of Sec. III, which generates all the terms of the small-n expansion, simplifies greatly in the scaling regime (c large and $n\sim c^{-1/2}$), allowing us to calculate all the terms of the expansion and to establish Eq. (8).

II. A QUANTUM SYSTEM OF *n* PARTICLES WITH δ INTERACTIONS

Let us start with a case slightly more general than Eq. (2) where the noise $\eta(x,t)$ in Eq. (1) is a Gaussian noise δ-correlated in time but with some given correlation v in space,

$$\langle\,\eta(x,t)\,\eta(x',t')\rangle=\gamma v(x-x')\,\delta(t-t'). \tag{9}$$

If we consider the correlation function $\langle Z(x_1,t)\,Z(x_2,t)\cdots Z(x_n,t)\rangle$ of the partition function $Z(x,t)$ at points $x_1, x_2,\ldots,x_n$, one can check [3] from Eqs. (1) and (9) that it satisfies

$$\frac{d}{dt}\langle Z(x_1,t)Z(x_2,t)\cdots Z(x_n,t)\rangle = -\tilde{\mathcal{H}}\langle Z(x_1,t)Z(x_2,t)\cdots Z(x_n,t)\rangle, \tag{10}$$

where the Hamiltonian $\tilde{\mathcal{H}}$ is given by

$$\tilde{\mathcal{H}}=-\frac{1}{2}\sum_\alpha\frac{\partial^2}{\partial x_\alpha^2}-\gamma\sum_{\alpha<\beta}v(x_\alpha-x_\beta)-\gamma\frac{n}{2}v(0), \tag{11}$$

and where, because of the cylinder geometry in the directed polymer problem, we have $x_\alpha\equiv x_\alpha+L$ for $1\leqslant\alpha\leqslant n$.

This implies that in the long-time limit,

$$\langle Z(x_1,t)Z(x_2,t)\cdots Z(x_n,t)\rangle\sim e^{-t\tilde{E}(n,L,\gamma)}, \tag{12}$$

where $\tilde{E}(n,L,\gamma)$ is the ground-state energy of Eq. (11).

If one takes the limit $v(x-x')\to\delta(x-x')$, the energy $\tilde{E}(n,L,\gamma)$ becomes infinite because of the constant part $nv(0)/2$ in Eq. (11). This divergence disappears, however, if we consider the ratio $\langle Z(x_1,t)Z(x_2,t)\cdots Z(x_n,t)\rangle/\Pi_\alpha\langle Z(x_\alpha,t)\rangle$, and one can see that in the long-time limit,

$$\frac{\langle Z(x_1,t)Z(x_2,t)\cdots Z(x_n,t)\rangle}{\langle Z(x_1,t)\rangle\langle Z(x_2,t)\rangle\cdots\langle Z(x_n,t)\rangle}\sim e^{-tE(n,L,\gamma)}, \tag{13}$$

where $E(n,L,\gamma)$ is the ground-state energy of the Hamiltonian

$$\mathcal{H}=-\frac{1}{2}\sum_\alpha\frac{\partial^2}{\partial x_\alpha^2}-\gamma\sum_{\alpha<\beta}\delta(x_\alpha-x_\beta), \tag{14}$$

where the positions x_α of the n particles are on a ring of length L.

Lieb and Liniger have shown that the Bethe ansatz allows us to calculate the ground-state energy $E(n,L,\gamma)$ of this one-dimensional quantum Hamiltonian exactly [18–24]. The Bethe ansatz consists in looking for a ground-state wave function $\Psi(x_1,\ldots,x_n)$ of Eq. (14) of the form

$$\Psi(x_1,\ldots,x_n)=\sum_P a_P e^{2(q_1x_{P(1)}+\cdots+q_nx_{P(n)})/L} \tag{15}$$

in the region $0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant L$. The sum in Eq. (15) runs over all the permutations P of $\{1,\ldots,n\}$ and the value of Ψ in other regions can be deduced from Eq. (15) by symmetries. One can show [22–24,17] that Eq. (15) is the ground-state wave function of Eq. (14) at energy

$$E(n,L,\gamma) = -\frac{2}{L^2}\sum_{1\leqslant\alpha\leqslant n} q_\alpha^2, \tag{16}$$

if the q_α are the solutions of the n coupled equations

$$e^{2q_\alpha} = \prod_{\beta\neq\alpha}\frac{q_\alpha - q_\beta + c}{q_\alpha - q_\beta - c}, \tag{17}$$

obtained by continuity from the solution $\{q_\alpha\}=\{0\}$ at $c=0$, where

$$c = \frac{\gamma L}{2}. \tag{18}$$

Moreover, the q_α are all different and the ground state is symmetric ($\{q_\alpha\}=\{-q_\alpha\}$). [See, for instance, [22]. Note that ik_j and c in [22] are here $(2/L)q_j$ and $-\gamma$; so our c defined by Eq. (18) and the c in [22] are different.]

If we introduce the polynomial $P(X)$,

$$P(X) = \prod_{q_\alpha}(X - q_\alpha), \tag{19}$$

the system of equations (17) becomes

$$e^{q_\alpha}P(q_\alpha - c) + e^{-q_\alpha}P(q_\alpha + c) = 0 \tag{20}$$

for any $1\leqslant\alpha\leqslant n$, and we have from the symmetry of the ground state

$$P(-X) = (-1)^n P(X). \tag{21}$$

The knowledge of the polynomial $P(X)$ determines the energy (16) as

$$P(X) = X^n - \frac{1}{2}\left(\sum_{1\leqslant\alpha\leqslant n} q_\alpha^2\right)X^{n-2} + \cdots \tag{22}$$

[using Eq. (19) and the fact that $\Sigma q_\alpha = 0$].

For small c, it is possible to solve directly Eq. (20) and to determine the q_α (see Appendix D). This leads to the following expression of the ground-state energy (16):

$$E(n,L,\gamma) = -\frac{2}{L^2}n(n-1)\left(\frac{c}{2} + \frac{c^2}{12} + \frac{nc^3}{180} + O(c^4)\right). \tag{23}$$

We see that the first coefficients of the small-c expansion are polynomial in n. In fact, following the approach of Appendix D, one can see that each coefficient of the small-c expansion of $E(n,L,\gamma)$ is polynomial in n, allowing us to define, at least perturbatively in c, the ground-state energy $E(n,L,\gamma)$ for noninteger n. The approach of Appendix D becomes, however, quickly complicated. This is why in the next section we develop a different approach [17] based on the integral equation (26).

III. SOLUTION OF THE BETHE ANSATZ USING AN INTEGRAL EQUATION

In this section we recall the approach developed in our previous work [17], which consists in writing an integral equation where c and n appear as continuous parameters and which allows us to expand the energy in powers of c as well as in powers of n.

Let us introduce the following function of $\{q_\alpha\}$:

$$B(u) = \frac{1}{n}e^{c(u^2-1)/4}\sum_{q_\alpha}\rho(q_\alpha)e^{q_\alpha(u-1)}, \tag{24}$$

where the parameters $\rho(q_\alpha)$ are defined by

$$\rho(q_\alpha) = \prod_{q_\beta\neq q_\alpha}\frac{q_\alpha - q_\beta + c}{q_\alpha - q_\beta}. \tag{25}$$

If the $\{q_\alpha\}$ are given by the solution of Eq. (17), which corresponds to the ground state, one can show (see Appendix A) that the function $B(u)$ satisfies the integral equation

$$B(1+u) - B(1-u) = nc\int_0^u dv\, e^{-c(v^2-uv)/2} \times B(1-v)B(1+u-v) \tag{26}$$

and the following two conditions:

$$B(1) = 1, \tag{27}$$

$$B(u) = B(-u). \tag{28}$$

Moreover, the energy (16) can be extracted from the knowledge of $B(u)$ through

$$E(n,L,\gamma) = \frac{2}{L^2}\left[\frac{n^3c^2}{6} + \frac{nc^2}{12} + \frac{nc}{2} - nB''(1)\right]. \tag{29}$$

The derivation of Eqs. (26)–(29) is given in Appendix A. We are now going to see how one can find perturbatively in c or in n the solution of Eqs. (26)–(28) and, consequently, the ground-state energy (29).

