

Relative phase of two Bose-Einstein condensates

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We show that two independent Bose-Einstein condensates, each initially containing a well-defined number of atoms, will appear coherent in an experiment that measures the beat note between these condensates. We investigate the role played by atomic interactions within each condensate in the time evolution of their relative phase. [S1050-2947(97)03606-8]

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Since the recent observations of the Bose-Einstein condensation (BEC) of a dilute atomic gas [1–3], the problem of the phase of an atomic sample has been raised with renewed interest. Theoretically, this phase appears naturally as a result of a broken symmetry in the theory of BEC [4,5]. At zero temperature, the atomic sample is described by a coherent state, i.e., an eigenstate of the annihilation operator for a particular state of the one-atom Hilbert space. A classical field $|\psi_0\rangle e^{i\phi}$ with a well defined amplitude $|\psi_0\rangle$ and phase ϕ is associated with this coherent state. Experimentally, however, one can, in principle, measure the exact number of trapped atoms. The condensate is then described by a Fock state (or number state), and no definite phase can be attributed to the gas. The question then arises of whether these two different descriptions lead to identical predictions for a given experimental setup.

To investigate this problem, we consider the following Gedanken experiment, using two trapped condensates of the same atomic species. The trapping potentials are isotropic and harmonic, except for a finite barrier in a given direction, through which the atoms can tunnel (Fig. 1). The phase between the two emerging beams can be probed by “beating” them together, i.e., by mixing them with a 50-50 atomic beam splitter [6].

If each condensate is in a coherent state with the same average number of atoms, the beams incident on the beam splitter are described by the two fields, $|\psi_0\rangle e^{i\phi_A}$ and $|\psi_0\rangle e^{i\phi_B}$. The intensities in the two outputs of the beam splitter are then

$$I_+ = 2|\psi_0|^2 \cos^2 \phi, \quad I_- = 2|\psi_0|^2 \sin^2 \phi, \quad (1)$$

where $\phi = (\phi_A - \phi_B)/2$. The recording of I_{\pm} allows one to determine the absolute value of the relative phase 2ϕ . Note that ϕ is an unpredictable random variable, which takes a different value for any new realization of the experiment.

In a description of the system in terms of Fock states, one supposes that the system is initially in the state $|N_A, N_B\rangle$, i.e., there are $N_{A/B}$ particles in the condensates A/B . Our purpose is to show that the predictions corresponding to a statistical mixture of states $|N_A, N_B\rangle$ with a Poissonian distribution for $N_{A/B}$ are identical to Eq. (1). The notion of phase-broken symmetry is therefore not indispensable in order to understand the beating of two condensates [7]. On the other hand,

it provides a simple way of analyzing such an experiment, while, as we see below, Fock states are more difficult to handle in such a situation.

The problem that we are facing here is analogous to the question raised by P. W. Anderson [8]: Do two superfluids that have never “seen” one another possess a definite relative phase? As pointed out in [4], the question is meaningless as long as no measurement is performed on the system. J. Javanainen and S. M. Yoo recently addressed a similar question by considering the spatial interferences of two condensates prepared in the state $|N, N\rangle$ and arriving on a given array of detectors [9]. He showed numerically that, after the detection of all the atoms of the two condensates, the count distribution on the set of detectors was similar to the one predicted from a phase broken symmetry state.

The paper is organized as follows. In the Sec. I, we address the simple particular case where all the detected particles are bunched in the same output channel of the beam splitter. In Sec. II, we present a general reasoning showing that the descriptions in terms of coherent or Fock states lead to identical predictions for any type of measurements performed on the system. In Sec. III, assuming an initial Fock state for the system, we investigate the buildup of a relative phase between the two condensates as the measurements proceed. In Sec. IV, we add a device, shifting the atomic phase in one of the channels of Fig. 1, in order to perform multi-channel detection; we then recover analytically the numerical results of [9]. Finally, we include the effect of the atomic interactions on the distribution of the relative phase between the two condensates. We predict collapses and revivals for this distribution with time scales that should be experimentally accessible.

I. A PARTICULAR CASE: ALL THE DETECTIONS IN ONE CHANNEL

We assume that k atoms are detected on D_{\pm} . For simplicity we consider in this section the situation where all the k detections occur in the (+) channel. If the system is initially in a coherent state, the probability for such a sequence (given that k atoms have been detected) is $\cos^{2k} \phi$. The average over the unknown relative phase 2ϕ gives

$$W_k = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \cos^{2k} \phi = \frac{(2k)!}{(2^k k!)^2} \underset{k \gg 1}{\sim} \frac{1}{\sqrt{\pi k}} \quad (2)$$

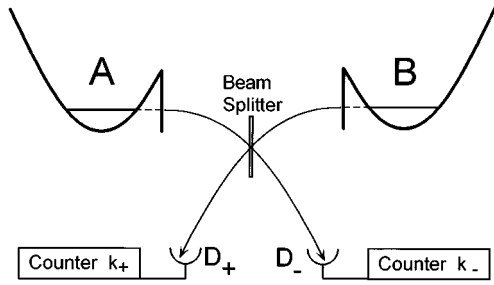


FIG. 1. A Gedanken experiment: atoms leaking from two trapped condensates, A and B , are detected in the output channels (\pm) of a 50-50 beam splitter.

For $k=100$, this probability of getting all counts in the ($+$) channel is $\sim 6\%$.