A. Expansion in powers of c

To obtain the small-c expansion of $B(u)$ for arbitrary n, we write

$$B(u) = B_0(u) + cB_1(u) + c^2B_2(u) + \cdots. \tag{30}$$

Conditions (27) and (28) impose that $B_0(0)=1$ and all $B_k(1)=0$ for $k>0$, and that the $B_k(u)$ are all even. Moreover, as can be seen directly from Eq. (17), the q_α scale like $\sqrt{c}$ when c is small. (Appendix D shows how to obtain the small-c expansion of the q_α.) This implies from the definition (24) of $B(u)$ that all the $B_k(u)$ are polynomials in u.

At zeroth order in c, Eq. (26) becomes

$$B_0(1+u) - B_0(1-u) = 0. \tag{31}$$

The only polynomial solution of Eq. (31) consistent with Eqs. (27) and (28), i.e., $B_0(u)=B_0(-u)$ and $B_0(1)=1$, is simply

$$B_0(u)=1 \tag{32}$$

for any u. We put this back into Eq. (26) and we get at first order in c

$$B_1(1+u)-B_1(1-u)=nu. \tag{33}$$

Again, there is a unique polynomial solution which satisfies the facts that $B_1(u)$ is even and that $B_1(1)=0$:

$$B_1(u)=\frac{n}{4}(u^2-1). \tag{34}$$

It is easy to see from Eq. (26) that at any order in c, we have to solve

$$B_k(1+u)-B_k(1-u)=\phi_k(u), \tag{35}$$

where $\phi_k(u)$ is a polynomial odd in u. There is a unique even polynomial $B_k(u)$ solution of Eq. (35) satisfying $B_k(1)=0$: it is one degree higher than $\phi_k(u)$ and can be determined by equating each power of u in both sides of Eq. (35). (Alternatively, we found a way of writing the solution for any $\phi_k(u)$:

$$B_k(u)=\Bigg[s_0\int_1^u dv\ \phi_k(v)+s_1[\phi_k'(u)-\phi_k'(1)] + s_2[\phi_k'''(u)-\phi_k'''(1)]+\cdots + s_p[\phi_k^{(2p-1)}(u)-\phi_k^{(2p-1)}(1)]+\cdots\Bigg]\Bigg/ 2, \tag{36}$$

where the s_k are the coefficients of the expansion of $x/\sinh x$ in powers of x (i.e., as $x/\sinh x=1-x^2/6+7x^4/360+\cdots$, one has $s_0=1,\ s_1=-1/6,\ s_2=7/360,\ldots$).)

This procedure gives for the first terms

$$B(u)=1+\frac{cn(u^2-1)}{4}+\frac{c^2n(2n+1)(u^2-1)^2}{96} + \frac{c^3n(u^2-1)^2(5n^2(u^2-1)+4n(2u^2-1)+2(u^2-3))}{5760}+O(c^4). \tag{37}$$

The energy can then be deduced from Eq. (29):

$$E(n,L,\gamma)=-2\frac{n(n-1)}{L^2}\left[\frac{c}{2}+\frac{c^2}{12}+\frac{n}{180}c^3 + \left(\frac{n^2}{1512}-\frac{n}{1260}\right)c^4+\cdots\right]. \tag{38}$$

[For Eq. (38), we used more terms than given above in $B(u)$.] Of course, this expression agrees with Eq. (23) obtained directly by expanding the q_α.

B. Expansion in powers of n

The number of particles n is *a priori* an integer. However, when we look at the small-c expansion (37) of $B(u)$ or Eq. (38) of the energy, we see that at any given order in c the expression is polynomial in n. Therefore, one can extend the definition of the small-c expansion of $B(u)$ or of $E(n,L,\gamma)$ to noninteger n. We can also collect in the small-c expansion of $B(u)$ all the terms proportional to n and call this series $b_1(u)$. From Eq. (37) we see that

$$b_1(u)=\frac{(u^2-1)}{4}c+\frac{(u^2-1)^2}{96}c^2 + \frac{(u^2-1)^2(u^2-3)}{2880}c^3+O(c^4). \tag{39}$$

More generally, we can collect all the terms proportional to n^k in the small-c expansion and call the series $b_k(u)$. This means that we can write $B(u)$ as a power series in n,

$$B(u)=1+nb_1(u)+n^2b_2(u)+\cdots, \tag{40}$$

where all the $b_k(u)$ are defined perturbatively in c. Conditions (27) and (28) impose that all the $b_k(u)$ are even and that $b_k(1)=0$ for all $k\geqslant 1$. We define $b_0(u)=1$ for consistency. [It is easy to see in the small-c expansion that if $n=0$, then $B(u)=1$.]

We are now going to describe the procedure we used [17] to determine the whole function $b_1(u)$ and eventually all the $b_k(u)$. If we insert Eq. (40) into Eq. (26) we get, at first order in n,

$$b_1(1+u)-b_1(1-u)=c\int_0^u e^{-c(v^2-uv)/2}\,dv. \tag{41}$$

It is easy to check that a solution of Eq. (41) compatible with the conditions $b_1(1)=0$ and $b_1(u)=b_1(-u)$ is

$$b_1(u)=\sqrt{c}\int_0^{+\infty}d\lambda\,\frac{\cosh\dfrac{\lambda u\sqrt{c}}{2}-\cosh\dfrac{\lambda\sqrt{c}}{2}}{\sinh\dfrac{\lambda\sqrt{c}}{2}}\,e^{-\lambda^2/2}. \tag{42}$$

There are, however, many other solutions of Eq. (41), which can be obtained by adding to Eq. (42) an arbitrary function $F(u,c)$ even and periodic in u of period 2 and vanishing at $u=1$. If we require that each term in the small-c expansion of $b_1(u)$ is polynomial in u (as justified in Sec. III A), we see that all the terms of the small-c expansion of $F(u,c)$ must be identically zero. This already shows that Eq. (42) has the same small c expansion (39) as one would get by collecting all the terms proportional to n in the small-c expansion of Sec. III A.

If the solution (42) of Eq. (41) had a nonzero radius of convergence in c, it would be natural to choose this solution and set $F(u,c)=0$. However, it is easy to see that Eq. (42) has a zero radius of convergence in c: by making the change of variable $\lambda^2=2\nu$, it is easy to see that Eq. (42) is the Borel sum of a divergent series [25].

Apart from being the Borel sum of its expansion in powers of c, we did not find definitive reasons why Eq. (42) is the solution of Eq. (41) we should select. However, we can notice that for integer n, all the q_α are real and $B(u)$ defined by Eq. (24) is analytic in u and remains bounded as $|\mathrm{Im}\,u|$

$\to\infty$. The solution $b_1(u)$ given by Eq. (42) is also analytic in u and grows as $\ln(u)$ as $|\mathrm{Im}\,u|\to\infty$. Adding any function $F(u,c)$ periodic and analytic in u to Eq. (42) would produce a much faster growth.

If we insert Eq. (40) into Eq. (26), we have to solve at order n^k

$$b_k(1+u)-b_k(1-u)=\varphi_k(u), \tag{43}$$

where $\varphi_k(u)$ is some function odd in u which can be calculated if we know the previous orders $b_1(u),\ldots,b_{k-1}(u)$,

$$\varphi_k(u)=c\sum_{i=0}^{k-1}\int_0^u dv\, e^{-c(v^2-uv)/2}b_i(1-v) \times b_{k-i-1}(1+u-v). \tag{44}$$

We see that the difficulty of selecting a solution of a difference equation appears at all orders in the expansion in powers of n, and we are now going to explain the procedure we have used to select one solution.

If we write, as $\varphi_k(u)$ is an odd function of u,

$$\varphi_k(u)=2\int_0^{+\infty}d\lambda\,\sinh\frac{\lambda u\sqrt{c}}{2}\;a_k(\lambda), \tag{45}$$

which is equivalent, by inverting when u is imaginary the Fourier transform in Eq. (45), to define $a_k(\lambda)$ by

$$a_k(\lambda)=\frac{1}{2i\pi}\int_0^{+\infty}du\,\sin\frac{\lambda u}{2}\;\varphi_k\left(\frac{iu}{\sqrt{c}}\right), \tag{46}$$

then the solution for $b_k(u)$ we select is given by

$$b_k(u)=\int_0^{+\infty}d\lambda\,\frac{\cosh\frac{\lambda u\sqrt{c}}{2}-\cosh\frac{\lambda\sqrt{c}}{2}}{\sinh\frac{\lambda\sqrt{c}}{2}}a_k(\lambda). \tag{47}$$

Indeed, $b_k(u)$ is an even function, vanishes at $u=1$, and one can check using Eq. (45) that Eq. (47) solves Eq. (43).

The integrals in Eqs. (45)–(47) are convergent [17] and Eqs. (44)–(47) give an automatic way of calculating the $b_k(u)$ up to any desired order.

This procedure is the direct generalization of the choice (42) we did to solve Eq. (41). In fact, for $k=1$, Eqs. (44) and (46) give (for $\lambda\geqslant 0$) $a_1(\lambda)=\sqrt{c}\exp(-\lambda^2/2)$ and Eq. (47) is identical to Eq. (42).

As for Eq. (42), the solution (47) is not the only solution of Eq. (43). At any order k, we could add an arbitrary even periodic function $F(u,c)$ of period 2, the expansion of which vanishes to all orders in c. As for $b_1(u)$, we did not find an unquestionable justification of our choice. One can notice, nevertheless, that Eq. (47) is the solution of Eq. (43) analytic in u and with the slowest growth with u in the imaginary direction.

At order n^2, the procedure (44) and (46) gives

$$a_2(\lambda)=ce^{-\lambda^2/2}\left[\int_0^{\lambda}d\mu\, e^{-\mu^2/2}\frac{2\cosh\frac{\lambda\mu}{2}-2}{\tanh\frac{\mu\sqrt{c}}{2}} + \int_{\lambda}^{+\infty}d\mu\, e^{-\mu^2/2}\frac{e^{-\lambda\mu/2}-2}{\tanh\frac{\mu\sqrt{c}}{2}}\right], \tag{48}$$

with $b_2(u)$ given by Eq. (47). Writing down $b_3(u)$ or $a_3(u)$ would take here about half a column.

We can now give the first terms in the small-n expansion of the energy. Using relation (29), we find

$$\frac{L^2}{2}E(n,L,\gamma)=n\left(\frac{c}{2}+\frac{c^2}{12}\right)-n^2\frac{c^{3/2}}{4}\int_0^{+\infty}d\lambda\,\frac{\lambda^2}{\tanh\frac{\lambda\sqrt{c}}{2}}e^{-\lambda^2/2}-n^3\frac{c^2}{4}\int_0^{+\infty}d\lambda\,\frac{\lambda^2}{\tanh\frac{\lambda\sqrt{c}}{2}}e^{-\lambda^2/2} \times\left(\int_0^{\lambda}d\mu\, e^{-\mu^2/2}\frac{2\cosh\frac{\lambda\mu}{2}-2}{\tanh\frac{\mu\sqrt{c}}{2}}+\int_{\lambda}^{+\infty}d\mu\, e^{-\mu^2/2}\frac{e^{-\frac{\lambda\mu}{2}}-2}{\tanh\frac{\mu\sqrt{c}}{2}}\right)+\frac{n^3c^2}{6}+O(n^4). \tag{49}$$

By making the change of variable $\lambda^2=2\nu$, the terms of order n^2 and n^3 appear as Borel transforms of series in c with a finite radius of convergence. We conclude that these terms both have a zero radius of convergence in c.

This small-n expansion gives quickly very complicated expressions of $b_k(u)$. It turns out, as we shall see in the next section, that for large c, the expressions of the $b_k(u)$ get simpler and the energy $E(n,L,\gamma)$ can be calculated to all orders in powers of n.

IV. EXPANSION IN POWERS OF n IN THE REGIME $c\to\infty$

In the preceding section, we have developed a procedure allowing to get the small-n expansion of the energy by solving the problem (26)–(28). Here, we show how this procedure becomes greatly simplified for large c.

The expansion in powers of n of the preceding section can be summarized as follows: if we use Eq. (40) and we write

$$a(\lambda)=na_1(\lambda)+n^2a_2(\lambda)+\cdots, \tag{50}$$

the $b_k(u)$ and $a_k(\lambda)$ can be obtained by expanding in powers of n the following two equations:

$$B(u)=1+\int_0^{+\infty}d\lambda\,\frac{\cosh\dfrac{\lambda u\sqrt{c}}{2}-\cosh\dfrac{\lambda\sqrt{c}}{2}}{\sinh\dfrac{\lambda\sqrt{c}}{2}}\,a(\lambda) \tag{51}$$

[this is a rewriting of Eq. (47)] and

$$a(\lambda)=\frac{nc}{2i\pi}\int_0^{+\infty}du\,\sin\frac{\lambda u}{2}\int_0^{iu/\sqrt{c}}dv\,e^{-c(v^2-iuv/\sqrt{c})/2}\times B(1-v)B\left(1+\frac{iu}{\sqrt{c}}-v\right). \tag{52}$$

[This is a rewriting of Eqs. (44) and (46).] It will be convenient in the following to replace Eq. (52) by its Fourier transform,

$$2\int_0^{+\infty}d\lambda\,\sinh\frac{\lambda u\sqrt{c}}{2}\,a(\lambda)=nc\int_0^{u}dv\,e^{-c(v^2-uv)/2}B(1-v)B(1+u-v). \tag{53}$$

[This is a rewriting of Eqs. (44) and (45).]

We are going to see how one can simplify Eqs. (51)–(53) when c is large. First we observe that for large c and u fixed of order 1, the expression $b_1(u)$ takes the scaling form

$$b_1\left(1+\frac{u}{\sqrt{c}}\right)\simeq\sqrt{c}\int_0^{+\infty}(e^{\lambda u/2}-1)e^{-\lambda^2/2}\,d\lambda. \tag{54}$$

One can check from Eqs. (44), (46), and (47) that this scaling form is present at any order in the small-n expansion. Indeed, Eq. (51) becomes in the large-c limit

$$B\left(1+\frac{u}{\sqrt{c}}\right)=1+\int_0^{+\infty}d\lambda\,(e^{\lambda u/2}-1)a(\lambda), \tag{55}$$

and using Eq. (53) we find

$$2\int_0^{+\infty}d\lambda\,\sinh\frac{\lambda u}{2}a(\lambda)=n\sqrt{c}\int_0^{u}dv\,e^{-(v^2-uv)/2}B\left(1-\frac{v}{\sqrt{c}}\right)\times B\left(1+\frac{u-v}{\sqrt{c}}\right). \tag{56}$$

It is apparent from Eqs. (55) and (56) that in the large-c limit the function $B(1+u/\sqrt{c})$ depends only on u and $n\sqrt{c}$, and $a(\lambda)$ depends only on λ and $n\sqrt{c}$. Let us introduce the constant K,

$$K=1-\int_0^{+\infty}d\lambda\,a(\lambda). \tag{57}$$

Equation (55) becomes

$$B\left(1+\frac{u}{\sqrt{c}}\right)=K+\int_0^{+\infty}d\lambda\,e^{\lambda u/2}a(\lambda). \tag{58}$$