We suppose now that the system is in a Fock state and for simplicity we take $N_B = N_A \gg k$. A naive argument could consist of saying that since $k \ll N_A$, the probability of detecting the n th atom ($n \leq k$) in the ($+$) channel is nearly independent of the $n-1$ previous detection results. The probability of k detections in the ($+$) channel should then be 2^{-k} . This is obviously very different from the result W_k obtained from the coherent state point of view ($2^{-k} < 10^{-30}$ for $k=100$).

However, the latter reasoning is wrong; the first detection of an atom in the ($+$) channel projects the atom in a state proportional to

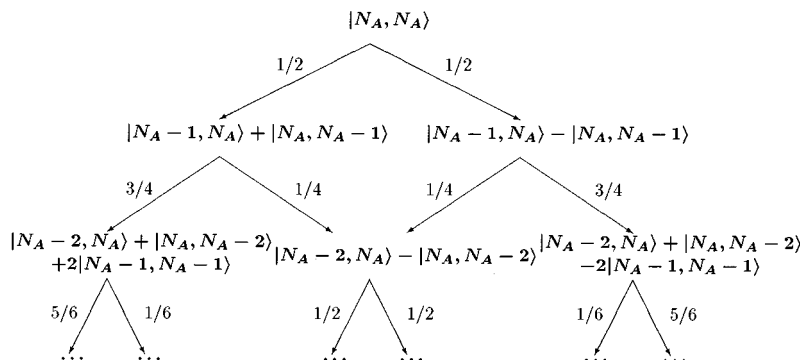
$$(\hat{a} + \hat{b})|N_A, N_A\rangle \propto |\Psi\rangle = |N_A, N_A - 1\rangle + |N_A - 1, N_A\rangle, \quad (3)$$

where \hat{a} (\hat{b}) annihilates a particle in the condensate A (B). To calculate the probability of detecting a second atom in the ($+$) channel, we have to compare the squared norm of the two vectors corresponding to a detection in the (\pm) channels:

$$\begin{aligned} (+): (\hat{a} + \hat{b})|\Psi\rangle &= \sqrt{N_A - 1}(|N_A - 2, N_A\rangle + |N_A, N_A - 2\rangle) \\ &\quad + 2\sqrt{N_A}|N_A - 1, N_A - 1\rangle, \end{aligned} \quad (4)$$

$$(-): (\hat{a} - \hat{b})|\Psi\rangle = \sqrt{N_A - 1}(|N_A - 2, N_A\rangle - |N_A, N_A - 2\rangle). \quad (5)$$

For $N_A = 1$, we recover the well-known interference effect leading to a bunching of the two bosons in a single output of the beam splitter [11]. For $N_A \gg 1$, the squared norms of



these two vectors are in the ratio 3:1. This indicates that once a first atom has been detected in the ($+$) channel, the probability of detecting the second atom in the same channel is $3/4$, while the probability of detecting this second atom in the ($-$) channel is only $1/4$. This somewhat counterintuitive result shows clearly that the successive detection probabilities are strongly correlated in the case of an initial Fock state, even if the number of detected atoms is very small compared to the number of atoms present in the condensates. The reasoning can be extended to k detections (see Fig. 2) and we find that the probability of detecting respectively, $k_+ = k$ and $k_- = 0$ atoms in the two channels is

$$\mathcal{P}(k, 0) = \frac{1}{2} \frac{3}{4} \cdots \frac{2k-1}{2k}, \quad (6)$$

which is equal to W_k for any k . Note that the explicit average over N_A and N_B is correctly omitted in this last calculation in the limit $\bar{N}_A = \bar{N}_B \gg 1$, where the Poissonian fluctuations have a negligible effect.

The predictions for an initial Fock state and for an initial coherent state with random phase are therefore equivalent, but the result for the coherent state is obtained in a much more straightforward and intuitive manner than for the Fock state.

II. ENSEMBLE AVERAGE WITH AND WITHOUT PHASE-BROKEN SYMMETRY

This equivalence between the Fock-state and the coherent state descriptions is actually not restricted to the particular detection scheme considered in this paper. It is a consequence of the identity of the density operators of the total system in those two descriptions. To prove this identity, we first consider the coherent state with well-defined phases ϕ_A and ϕ_B :

$$\begin{aligned} &|\bar{N}_A|^{1/2} e^{i\phi_A} |\bar{N}_B|^{1/2} e^{i\phi_B} \\ &= \sum_{N_A, N_B} \frac{\bar{N}_A^{N_A/2} \bar{N}_B^{N_B/2}}{\sqrt{N_A!} N_B!} e^{i(N_A \phi_A + N_B \phi_B)} |N_A, N_B\rangle \\ &\quad \times e^{-(\bar{N}_A + \bar{N}_B)/2}, \end{aligned} \quad (7)$$

where \bar{N}_A and \bar{N}_B are the mean number of particles in the condensates A and B . In the coherent-state description, the

FIG. 2. Possible outcomes (and the corresponding branching ratios) of the first three detections in the output channels of the beam splitter. Initially, the system is in a Fock state, with the same numbers of particles $N_A \gg 1$ in the two condensates.