In Eq. (56), if we write the integral from 0 to u as the difference between an integral from 0 to $+\infty$ and an integral from u to $+\infty$, and if we change the variable in the second integral to shift it to 0 to $+\infty$, we obtain

$$2\int_0^{+\infty}d\lambda\,\sinh\frac{\lambda u}{2}a(\lambda)=n\sqrt{c}\int_0^{+\infty}dv\,e^{-v^2/2}B\left(1-\frac{v}{\sqrt{c}}\right)\times\left[e^{uv/2}B\left(1+\frac{u-v}{\sqrt{c}}\right)-e^{-uv/2}\times B\left(1-\frac{u+v}{\sqrt{c}}\right)\right]. \tag{59}$$

If we replace $B[1+(u-v)/\sqrt{c}]$ and $B[1-(u+v)/\sqrt{c}]$ by their expression (58), we get after some rearrangements

$$2\int_0^{+\infty}d\lambda\,\sinh\frac{\lambda u}{2}a(\lambda)=n\sqrt{c}\int_0^{+\infty}dv\,e^{-v^2/2}B\left(1-\frac{v}{\sqrt{c}}\right)\times\left[2K\sinh\frac{uv}{2}+\int_0^{+\infty}d\mu\,a(\mu)\times e^{-\mu v/2}2\sinh\left(u\,\frac{v+\mu}{2}\right)\right]. \tag{60}$$

Taking the Fourier transform of this expression for imaginary u, we get for $\lambda\geqslant 0$

$$a(\lambda)=n\sqrt{c}\int_0^{+\infty}dv\,e^{-v^2/2}B\left(1-\frac{v}{\sqrt{c}}\right)\times\left[K\delta(\lambda-v)+\int_0^{+\infty}d\mu\,a(\mu)e^{-\mu v/2}\delta(\lambda-v-\mu)\right]. \tag{61}$$

This last expression can be used to calculate $B(1+u/\sqrt{c})$ using Eq. (58):

$$B\left(1+\frac{u}{\sqrt{c}}\right)=K+n\sqrt{c}\int_0^{+\infty}dv\, e^{-v^2/2}B\left(1-\frac{v}{\sqrt{c}}\right)\times\left[Ke^{vu/2}+\int_0^{+\infty}d\mu\, a(\mu)e^{-\mu v/2}e^{(v+\mu)u/2}\right]. \tag{62}$$

Finally, using Eq. (58), we recognize the relation

$$B\left(1+\frac{u}{\sqrt{c}}\right)=K+n\sqrt{c}\int_0^{+\infty}dv\, e^{-v^2/2}B\left(1-\frac{v}{\sqrt{c}}\right)\times e^{vu/2}B\left(1+\frac{u-v}{\sqrt{c}}\right). \tag{63}$$

We see that, in the large-c limit, Eqs. (51) and (52) reduce to this single equation (63). We are now going to see that Eq. (63) can be solved to all orders in the parameter $n\sqrt{c}$. If we introduce the function $\beta(u)$ and the parameter ϵ defined by

$$\beta(u)=\frac{1}{2K\sqrt{\pi}}e^{-u^2/4}B\left(1+\frac{u}{\sqrt{c}}\right) \tag{64}$$

and

$$\epsilon=2nK\sqrt{\pi c}, \tag{65}$$

then Eq. (63) simply becomes

$$\beta(u)=\frac{1}{2\sqrt{\pi}}e^{-u^2/4}+\epsilon\int_0^{+\infty}dv\,\beta(u-v)\beta(-v). \tag{66}$$

Using Eqs. (27), (29), and (64), we can express the ground-state energy $E(n,L,\gamma)$ in terms of $\beta(u)$:

$$E(n,L,\gamma)=\frac{2}{L^2}\left[\frac{n^3c^2}{6}+\frac{nc^2}{12}-nc\frac{\beta''(0)}{\beta(0)}\right]. \tag{67}$$

It is clear that relation (66) alone determines $\beta(u)$, at least perturbatively in ϵ. So, from Eq. (67), we only need to extract $\beta(0)$ and $\beta''(0)$ from Eq. (66).

It is easy to do it for the first orders in ϵ directly from Eq. (66). Moreover, we have found a way of calculating $\beta(0)$ and $\beta''(0)$, and hence the energy, to all orders in ϵ. This calculation is technical and we present it in Appendix B. The final result can be written as

$$n\sqrt{c}=\frac{1}{2\sqrt{\pi}}\sum_{k=1}^{+\infty}\frac{\epsilon^k}{k^{3/2}}, \tag{68}$$

$$E(n,L,\gamma)=\frac{2}{L^2}\left[\frac{nc^2}{12}+\frac{\sqrt{c}}{4\sqrt{\pi}}\sum_{k=1}^{+\infty}\frac{\epsilon^k}{k^{5/2}}\right]. \tag{69}$$

We see that the energy is defined in an implicit way: expression (68) allows us to calculate ϵ as a function of $n\sqrt{c}$, and Eq. (69) gives the energy as a function of ϵ. If we substitute c using Eq. (18), we obtain the result announced in Eq. (8).

For small n, one can eliminate ϵ from Eqs. (68) and (69). We get

$$\frac{L^2}{2}E(n,L,\gamma)-\frac{nc^2}{12}=\frac{\sqrt{c}}{4\sqrt{\pi}}\left[2n\sqrt{c\pi}-\frac{\sqrt{2}}{8}(2n\sqrt{c\pi})^2+\left(\frac{1}{8}-\frac{2\sqrt{3}}{27}\right)(2n\sqrt{c\pi})^3+O((n\sqrt{c})^4)\right]. \tag{70}$$

V. CONCLUSION

In this paper, we have calculated, using the replica method, the first cumulants (13) and (49) of the free energy of a directed polymer in a random medium (1) for a cylinder geometry. We used the integral equation (26) of [17] which together with conditions (27) and (28) allowed us to expand the moments $\langle Z^n\rangle$ of the partition function in powers of the strength c of the disorder or in powers of the number n of the replica. All the coefficients of the small-c expansion (38) are polynomial in n, allowing us to define the expansions for noninteger n. On the other hand, the coefficients of the expansion (49) in powers of n are complicated functions of c, with in general a zero radius of convergence at $c=0$. As already mentioned in [17], we think that weak disorder expansions of the moments $\langle Z^n\rangle$ have generically a zero radius of convergence for noninteger n when the disorder is Gaussian; this is already the case for a single Ising spin in a Gaussian random field.

To obtain our small-n expansion, we solved a difference equation (26) which at each order in powers of n has several solutions. We selected the particular solution which has the slowest growth in the imaginary u direction and has the right small-c expansion, but we could not exclude other solutions. A different approach, with a direct calculation of the first cumulants of the free energy, and not based on the replica, would therefore be very useful to test the validity of our expressions (49), which we have been able to derive only perturbatively to all orders in c.

Although our expansion in powers of n becomes quickly very complicated, it simplifies when c is large and we could write in this limiting case all the terms of the small-n expansion (68) and (69). The expression (8) we obtain of the energy $E(n,L,\gamma)$ [that is, through Eq. (3), the expression of $\langle Z^n\rangle$] is given exactly by the same scaling function as found for the ASEP. The present work therefore gives additional evidence that the scaling function $G(\beta)$ given by Eqs. (6) and (7) is characteristic of the long-time behavior of the KPZ equation in $1+1$ dimensions on a ring and that the probability distribution of the free energy for a very long directed polymer on a ring should have a universal shape in the range where the fluctuations per unit length of the free energy are of order $1/L$. Other universal distributions for the free energy of a directed polymer have been found recently for different geometries [26–30]. Our present approach, based on the Bethe ansatz, is, at the moment, unable to recover these other distributions. One can try, however, to extend it to open boundary conditions (in this case too, the Bethe ansatz can be used [24]) instead of periodic boundary conditions and see how this change of boundary conditions affects the distribution of $\ln Z$. Of course, it would be very nice to find a simpler approach which would somehow unify all these results and allow us to relate all these universal distributions corresponding to the possible geometries, in the spirit of critical phenomena in two dimensions where conformal invariance [31] allows us to connect the properties of different geometries.

Technically, the approach followed in the present work is simply to try to find the q_α solution of Eq. (17) and to calculate the energy (16), which is a symmetric function of the roots q_α, in such a way that n becomes a continuous variable. One could do the same in all kinds of situations. For example, in Appendix C, we show how to define and calculate symmetric functions of the roots of Hermite polynomials when the degree of the polynomial becomes noninteger.

Another interesting extension of the present work would be to consider more general correlations of the noise (9). The corresponding quantum problem becomes then the general problem of quantum particles interacting with an arbitrary pair potential. If the interactions are short ranged, one expects the universality class of the KPZ equation to hold, so one could try to repeat our expansion in powers of c for a general potential (without the use of the Bethe ansatz) simply by a standard perturbation theory in the strength of the potential. We believe that at any order in the strength of the potential, the ground-state energy is polynomial in n allowing us to define the perturbation expansion for noninteger n as we did here. If, with such an approach based on perturbation theory, one could recover the scaling function G of Eqs. (6) and (7), one could try to extend the approach to higher dimension as the relation between the directed polymer problem and the quantum Hamiltonian is valid in any dimension.

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APPENDIX A: DERIVATION OF EQS. (26)–(29)

Let us first establish some useful properties of the numbers $\rho(q_\alpha)$ defined by Eq. (25). If the q_α are the n roots of the polynomial $P(X)$,

$$P(X)=\prod_{q_\alpha}(X-q_\alpha), \tag{A1}$$

it is easy to see that the $\rho(q_\alpha)$ defined in Eq. (25) satisfy

$$\frac{P(X+c)}{P(X)}=1+c\sum_{q_\alpha}\frac{\rho(q_\alpha)}{X-q_\alpha}. \tag{A2}$$

(The two sides have the same poles with the same residues and coincide at $X\to\infty$.) Expanding the right-hand side of Eq. (A2) for large X, we get

$$\frac{P(X+c)}{P(X)}=1+c\sum_{q_\alpha}\frac{\rho(q_\alpha)}{X}\left(1+\frac{q_\alpha}{X}+\frac{q_\alpha^2}{X^2}\right)+O\left(\frac{1}{X^4}\right). \tag{A3}$$

On the other hand, using Eqs. (16) and (A1) and the symmetry $\{q_\alpha\}=\{-q_\alpha\}$, we have

$$P(X)=X^n+\frac{L^2}{4}E(n,L,\gamma)X^{n-2}+O(X^{n-4}), \tag{A4}$$

so that

$$\frac{P(X+c)}{P(X)}=1+\frac{nc}{X}+\frac{c^2\binom{n}{2}}{X^2}+\frac{c^3\binom{n}{3}-cE(n,L,\gamma)L^2/2}{X^3}+O\left(\frac{1}{X^4}\right). \tag{A5}$$

Comparing Eqs. (A3) and (A5), we get the relations

$$\sum_{q_\alpha}\rho(q_\alpha)=n, \tag{A6}$$

$$\sum_{q_\alpha}q_\alpha\rho(q_\alpha)=c\binom{n}{2}, \tag{A7}$$

$$\sum_{q_\alpha}q_\alpha^2\rho(q_\alpha)=c^2\binom{n}{3}-\frac{E(n,L,\gamma)L^2}{2}. \tag{A8}$$

Moreover, by letting $X=\pm q_\beta-c$ in Eq. (A2), we get for any q_β root of $P(X)$

$$\frac{1}{c}=\sum_{q_\alpha}\frac{\rho(q_\alpha)}{q_\alpha-q_\beta+c}=\sum_{q_\alpha}\frac{\rho(q_\alpha)}{q_\alpha+q_\beta+c}. \tag{A9}$$

Lastly, using the symmetry $\{q_\alpha\}=\{-q_\alpha\}$ and the definition (25), the Bethe ansatz equations (17) reduce to

$$e^{q_\alpha}\rho(-q_\alpha)-e^{-q_\alpha}\rho(q_\alpha)=0. \tag{A10}$$

From the definition (24) of $B(u)$ and the properties (A6)–(A10), it is straightforward to establish Eqs. (26)–(29): the integral equation (26) is a direct consequence of Eqs. (24) and (A9). Properties (27) and (28) follow from Eqs. (24) and (A6) and Eqs. (24) and (A10), respectively. Lastly, Eq. (29) is a consequence of Eqs. (24) and (A6)–(A8).

APPENDIX B: THE ENERGY IN THE SCALING REGIME

In this appendix, we show how to calculate the energy from the integral equation (66). This equation is of the form

$$\beta(u)=H(u)+\epsilon\int_0^{+\infty}dv\ \beta(u-v)\beta(-v), \tag{B1}$$

where, in our case, $H(u)$ is given by

$$H(u)=\frac{1}{2\sqrt{\pi}}e^{-u^2/4}. \tag{B2}$$

We are going to do our calculations for an arbitrary function $H(u)$, even in u and decreasing fast enough (to make all the integrals converge) when $|u|\to\infty$.

To find the energy, we see from Eq. (67) that we have to calculate from Eq. (B1) the quantities $\beta(0)$ and $\beta''(0)$ as functions of ϵ. We first show that Eq. (B1) is equivalent to

$$\beta(u) = H(u) + \epsilon \int_0^{+\infty} dv\, H(u-v)\beta(v), \tag{B3}$$

as long as $H(u)$ is even and decreases fast enough. Then, we will introduce a new function $\beta^*(u)$ which is easy to calculate, and relate the derivatives of $\beta(u)$ and $\beta^*(u)$ at $u=0$.

1. Equivalence between Eqs. (B1) and (B3)

The solution of Eq. (B3) can be written as

$$\beta(u) = \beta_0(u) + \epsilon\beta_1(u) + \epsilon^2\beta_2(u) + \cdots, \tag{B4}$$

where

$$\begin{aligned} \beta_0(u) &= H(u), \\ \beta_1(u) &= \int_0^{+\infty} H(u-v_1)H(v_1)\, dv_1, \\ \beta_2(u) &= \iint_0^{+\infty} H(u-v_1)H(v_1-v_2)H(v_2)\, dv_1 dv_2, \\ &\cdots \\ \beta_k(u) &= \int \cdots \int_0^{+\infty} H(u-v_1)H(v_1-v_2)\cdots \\ &\quad \cdots H(v_{k-1}-v_k)H(v_k)\, dv_1 \cdots dv_k . \end{aligned} \tag{B5}$$

For a given $k>0$, the integration range of $\beta_k(u)$ can be divided into k parts: the region where v_1 has the lowest value of all the $\{v_i\}$, the region where v_2 has the lowest value, . . . , and the region where v_k has the lowest value. Let us consider, for some j such that $1 \leqslant j \leqslant k$, the region where v_j has the lowest value. All the other integrals then run from v_j to $+\infty$. If we translate those to integrals running from 0 to $+\infty$ by changing v_i into $v_i + v_j$, we get

$$\begin{aligned} &\int_0^{+\infty} dv_j \int_0^{+\infty} dv_1 \cdots dv_{j-1}\, H(u-v_1-v_j) \\ &\times H(v_1-v_2)\cdots H(v_{j-1}) \int_0^{+\infty} dv_{j+1} \cdots dv_k \\ &\times H(-v_{j+1})H(v_{j+1}-v_{j+2})\cdots H(v_k+v_j). \end{aligned} \tag{B6}$$

Using the fact that $H(u)=H(-u)$, we see that Eq. (B6) is equal to

$$\int_0^{+\infty} dv_j\, \beta_{j-1}(u-v_j)\beta_{k-j}(-v_j). \tag{B7}$$

By summing over j, we therefore have

$$\beta_k(u) = \int_0^{+\infty} dv \sum_{j=1}^{k} \beta_{j-1}(u-v)\beta_{k-j}(-v). \tag{B8}$$

Finally, if we multiply by ϵ^k and if we sum over k all these terms (keeping apart the term for $k=0$), we obtain Eq. (B1).