density operator ρ of the system is then obtained by a statistical average over the phases ϕ_A and ϕ_B :

$$\rho = \int_0^{2\pi} \int_0^{2\pi} \frac{d\phi_A}{2\pi} \frac{d\phi_B}{2\pi} |\bar{N}_A^{1/2} e^{i\phi_A}, \bar{N}_B^{1/2} e^{i\phi_B}\rangle \times \langle \bar{N}_A^{1/2} e^{i\phi_A}, \bar{N}_B^{1/2} e^{i\phi_B} |. \quad (8)$$

Using the explicit expression (7) of the coherent states, we find that all off-diagonal terms in the Fock-state basis are suppressed after the integration over ϕ_A, ϕ_B ,

$$\rho = \sum_{N_A, N_B} \frac{\bar{N}_A^{N_A} \bar{N}_B^{N_B}}{N_A! N_B!} |N_A, N_B\rangle \langle N_A, N_B| e^{-(\bar{N}_A + \bar{N}_B)}, \quad (9)$$

which coincides with the Poissonian statistical mixture considered in the Fock-state description [7,12].

From the identity of the density operators ρ we can conclude that no measurement (or no series of measurements) performed on the system can allow one to distinguish between the coherent state and the Fock-state descriptions. Indeed, in a sequence of measurements, the probability of getting a given set of results can always be expressed as the expectation value $\text{Tr}(\hat{O}\rho)$ of an operator \hat{O} [13]. For example, the probability that at least k atoms are detected and that the first k detections occur in the (+) channel is obtained from (see Appendix)

$$\hat{O} = \frac{1}{2^k} (\hat{a}^\dagger + \hat{b}^\dagger)^k \frac{1}{(\hat{N} + 1) \cdots (\hat{N} + k)} (\hat{a} + \hat{b})^k, \quad (10)$$

where $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$. It is shown in the Appendix that $\text{Tr}(\hat{O}\rho)$ is nearly equal to $\mathcal{P}(k, 0)$. The difference between the two quantities is due to the fact that $\mathcal{P}(k, 0)$ is a *conditional* probability that the first k detections occur in the (+) channel, knowing that at least k particles have been detected. This difference is negligible in the limit where the mean numbers of atoms $\bar{N}_A = \bar{N}_B$ are much larger than k , since the probability that at least k detections occur after an arbitrarily long time approaches 1 in this case.

III. PROBABILITY OF A GENERAL (k_+, k_-) DETECTION RESULT

We now generalize the discussion of Sec. I to the general case of k_\pm detected atoms in the (\pm) channels for a fixed number of measurements $k = k_+ + k_-$. We first address the case of an initial coherent state. We then define the so-called phase states, which correspond to a well defined total number of particles and a well defined relative phase between the two condensates. Finally, starting from the system in a Fock state, we expand the state vector on those phase states as the measurements proceed, to show the emergence of a relative phase.

A. Case of an initial coherent state

We assume that the system is initially in the coherent state, Eq. (7). As the measurements proceed, the state of the system remains coherent, with the same relative phase

$\phi = (\phi_A - \phi_B)/2$. Each count occurs with probabilities $\cos^2\phi$ and $\sin^2\phi$ in the (+) and (-) channels. Given that k particles have been detected, the distribution of counts in the (\pm) channels is binomial and the probability for the result (k_+, k_-) is

$$\mathcal{P}(k_+, k_-, \phi) = \frac{k!}{k_+! k_-!} (\cos\phi)^{2k_+} (\sin\phi)^{2k_-}. \quad (11)$$

The number of counts k_+ in the (+) channel has, therefore, a mean value $k \cos^2\phi$ and a standard deviation (shot noise) $\sigma[k_+] = \sqrt{k} |\cos\phi \sin\phi|$.

In the limit $k_\pm \gg 1$, using $\ln n! \sim n \ln n - n$ for $n \gg 1$, we find from Eq. (11) that $\mathcal{P}(k_+, k_-, \phi)$ is maximal for $k_-/k_+ = \tan^2\phi$, as expected from Eq. (1). In other words, for $k \gg 1$, the mean and most probable intensities coincide, since the shot noise on the signal in the two channels (\pm) becomes negligible.

B. Phase states

For an initial state $|\Psi\rangle$ with a well-defined total number of particles N , the evolution due to the sequence of measurements is conveniently analyzed by expanding $|\Psi\rangle$ onto the overcomplete set of *phase states* $|\phi\rangle_N$ [4]:

$$|\phi\rangle_N = \frac{1}{\sqrt{2^N N!}} (\hat{a}^\dagger e^{i\phi} + \hat{b}^\dagger e^{-i\phi})^N |0\rangle, \quad (12)$$

where $|0\rangle$ stands for the vacuum. If the system is in a given state $|\phi\rangle_N$, there exists a well defined relative phase ϕ between A and B : if a device shifting the phase of the matter wave by 2ϕ were placed in front of the B input of the beam splitter, all the atoms would be detected in the (+) output of the beam splitter.