Equations (B1) and (B3) are thus equivalent and Eqs. (B4) and (B5) give the solution of Eq. (B1) to any order in ϵ.

2. Calculation of the derivatives of $\beta(u)$

If we look at the expression (B5) of $\beta(u)$ in powers of ϵ, the calculation of $\beta(0)$ and $\beta''(0)$ looks simple, especially when $H(u)$ is given by Eq. (B2). However, when we try to actually do the calculation, the expressions become quickly complicated with error functions, primitives of error functions, etc. It would be much easier if the integrals in Eq. (B5) were running from $-\infty$ to $+\infty$ instead of 0 to $+\infty$. This is why we introduce the even function

$$\beta^*(u) = \beta_0^*(u) + \epsilon\beta_1^*(u) + \epsilon^2\beta_2^*(u) + \cdots, \tag{B9}$$

where, for $k>0$,

$$\beta_k^*(u) = \frac{1}{k+1}\int \cdots \int_{-\infty}^{+\infty} H(u-v_1)\cdots H(v_k)\, dv_1 \cdots dv_k \tag{B10}$$

and $\beta_0^*(u) = H(u)$. One can see easily that

$$\beta^*(u) = \frac{-1}{2\pi\epsilon}\int_{-\infty}^{+\infty} dq\, e^{-iqu} \ln[1-\epsilon\hat{H}(q)], \tag{B11}$$

where we have defined

$$\hat{H}(q) = \int_{-\infty}^{+\infty} du\, e^{iqu} H(u). \tag{B12}$$

The Wiener-Hopf technique [32] allows us to relate $\beta(u)$ and $\beta^*(u)$. More specifically, we are going to show that for any $X>0$,

$$\epsilon \int_0^{+\infty} du\, e^{-uX}\beta^*(u) = \ln\left(1+\epsilon\int_0^{+\infty} du\, e^{-uX}\beta(u)\right). \tag{B13}$$

This relation allows us to relate the derivatives of $\beta(u)$ and $\beta^*(u)$ at $u=0$: indeed, if X is large in Eq. (B13), we get

$$\int_0^{+\infty} du\, e^{-uX}\beta(u) = \frac{\beta(0)}{X} + \frac{\beta'(0)}{X^2} + \frac{\beta''(0)}{X^3} + \cdots \tag{B14}$$

and a similar expression for $\beta^*(u)$. Comparing both sides of Eq. (B13) gives

$$\begin{aligned} \beta(0) &= \beta^*(0), \\ \beta'(0) &= \frac{\epsilon}{2}\beta(0)^2, \\ \beta''(0) &= \beta^{*\prime\prime}(0) + \frac{\epsilon^2}{6}\beta(0)^3. \end{aligned} \tag{B15}$$

[We have used the fact that $\beta^{*\prime}(0)=0$ because $\beta^*(u)$ is an even function.]

In order to prove Eq. (B13), the first thing to note is that, as $H(u)$ decreases fast when $u\to\pm\infty$, then also does $\beta(u)$. This allows us to define the two ''partial'' Fourier transforms

$$\hat\beta_+(q)=\int_0^{+\infty}du\, e^{iqu}\beta(u), \tag{B16}$$

$$\hat\beta_-(q)=\int_{-\infty}^{0}du\, e^{iqu}\beta(u). \tag{B17}$$

It is easy to see that $\hat\beta_+(q)$ is analytic in the upper half-plane $(\operatorname{Im} q\geqslant 0)$. Moreover, in this half-plane, $\hat\beta_+(q)$ is bounded and vanishes when $|q|\to\infty$. Conversely, $\hat\beta_-(q)$ is analytic, bounded, and decreases to 0 at infinity when $\operatorname{Im} q\leqslant 0$.

The function $\beta(u)$ can be written in terms of $\hat\beta_+(q)$ and $\hat\beta_-(q)$:

$$\beta(u)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq\, e^{-iqu}[\hat\beta_+(q)+\hat\beta_-(q)], \tag{B18}$$

which allows us to express the right-hand side of Eq. (B13) when x is positive,

$$\int_0^{+\infty}du\, e^{-uX}\beta(u)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq\,\frac{\hat\beta_+(q)}{X+iq}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq\,\frac{\hat\beta_-(q)}{X+iq}. \tag{B19}$$

We calculate the two integrals in the right-hand side of Eq. (B19) by the residue theorem. As $\hat\beta_+(q)$ is analytic and decreases at infinity in the upper half-plane, the first integral can be written as a contour integral around the upper half-plane. The only contribution to the first integral comes, using Cauchy's theorem, from the pole $q=iX$. One can also check that the second integral vanishes [using a contour around the lower half-plane and the fact that $\hat\beta_-(q)$ has no pole]. Therefore, Eq. (B19) gives

$$\int_0^{+\infty}du\, e^{-uX}\beta(u)=\hat\beta_+(iX). \tag{B20}$$

Now, if we multiply Eq. (B3) by exp(iqu) and if we integrate over u, we easily get for any real q

$$\hat\beta_+(q)+\hat\beta_-(q)=\hat H(q)+\epsilon\hat H(q)\hat\beta_+(q). \tag{B21}$$

This relation between $\hat H(q)$, $\hat\beta_-(q)$, and $\hat\beta_+(q)$, together with Eq. (B11), gives

$$\beta^*(u)=\frac{1}{2\pi\epsilon}\int_{-\infty}^{+\infty}dq\, e^{-iqu}\{\ln[1+\epsilon\hat\beta_+(q)]-\ln[1-\epsilon\hat\beta_-(q)]\}. \tag{B22}$$

Using again that, in the upper half-plane, $\hat\beta_+(u)$ is analytic and vanishes at infinity, we see that, for a *small enough* ϵ, the quantity $\ln[1+\epsilon\hat\beta_+(q)]$ is also analytic and decreases to 0 at infinity when $\operatorname{Im} q\geqslant 0$. Similarly, $\ln[1-\epsilon\hat\beta_-(q)]$ has the same properties for $\operatorname{Im} q\leqslant 0$. This allows us to calculate the left-hand side of Eq. (B13) as we did for the right-hand side. We find

$$\int_0^{+\infty}du\, e^{-uX}\beta^*(u)=\frac{1}{\epsilon}\ln[1+\epsilon\hat\beta_+(iX)]. \tag{B23}$$

Comparing Eqs. (B20) and (B23) completes the proof of Eq. (B13).

We can now give an expression of the energy. If we use the definition (B2) of $H(u)$ in Eqs. (B11) and (B12), we find

$$\beta^*(u)=\frac{1}{2\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{\epsilon^k}{(k+1)^{3/2}}\, e^{-u^2/[4(k+1)]}. \tag{B24}$$

This gives

$$\beta^*(0)=\frac{1}{2\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{\epsilon^k}{(k+1)^{3/2}}, \tag{B25}$$

$$\beta^{*\prime\prime}(0)=-\frac{1}{4\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{\epsilon^k}{(k+1)^{5/2}}, \tag{B26}$$

and, together with Eq. (B15), these equations allow us to give an expression of $\beta(0)$ and $\beta''(0)$.

From Eqs. (27), (64), and (65), we see that

$$\epsilon\beta(0)=n\sqrt{c}. \tag{B27}$$

Then, using Eq. (B15), we get

$$\epsilon\beta^*(0)=n\sqrt{c}, \tag{B28}$$

$$\frac{\beta''(0)}{\beta(0)}=\frac{\epsilon}{n\sqrt{c}}\beta^{*\prime\prime}(0)+\frac{n^2c}{6}.$$

The energy is given by Eq. (67). We get

$$E(n,L,\gamma)=\frac{2}{L^2}\left[\frac{nc^2}{12}-\epsilon\sqrt{c}\beta^{*\prime\prime}(0)\right]. \tag{B29}$$

And, finally, using relations (B25) and (B26), we obtain Eqs. (68) and (69).

APPENDIX C: HERMITE POLYNOMIALS WITH A NONINTEGER NUMBER OF ROOTS

What we try to do in this whole paper is essentially to calculate $\Sigma_\alpha q_\alpha^2$ (the energy) where $\{q_\alpha\}$ is a solution of Eq. (17), in such a way that n appears as a continuous parameter. This allows us to obtain expressions of the energy for noninteger n.

One can use the same procedure in other kinds of situations. A simple example which illustrates our calculations is the case of the zeros of Hermite polynomials.

The nth Hermite polynomial $H_n(X)$ is the solution polynomial in X with leading coefficient 1 of the differential equation [33]

$$\frac{1}{2}H_n''(X)-XH_n'(X)+nH_n(X)=0. \tag{C1}$$

The polynomial $H_n(x)$ is of degree n and has the symmetry $H_n(X)=(-)^n H_n(-X)$. For example, we have $H_4(X)=X^4-3X^2+\frac{3}{4}$. The n roots $\{h_\alpha\}$ $(1\leqslant\alpha\leqslant n)$ of $H(X)$ are real and distinct [34].

By deriving Eq. (C1) p times with respect to X, we see that, for all p,

$$XH_n^{(p+1)}(X)=\frac{1}{2}H_n^{(p+2)}(X)+(n-p)H_n^{(p)}(X). \tag{C2}$$

This shows that the $(n-p)$th Hermite polynomial is, up to a constant factor, equal to the pth derivative of $H_n(X)$. (This property will be used a lot in Appendix D.)

Equation (C1) can be used directly to calculate the first coefficients of $H_n(X)$,

$$H_n(X)=X^n-\frac{1}{2}\binom{n}{2}X^{n-2}+\frac{3}{4}\binom{n}{4}X^{n-4}+\cdots. \tag{C3}$$

Using Eq. (C3), the symmetry of $H(X)$, and the large X expansion,

$$\frac{H_n'(X)}{H_n(X)}=\sum_{p\geqslant 0}\frac{1}{X^{p+1}}\left(\sum_\alpha h_\alpha^p\right), \tag{C4}$$

we can calculate the moments of the roots $\{h_\alpha\}$ of $H(X)$:

$$\sum_\alpha h_\alpha^2=\frac{n(n-1)}{2}, \tag{C5}$$

$$\sum_\alpha h_\alpha^4=\frac{n(n-1)}{4}(2n-3), \tag{C6}$$

and so on. These moments are *a priori* defined only for integer n but as the expressions are polynomial in n, one can obviously extend their definition to noninteger n [similarly to what we do in the small-c expansion of $B(u)$ in Sec. (III B)].

To generate all the moments of the roots h_α, it is convenient to consider the generating function

$$Q(u)=\sum_{h_\alpha}e^{h_\alpha u}, \tag{C7}$$

which is quite reminiscent of the quantity $\beta(u)$ defined in our quantum problem. [Using Eqs. (24) and (64), we can check that $\beta(u)\propto\exp(u\sqrt{c}/2)\Sigma\rho(q_\alpha)\exp(q_\alpha u/\sqrt{c})$.]

The function $Q(u)$ is hard to calculate for general n but we can expand it in powers of n. This can be done by considering

$$\Psi(X)=\frac{H_n'(X)}{H_n(X)}=\int_0^{+\infty}du\,Q(u)e^{-uX}, \tag{C8}$$

which is defined only for X positive and large enough to make the integral converge. This function $\Psi(X)$ is solution of a differential equation which follows from Eq. (C1):

$$\frac{1}{2}\Psi'(X)+\frac{1}{2}\Psi(X)^2-X\Psi(X)+n=0. \tag{C9}$$

To obtain an expansion in powers of n, we write

$$\Psi(X)=n\Psi_1(X)+n^2\Psi_2(X)+\cdots. \tag{C10}$$

Thus $\Psi_1(X)$ satisfies

$$\frac{1}{2}\Psi_1'(X)-X\Psi_1(X)+1=0. \tag{C11}$$

This differential equation can easily be solved, and the integration constant can be fixed using the requirement (C8) that, for large X, $\Psi(X)\simeq n/X$,

$$\Psi_1(X)=\int_0^{+\infty}du\,e^{-uX-(u^2/4)}. \tag{C12}$$

Then order n^2 of Eq. (C9) gives

$$\frac{1}{2}\Psi_2'(X)-X\Psi_2(X)+\frac{1}{2}\Psi_1(X)^2=0, \tag{C13}$$

the solution of which can be written as

$$\Psi_2(X)=2\int_0^{+\infty}du\,e^{-uX-u^2/4}\int_0^{+\infty}dt\,\frac{\cosh\frac{ut}{\sqrt{2}}-1}{t}e^{-t^2}. \tag{C14}$$

The procedure can be iterated to any order in n (of course expressions become more and more complicated). Using Eq. (C8) and the expressions of $\Psi_1(X)$ and $\Psi_2(X)$, we can give an expression of $Q(u)$:

$$Q(u)=ne^{-u^2/4}+2n^2e^{-u^2/4}\int_0^{+\infty}dt\,\frac{\cosh\frac{ut}{\sqrt{2}}-1}{t}\times e^{-t^2}+O(n^3). \tag{C15}$$

Expanding this expression in powers of u, one calculates from this expression and from Eq. (C7) the terms linear and quadratic in n of all the moments of the h_α. [The results agree for the second and the fourth moments with Eqs. (C5) and (C6).]

We noticed that for small n, the expression (C15) corresponds to n roots h_α distributed along the imaginary axis with a Gaussian distribution. We do not know whether this is general and whether there exists, for general noninteger n, a distribution of the roots h_α in the complex plane which gives all moments calculated as in Eqs. (C5) and (C6).

It is interesting to notice the similarity between $Q(u)$ and $\beta(u)$ defined in Sec. IV.

APPENDIX D: THE EXPANSION IN POWERS OF *c* USING HERMITE POLYNOMIALS

In this appendix we show how to expand the solution $\{q_\alpha\}$ of Eq. (17) in powers of c for integer n. One can see from Eq. (17) that the roots q_α scale for small c like $\sqrt{c}$. It is thus convenient to rescale the polynomial $P(X)$ defined in Eq. (19) and the q_α in the following way:

$$q_\alpha = r_\alpha \sqrt{c}, \qquad P(X\sqrt{c}) = c^{n/2} R(X). \tag{D1}$$

[$\{r_\alpha\}$ are thus the roots of $R(X)$.] With these new variables, Eq. (20) becomes

$$e^{r_\alpha \sqrt{c}} R(r_\alpha - \sqrt{c}) + e^{-r_\alpha\sqrt{c}} R(r_\alpha + \sqrt{c}) = 0. \tag{D2}$$

As the roots r_α of $R(X)$ are all distinct, this equation is obviously equivalent to

$$e^{X\sqrt{c}} R(X-\sqrt{c}) + e^{-X\sqrt{c}} R(X+\sqrt{c}) = 2[\cosh X\sqrt{c} + f(X)] R(X), \tag{D3}$$

where $f(X)$ is *analytic* [this follows from the fact that as $R(X)$ is polynomial, $f(X)$ defined by Eq. (D3) is obviously meromorphic; moreover, as the left-hand side of Eq. (D3) vanishes at all the roots of $R(X)$, $f(X)$ has no pole]. We are now going to solve Eq. (D3) as a power series in c [i.e., find both $f(X)$ and $R(X)$ as power series in c].