Any state $|\Psi\rangle$ with N particles can be expanded in the set of phase states:

$$|\Psi\rangle = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} c(\phi) |\phi\rangle_N, \quad (13)$$

where the phase amplitude $c(\phi)$ is obtained as

$$c(\phi) = 2^{N/2} \sum_{N_A=0}^N \left(\frac{N_A! (N-N_A)!}{N!} \right)^{1/2} e^{i(N-2N_A)\phi} \times \langle N_A, N-N_A | \Psi \rangle. \quad (14)$$

In what follows, we will use the quasiorthogonality of the phase states valid for large N and for $-\pi/2 \leq \phi, \phi' < \pi/2$:

$$\begin{aligned} {}_N \langle \phi | \phi' \rangle_N &= \cos^N(\phi - \phi') \simeq e^{-N(\phi' - \phi)^2/2} \\ &\simeq \sqrt{2\pi/N} \delta(\phi - \phi'). \end{aligned} \quad (15)$$

As an illustration of the relevance of the phase states we now derive the probability $\mathcal{P}(k_+, k_-)$ for the system in the initial state $|N/2, N/2\rangle$. We show that it is approximately equal to the result obtained for a statistical mixture of coherent states, as expected from the general discussion of Sec. II.

Using the formula found in Eq. (A4) of the Appendix, we get as a starting point

$$\mathcal{P}(k_+, k_-) = \frac{(N-k)!}{2^k N!} \frac{k!}{k_+! k_-!} \times ||(\hat{a} + \hat{b})^{k_+} (\hat{a} - \hat{b})^{k_-} |N/2, N/2\rangle||^2. \quad (16)$$

We expand the state vector over the set of phase states:

$$|N/2, N/2\rangle = c_0 \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} |\phi\rangle_N, \quad (17)$$

$$c_0 = 2^{N/2} \frac{(N/2)!}{\sqrt{N!}} \sim (\pi N/2)^{1/4}. \quad (18)$$

We calculate first the action of the annihilation operators in Eq. (16) on the phase states:

$$(\hat{a} + \hat{b})^{k_+} (\hat{a} - \hat{b})^{k_-} |\phi\rangle_N = \left(\frac{N! 2^k}{(N-k)!} \right)^{1/2} e^{ik_- \pi/2} (\cos\phi)^{k_+} (\sin\phi)^{k_-} |\phi\rangle_{N-k}, \quad (19)$$

with $k = k_+ + k_-$. The quasiorthogonality [Eq. (15)] of the phase states in the limit of large N then gives

$$\begin{aligned} \mathcal{P}(k_+, k_-) &\sim \frac{k!}{k_+! k_-!} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} (\cos\phi)^{2k_+} (\sin\phi)^{2k_-} \\ &= \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \mathcal{P}(k_+, k_-, \phi), \end{aligned} \quad (20)$$

which shows the announced equivalence.

C. Emergence of the relative phase for an initial Fock state.

For an initial Fock state $|N/2, N/2\rangle$, which has a flat phase probability distribution $|c(\phi)|^2$, we now investigate the emergence of a relative phase between the two condensates during the sequence of measurements.

After a sequence of $(k_+, k_- = k - k_+)$ detections, the state of the system is obtained from Eqs. (17) and (19):

$$\begin{aligned} |\Psi(k_+, k_-)\rangle &\propto (\hat{a} + \hat{b})^{k_+} (\hat{a} - \hat{b})^{k_-} |N/2, N/2\rangle \\ &\propto \int_{-\pi/2}^{\pi/2} d\phi (\cos\phi)^{k_+} (\sin\phi)^{k_-} |\phi\rangle_{N-k}. \end{aligned} \quad (21)$$

For $k_{\pm} \gg 1$, we use the stationary-phase method, which approximates the integrand $(\cos\phi)^{k_+} (\sin\phi)^{k_-}$ by a Gaussian around each of its maxima. The maxima in $[-\pi/2, \pi/2]$ are located in ϕ_0 and $-\phi_0$, with $0 \leq \phi_0 \leq \pi/2$ and

$$k_+ = k \cos^2 \phi_0, \quad (22)$$

$$k_- = k \sin^2 \phi_0. \quad (23)$$

We get, for instance, for ϕ close to ϕ_0 ,

$$\begin{aligned} (\cos\phi)^{k_+} (\sin\phi)^{k_-} &\approx \exp \left[\frac{1}{2} \left(k_+ \log \frac{k_+}{k} + k_- \log \frac{k_-}{k} \right) \right. \\ &\quad \left. - k(\phi - \phi_0)^2 \right]. \end{aligned} \quad (24)$$

We obtain therefore:

$$\begin{aligned} |\Psi(k_+, k_-)\rangle &\propto \int_{-\pi/2}^{\pi/2} d\phi [e^{-k(\phi - \phi_0)^2} \\ &\quad + (-1)^{k_-} e^{-k(\phi + \phi_0)^2}] |\phi\rangle_{N-k}. \end{aligned} \quad (25)$$

The interpretation of this result is quite clear: initially, the relative phase of the two condensates is indefinite, since the vector state of the system projects equally onto the various phase states [see Eq. (17)]. After $k \gg 1$ detections, the system has evolved into a state where the phase ϕ is well defined; more precisely, the phase distribution is a double Gaussian, centered on ϕ_0 and $-\phi_0$, with a standard deviation of $1/\sqrt{2k}$. This ambiguity between ϕ_0 and $-\phi_0$ also arises in the determination of ϕ from Eq. (1).

To summarize, we have two different points of view on the system: for an initial coherent state, the measurement ‘‘reveals’’ the pre-existing phase through $\tan^2 \phi = k_- / k_+$; for an initial Fock state, the detection sequence ‘‘builds up’’ the phase. A similar conclusion has been reached by a numerical analysis of quantum trajectories in the framework of continuous measurement theory [12]. It is not possible to favor one particular point of view, based on experimental results. If the same experimental sequence involving k detections is repeated, with the phase varying randomly from shot to shot in the coherent-state point of view, the predicted occurrence of a given result $k_+, k_- = k - k_+$ is identical in the two points of view.