1. Expansion of the polynomial $R(X)$

We only have the single equation (D3) to obtain two quantities [$R(X)$ and $f(X)$]; however, using the fact that $f(X)$ has no pole and $R(X)$ is a polynomial, both quantities can be determined in a small-c expansion. Let us write

$$R(X) = R_0(X) + cR_1(X) + c^2 R_2(X) + \cdots, \qquad f(X) = c f_1(X) + c^2 f_2(X) + \cdots, \tag{D4}$$

where the $f_i(X)$ have no pole, $R_0(X)$ is a polynomial of degree n [the term of highest degree in $R_0(X)$ is X^n], and all the $R_i(X)$ (for $i \geqslant 1$) are polynomials of degree less than n. At first order in c, we find that Eq. (D3) gives

$$\frac{1}{2} R_0'' - X R_0' = f_1 R_0. \tag{D5}$$

As $f_1(X)$ has no pole, it must be a polynomial. Because $R_0(X)$ is of degree n, we see by looking at both sides of Eq. (D5) that, necessarily, $f_1(X) = -n$. We recognize then the differential equation (C1) that defines Hermite polynomials. Therefore,

$$f_1(X) = -n, \qquad R_0(X) = H(X). \tag{D6}$$

We recover that way that the r_α are the zeros of the nth Hermite polynomial when c is very small [24].

At next order in c, Eq. (D3) gives

$$\frac{1}{2} R_1'' - X R_1' + nR_1 - f_2 H = \frac{X^3}{6} H' - \frac{X^2}{4} H'' + \frac{X}{6} H^{(3)} - \frac{1}{24} H^{(4)}. \tag{D7}$$

As R_1 and H are polynomials, Eq. (D7) tells us that $f_2 H$ is a polynomial, too. We also know that $f_2(X)$ has no pole, thus it must be a polynomial. $R_1(X)$ is of degree strictly less than n, so the expression $R_1''/2 - XR_1 + nR_1$ is of degree strictly less than n. As H is of degree n, we recognize in Eq. (D7) a Euclidian division of polynomials: $-f_2(X)$ is the quotient of the right-hand side of Eq. (D7) divided by $H(X)$, and the terms involving $R_1(X)$ form the remainder of this division. This ensures that there is only one possible function $f_2(X)$ which verifies Eq. (D7).

In practice, to perform this Euclidian division we can use the property (C2) of the Hermite polynomials as many times as needed in the right-hand side of Eq. (D7): for instance, we transform the term $X^3 H'/6$ into $nX^2 H/6 + X^2 H''/12$. We cannot change $X^2 H$ anymore, but we can apply Eq. (C2) to the term $X^2 H''$. When no more transformation is possible, we are left with

$$\frac{X^3}{6} H' - \frac{X^2}{4} H'' + \frac{X}{6} H^{(3)} - \frac{1}{24} H^{(4)} = \left(\frac{n}{6} X^2 - \frac{n(n-1)}{6} \right) H - \frac{1}{12} H''. \tag{D8}$$

The Euclidian division is then easy to perform,

$$f_2(X) = -\frac{n}{6} X^2 + \frac{n(n-1)}{6}, \qquad \frac{1}{2} R_1'' - X R_1' + nR_1 = -\frac{1}{12} H''. \tag{D9}$$

Using again Eq. (C2), the differential equation on R_1 can be solved; we find

$$R_1(X) = -\frac{1}{24} H''(X). \tag{D10}$$

As $R_1(X)$ is simply a derivative of $H(X)$, and as $f_1(X)$ is a known polynomial of X, we see that at the next order in c we will have to solve an equation of the form

$$\frac{1}{2} R_2'' - X R_2 + nR_2 - f_3 H = \sum X^j H^{(k)}. \tag{D11}$$

Using many times Eq. (C2), the right-hand side can be written in a "canonical form":

$$\sum X^j H^{(k)} = \sum X^j H + \sum H^{(k)}, \tag{D12}$$

which allows us to write f_3 as a polynomial in X and R_2 as a sum of derivatives of $H(X)$. It is easy to see recursively that at any order c^k in the expansion we can repeat this procedure to calculate $f_k(X)$ and $R_{k-1}(X)$. As a result we see that f_k is a polynomial in X and that R_{k-1} can be written as a sum of derivatives of $H(X)$.

It is worth noting that at each order the variable n comes from the previous orders and from transformations of the kind $XH'(X)\rightarrow\frac{1}{2}H''(X)+nH(X)$. Because those are the only two mechanisms by which n appears, it is easy to see that at each order the coefficients of the sum of derivatives of $H(X)$ that constitutes $R_{k-1}(X)$ are all *polynomials in* n.

A computer can easily do this tedious but straightforward task to any desired order. Up to c^3, we find

$$R=H-\frac{c}{24}H''-c^2\left(\frac{n}{360}H''-\frac{7}{5760}H^{(4)}\right)+c^3\left[\left(\frac{n}{2520}-\frac{n^2}{3024}\right)H''+\frac{11n}{60\,480}H^{(4)}-\frac{31}{967\,680}H^{(6)}\right]+O(c^4). \tag{D13}$$

2. Expansion of the roots r_α of $R(X)$

As seen in Eq. (D13), the polynomial $R(X)$ is to leading order in c given by $H(X)$. It is thus natural to write the roots r_α of $R(X)$ as

$$r_\alpha=h_\alpha+cx_\alpha+O(c^2). \tag{D14}$$

($\{h_\alpha\}$ are the roots of H.) Inserting Eq. (D14) into Eq. (D13), we find, at first order in c,

$$x_\alpha H'(h_\alpha)-\frac{1}{24}H''(h_\alpha)=0. \tag{D15}$$

Using the definition (C1) of Hermite polynomials, we have $H''(h_\alpha)=2h_\alpha H'(h_\alpha)$. This gives in turn $x_\alpha=\frac{1}{12}h_\alpha$. Repeating this procedure to any order in c, we generate terms of the form $h_\alpha^j H^{(k)}(h_\alpha)$ which can be reduced to terms of the form $h_\alpha^l H'(h_\alpha)$ by using Eq. (C2) as many times as necessary. It is then possible to divide the expression by $H'(h_\alpha)$ and we are left with an equation giving each new term in the expansion of r_α as a *polynomial in* h_α. Again, this can be programmed, and we get, up to the order c^2,

$$r_\alpha=\frac{q_\alpha}{\sqrt{c}}=h_\alpha+\frac{c}{12}h_\alpha+c^2\left[\left(\frac{n}{120}-\frac{11}{1440}\right)h_\alpha-\frac{1}{360}h_\alpha^3\right]+O(c^3). \tag{D16}$$

Using Eqs. (D1) and (16), this leads to

$$\frac{2}{L^2}E(n,L,\gamma)=-c\sum h_\alpha^2-\frac{c^2}{6}\sum h_\alpha^2-\frac{c^3}{360}\left((6n-3)\sum h_\alpha^2-2\sum h_\alpha^4\right)+O(c^4), \tag{D17}$$

which coincides with Eq. (38) when one uses the properties (C5) and (C6) of the roots h_α of the Hermite polynomials.

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