IV. MOST PROBABLE MEASUREMENT SEQUENCES IN A MULTICHANNEL DETECTION SCHEME

In this section, we analyze the results of a multichannel experiment where a device shifting the atomic phase by an adjustable quantity 2γ is introduced in one of the input channels of the beam splitter, sketched in Fig. 1. Our analysis also applies to the case of spatial interferences between two condensates arriving simultaneously on an array of atom detectors [9].

We imagine that the phase shift γ is tuned successively to the L different values $\gamma_j = j\pi/2L, j = 0, \dots, L-1$. We assume, for simplicity, that exactly $k \gg 1$ particles are detected for each value of γ . Our goal is to show that the signals in each $+$ and $-$ channel, $k_+(j)$ and $k_-(j)$, are equal (within shot noise) to $k \cos^2(\phi_0 - \gamma_j)$ and $k \sin^2(\phi_0 - \gamma_j)$, where the parameter ϕ_0 , varying randomly for any new realization of the whole experiment, is the same for all channels.

As emphasized in Sec. II, the probability for a given set of results $\{k_{\pm}(j)\}$ is given by the average of an operator \hat{O} over the density matrix of the system [Eq. (8)]. For the multichannel detection scheme considered here, the probability of observing this sequence, knowing that k counts have been ob-

tained in each channel, is obtained by a generalization of the result of the Appendix:

$$\begin{aligned} \mathcal{P}[\{k_+(j)\}] &= \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \prod_{j=0}^{L-1} \frac{k!}{k_+(j)! k_-(j)!} \\ &\times [\cos(\phi - \gamma_j)]^{2k_+(j)} [\sin(\phi - \gamma_j)]^{2k_-(j)}. \end{aligned} \quad (26)$$

To demonstrate this result, we have used the operators $\hat{a} \pm e^{2i\gamma_j} \hat{b}$ associated with a count in the (\pm) channel with a phase shift $2\gamma_j$.

We investigate first the case $L=1$ discussed in the preceding section, to identify a physically optimal parametrization of the problem. For $k \gg 1, k_{\pm} > 1$, we get an approximation of the right-hand side of Eq. (20) by using the stationary phase method for the integral [Eqs. (24) and (25)] and Stirling's formula for the binomial factor:

$$\frac{k!}{k_+! k_-!} \approx \left(\frac{k}{2\pi k_+ k_-} \right)^{1/2} \exp\left(-k_+ \ln \frac{k_+}{k} - k_- \ln \frac{k_-}{k} \right). \quad (27)$$

This leads to

$$\mathcal{P}(k_+, k_-) \approx \frac{1}{\pi \sqrt{k_+ k_-}}. \quad (28)$$

The remaining slow variation with k_{\pm} can be suppressed by characterizing the sequence of measurements by the angle ϕ_0 , defined in Eq. (22), rather than by k_{\pm} ; in the limit $k \gg 1$ we treat ϕ_0 as a continuous variable, and the Jacobian for the change of variables $k_+ \rightarrow \phi_0$ satisfactorily leads to a flat probability density for ϕ_0 :

$$\tilde{\mathcal{P}}(\phi_0) \equiv \mathcal{P}(k_+, k_-) \left| \frac{dk_+}{d\phi_0} \right| \approx \frac{2}{\pi}. \quad (29)$$

We now generalize this calculation to an arbitrary value for the number of phase bins L . A sequence of kL detections (with k detection in each bin) $\{k_+(j), j=0, \dots, L-1\}$ is characterized by the L angles θ_j , such that

$$k_+(j) = k \cos^2 \theta_j, \quad \theta_j \in [0, \pi/2], \quad j=0, \dots, L-1. \quad (30)$$

In the limit $k \gg 1$, we consider the θ_j 's as continuous variables; expanding the binomial factors in Eq. (26), using Eq. (27) as in the previous case $L=1$, we find for the probability density of the θ_j 's

$$\begin{aligned} \tilde{\mathcal{P}}(\theta_0, \dots, \theta_{L-1}) &\equiv \mathcal{P}(\{k_+(j)\}) \prod_{j=0}^{L-1} \left| \frac{dk_+(j)}{d\theta_j} \right| \\ &\approx \left(\frac{2k}{\pi} \right)^{L/2} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \exp[kS(\theta_0, \dots, \theta_{L-1}, \phi)], \end{aligned} \quad (31)$$

where we have introduced

$$\begin{aligned} S(\theta_0, \dots, \theta_{L-1}, \phi) &\equiv \sum_{j=0}^{L-1} \cos^2 \theta_j \ln \left(\frac{\cos^2(\phi - \gamma_j)}{\cos^2 \theta_j} \right) \\ &+ \sin^2 \theta_j \ln \left(\frac{\sin^2(\phi - \gamma_j)}{\sin^2 \theta_j} \right). \end{aligned} \quad (32)$$

We now look for the values $(\theta_0, \dots, \theta_{L-1})$ maximizing $\tilde{\mathcal{P}}(\theta_0, \dots, \theta_{L-1})$. We note the position ϕ_0 of the absolute maximum of S [15] as a function of ϕ , for given $(\theta_0, \dots, \theta_{L-1})$, and we perform the stationary-phase approximation in the integral [Eq. (31)]:

$$\tilde{\mathcal{P}}(\theta_0, \dots, \theta_{L-1}) \approx \frac{C}{\sqrt{|\partial_{\phi}^2 S|_{\phi_0}}} \exp[kS(\theta_0, \dots, \theta_{L-1}, \phi_0)], \quad (33)$$

where the normalization factor C depends on k and L only. If one neglects the slow variations of the prefactor, the maximum of $\tilde{\mathcal{P}}$ is obtained by maximizing S in Eq. (33) over the remaining variables $\theta_0, \dots, \theta_{L-1}$. This is equivalent to a global maximization of S in Eq. (32) over all the variables. We find [16] that the maximal value of S is 0 and that it is obtained for the measurement sequences

$$\begin{aligned} k_+(j) &= k \cos^2(\phi_0 - \gamma_j), \quad j=0, \dots, L-1, \\ -\pi/2 &\leq \phi_0 < \pi/2. \end{aligned} \quad (34)$$

The curve defined by Eq. (34) for the θ_j 's is the straight line $\theta_j = \pm(\phi_0 - \gamma_j)[\pi]$. Along this line the probability density is constant (and equal to $C/\sqrt{4L}$). Expanding S around the distributions [Eq. (34)], we find, after a rather involved calculation, that the probability density decreases away from this line as

$$\tilde{\mathcal{P}} \approx \frac{C}{\sqrt{4L}} \exp(-2kd^2), \quad (35)$$

where d is the Euclidian distance to the line.

These results can be understood in a simple and physical manner as follows. Assume that the system is initially in a coherent state [Eq. (7)], with a random relative phase $\phi_A - \phi_B \equiv 2\phi_0$ uniformly distributed in $[-\pi, \pi]$. For a total number of $k \gg 1$ counts, measured with a phase shift $2\gamma_j$ in the input channel B of the beam splitter, we use the results of Sec. III A, replacing ϕ by $\phi_0 - \gamma_j$ in Eq. (11); we find that the probability distribution for the angle θ_j in Eq. (30) is strongly peaked around $\phi_0 - \gamma_j$, with a standard deviation $1/\sqrt{4k}$, in agreement with Eqs. (34) and (35).

This exemplifies again the relevance of the coherent-state point of view in the description of the measured results for a single realization of the beating experiment.

V. ROLE OF ATOMIC INTERACTIONS: COLLAPSES AND REVIVALS

Up to now, we have neglected the time evolution of the system, except for the state projection consecutive to a detection on D_{\pm} . We now investigate the dynamics of the phase distribution $c(\phi, t)$ [Eq. (14)] for a state with N par-

ticles, including the effects of atomic interactions. We consider here the situation where no interaction takes place between A and B ; this situation differs from the one in [10] where the two condensates are spatially overlapping and the interferences are modified by their mutual interactions.

In our case, the Fock states $|N_A, N_B\rangle$ are eigenstates of the total Hamiltonian, with an energy $E(N_A, N_B)$. To express the phase distribution at time t in terms of $c(\phi, 0)$, we expand the initial state to the Fock states using Eqs. (12) and (13), we evolve the Fock states for the time t with the appropriate phase factors, and we calculate $c(\phi, t)$ from Eq. (14). This leads to

$$c(\phi, t) = \int_{-\pi/2}^{\pi/2} \frac{d\phi'}{\pi} K(\phi - \phi'; t) c(\phi', 0), \quad (36)$$

with

$$K(\phi; t) = \sum_{N_A=0}^N e^{i[(N_B - N_A)\phi - E(N_A, N_B)t/\hbar]}, \quad (37)$$

where $N_B = N - N_A$. Assuming a distribution of the Fock states peaked around $N_A = N_B = N/2$ with a width $\ll N$ [14], we expand

$$E(N_A, N_B) = E(N/2, N/2) + (\mu_B - \mu_A)(N_B - N_A)/2 + \hbar\kappa(N_B - N_A)^2 + \dots, \quad (38)$$

where $\mu_{A/B}$ are the chemical potentials for the condensates A/B . From Eq. (38), we find that the effect of the linear term is a mere phase drift, with a velocity $\dot{\phi} = (\mu_B - \mu_A)/(2\hbar)$. For an ideal gas, $\kappa = 0$ and this drift is the only possible evolution, with $\dot{\phi} = (3/4)(\Omega_B - \Omega_A)$, where $\Omega_{A/B}$ are the trap oscillation frequencies.

When atomic interactions are present, $\kappa \neq 0$, and the $(N_B - N_A)^2$ term in $E(N_A, N_B)$ is responsible for a phase spreading analogous to the spreading of the wave function of a free massive particle. This phenomenon is similar to the ‘‘phase diffusion’’ predicted in [17]. If we replace the sum over N_A in Eq. (37) with an integral, we find that an initial Gaussian phase distribution remains Gaussian; the variance for ϕ calculated in $|c(\phi, t)|^2$ then evolves as

$$\Delta\phi_t^2 = \Delta\phi_0^2 + (\kappa t / \Delta\phi_0)^2. \quad (39)$$

Therefore, a state with a well-defined initial phase ($\Delta\phi_0 \ll 1$) will be ‘‘dephased’’ in a time $\sim t_{\text{collapse}} \equiv \Delta\phi_0 / \kappa$. For times longer than t_{collapse} , we have to keep the discrete sum over N_A in Eq. (37). We find that this phase collapse is followed by revivals occurring at times $t_j = \pi j / 4\kappa$, j integer [18], with an average phase displaced by $\dot{\phi}t_j + j\pi/2$ from its initial value.

This discussion implies that the results derived in the first part of this paper are valid provided the measurement sequence is performed in a time short enough that the phase spreading or drift is small compared to the final phase width. As a typical situation, we consider a condensate in the Thomas-Fermi regime [19] for which

$$2\mu_i = \hbar\Omega_i(225 N_i^2 a^2 m \Omega_i / \hbar)^{1/5}, \quad i = A, B, \quad (40)$$

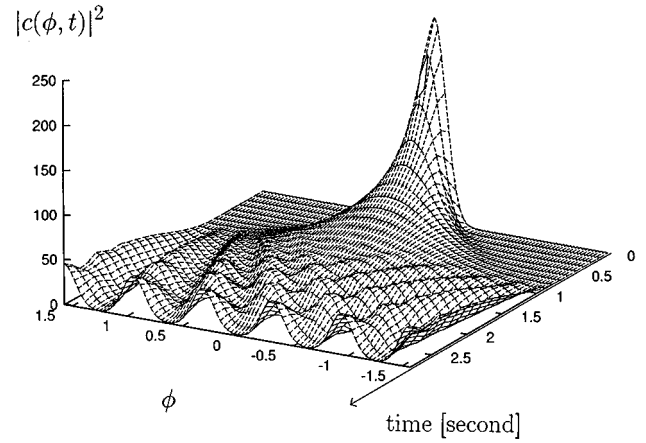


FIG. 3. Short time evolution of the phase probability distribution $|c(\phi, t)|^2$ for $N_A = N_B = 10^4$ rubidium atoms. The scattering length is $a = 10$ nm. The frequencies of the two traps are identical, $\Omega_{A/B} = 2\pi \times 100$ Hz. The initial phase distribution $c(\phi, t=0)$ is real Gaussian, with $|c(\phi, t=0)|^2$ leading to a standard deviation $\Delta\phi_0 = 1/10$. Note the emergence of fractional revivals after the collapse of the wave packet.

where m is the atomic mass and a is the s -wave scattering length; this leads to $\hbar\kappa = (\mu_A + \mu_B)/(10N)$. Using $\Omega_i = 2\pi \times 100$ Hz [1], we find $t_{\text{collapse}} = \Delta\phi_0 \times 18$ s for $N/2 = 10^4$ rubidium atoms ($a \sim 10$ nm). The phase collapse is followed by partial fractional revivals (see Fig. 3), a known phenomenon in quantum mechanics [20]. The first full revival occurs at a time ~ 14 s, with an average phase shifted by

$$\delta\phi = \frac{\pi}{2} + \frac{5\pi}{4} N \frac{\Omega_A^{6/5} - \Omega_B^{6/5}}{\Omega_A^{6/5} + \Omega_B^{6/5}}. \quad (41)$$

For $N \gg 1$, this shift is very sensitive to any asymmetry between the two traps.

To summarize, we have developed an approach to the problem of relative phase of two macroscopic entities that is based on microscopic measurements. In this way, quantitative predictions can be obtained about the phase distribution and its time evolution. This approach is complementary to the one dealing with a macroscopic variable, such as a Josephson current, connecting these two entities [4,21]. It can be extended to the case of more than two condensates in order to discuss the problem of an ‘‘atomic phase standard’’ [22]. We note, however, that the phase dynamics described above makes it difficult to establish a long-lived phase coherence between separate atomic samples.

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APPENDIX A: PROBABILITY FOR A GIVEN DETECTION SEQUENCE

Consider first a single condensate A leaking with a loss rate Γ towards an ideal detector. As usual, in the continuous measurement theory [23,24] the state of the system undergoes a sequence of quantum jumps; the operator characterizing these jumps is $\sqrt{\Gamma}\hat{a}$, where \hat{a} annihilates one particle in the condensate. Between these jumps, the evolution of the system is governed by the non-Hermitian Hamiltonian $-i\hbar\Gamma\hat{N}_A/2$, where $\hat{N}_A = \hat{a}^\dagger\hat{a}$. From [23], we find that the probability density that k counts occur at times $t_1 \leq t_2 \leq \dots \leq t_k$ (with no additional count between $t=0$ and $t=t_k$) is given by

$$\begin{aligned} Q(t_1, \dots, t_k) &= \Gamma^k \text{Tr}[\rho(0) e^{-\Gamma\hat{N}_A t_1/2} \hat{a}^\dagger \dots e^{-\Gamma\hat{N}_A(t_k - t_{k-1})/2} \\ &\quad \times \hat{a}^\dagger \hat{a} e^{-\Gamma\hat{N}_A(t_k - t_{k-1})/2} \dots \hat{a} e^{-\Gamma\hat{N}_A t_1/2}]. \end{aligned} \quad (\text{A1})$$

The probability of getting at least k counts for an arbitrarily long measurement time is obtained after some algebra:

$$\begin{aligned} Q(k) &= \int_0^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \dots \int_{t_{k-1}}^{+\infty} dt_k Q(t_1, \dots, t_k) \\ &= \text{Tr} \left[\rho(0) (\hat{a}^\dagger)^k \frac{1}{(\hat{N}_A + 1) \dots (\hat{N}_A + k)} \hat{a}^k \right], \end{aligned} \quad (\text{A2})$$

which, in the present case of a single condensate, reduces to the expectation value of the projector onto the states with at least k particles.

This can be generalized to the case of two condensates A and B , with identical loss rates Γ and whose outputs are mixed on a 50-50 beam splitter (see Fig. 1). The measurement process now involves two types of quantum jumps, characterized by the two operators $\sqrt{\Gamma/2}(\hat{a} \pm \hat{b})$. The non-Hermitian Hamiltonian governing the evolution between the quantum jumps is given by $-i\hbar\Gamma\hat{N}/2$, with $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$. We now define the probability density $Q(t_1 \eta_1, \dots, t_k \eta_k)$ that k counts occur at times $t_1 \leq t_2 \leq \dots \leq t_k$ (with no additional count between $t=0$ and $t=t_k$) in the output channels $\eta_1 = \pm, \dots, \eta_k = \pm$:

$$\begin{aligned} Q(t_1 \eta_1, \dots, t_k \eta_k) &= (\Gamma/2)^k \text{Tr}[\rho(0) e^{-\Gamma\hat{N}t_1/2} (\hat{a}^\dagger + \eta_1 \hat{b}^\dagger) \dots \\ &\quad \times e^{-\Gamma\hat{N}(t_k - t_{k-1})/2} (\hat{a}^\dagger + \eta_k \hat{b}^\dagger) \\ &\quad \times (\hat{a} + \eta_k \hat{b}) e^{-\Gamma\hat{N}(t_k - t_{k-1})/2} \dots \\ &\quad \times (\hat{a} + \eta_1 \hat{b}) e^{-\Gamma\hat{N}t_1/2}]. \end{aligned} \quad (\text{A3})$$

From this expression, we can determine the probability $Q(k_+, k_-)$ of getting at least $k = k_+ + k_-$ counts for an arbitrary

long measurement time, the first k counts involving k_\pm counts in the output channels \pm :

$$\begin{aligned} Q(k_+, k_-) &= \frac{1}{2^k} \frac{k!}{k_+! k_-!} \text{Tr} \left[\rho(0) (\hat{a}^\dagger + \hat{b}^\dagger)^{k_+} (\hat{a}^\dagger - \hat{b}^\dagger)^{k_-} \right. \\ &\quad \left. \times \frac{1}{(\hat{N} + 1) \dots (\hat{N} + k)} (\hat{a} - \hat{b})^{k_-} (\hat{a} + \hat{b})^{k_+} \right]. \end{aligned} \quad (\text{A4})$$

The operator \hat{O} introduced in Eq. (10) is readily obtained from this expression, with $k_+ = k, k_- = 0$.

We now calculate the probability $Q(k_+, k_-)$ when the system is in the coherent state Eq. (7), assuming $\bar{N}_A = \bar{N}_B$. The action of the annihilation and creation operators in Eq. (A4) is easily obtained and we are left with

$$\begin{aligned} Q(k_+, k_-) &= (2\bar{N}_A)^k \frac{k!}{k_+! k_-!} (\cos\phi)^{2k_+} (\sin\phi)^{2k_-} \\ &\quad \times \left\langle \frac{1}{(\hat{N} + 1) \dots (\hat{N} + k)} \right\rangle, \end{aligned} \quad (\text{A5})$$

where $\phi = (\phi_A - \phi_B)/2$ and where the average $\langle \dots \rangle$ is taken in the coherent state. Since the total number of particles has a Poissonian distribution with a mean value $\bar{N}_A + \bar{N}_B = 2\bar{N}_A$, this average is given by

$$\begin{aligned} &\left\langle \frac{1}{(\hat{N} + 1) \dots (\hat{N} + k)} \right\rangle \\ &= \sum_{N=0}^{\infty} \frac{1}{(N+1) \dots (N+k)} \frac{1}{N!} (2\bar{N}_A)^N e^{-2\bar{N}_A} \\ &= \frac{1}{(2\bar{N}_A)^k} [1 - \Pi(k, \bar{N}_A)], \end{aligned} \quad (\text{A6})$$

where we have introduced

$$\Pi(k, \bar{N}_A) = e^{-2\bar{N}_A} \sum_{N=0}^{k-1} \frac{(2\bar{N}_A)^N}{N!} \quad (\text{A7})$$

which is the probability that the total number of counts remains smaller than k for an arbitrarily long time. This quantity becomes exponentially small when $\bar{N}_A \gg k$.

For a statistical mixture of coherent states with random phases [see Eq. (8), with $\bar{N}_A = \bar{N}_B$], $Q(k_+, k_-)$ becomes

$$\begin{aligned} Q(k_+, k_-) &= [1 - \Pi(k, \bar{N}_A)] \frac{k!}{k_+! k_-!} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \\ &\quad \times (\cos\phi)^{2k_+} (\sin\phi)^{2k_-}. \end{aligned} \quad (\text{A8})$$

The conditional probability $\mathcal{P}(k_+, k_-)$ defined in the text is equal to $Q(k_+, k_-)/[1 - \Pi]$, since $1 - \Pi$ is the probability of getting at least k counts.

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