Strongly correlated states of a cold atomic gas from geometric gauge fields

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Using exact diagonalization for a small system of cold bosonic atoms, we analyze the emergence of strongly correlated states in the presence of an artificial magnetic field. This gauge field is generated by a laser beam that couples two internal atomic states, and it is related to Berry’s geometrical phase that emerges when an atom follows adiabatically one of the two eigenstates of the atom–laser coupling. Our approach allows us to go beyond the adiabatic approximation, and to characterize the generalized Laughlin wave functions that appear in the strong magnetic field limit.

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Trapped atomic gases provide a unique playground to address many-body quantum physics in a very controlled way [1][2]. Studies of ultracold atoms submitted to artificial gauge fields are particularly interesting in this context, since they establish a link with the physics of the quantum Hall effect. Especially intriguing is the possibility of realizing strongly correlated states of the gas, such as an atomic analog of the celebrated Laughlin state [3].

One way to simulate orbital magnetism is to rotate the trapped gas [3][4]. One uses in this case the analogy between the Coriolis force that appears in the rotating frame and the Lorentz force acting on a charged particle in a magnetic field. This technique allows one to nucleate vortices and observe their ordering in an Abrikosov lattice [4]. Another promising method takes advantage of Berry’s geometrical phase that appears when a moving atom with multiple internal levels follows adiabatically a non-trivial linear combination of these levels [5]. This can be achieved in practice by illuminating the gas with laser beams that induce a spatially varying coupling between atomic internal levels [5] (for a review of recent proposals, see e.g. [7]). Recently spectacular experimental progress has been made with this technique, leading here also to the observation of quantized vortices [8].

In this Letter we focus on the generation of strongly correlated states of the atomic gas with geometrical gauge fields. We show how Laughlin-type states emerge in a small quasi two-dimensional system of trapped bosonic atoms when two internal states are coupled by a spatially varying laser field. The key point of our approach is to go beyond the adiabatic approximation and study how the possibility of transitions between the internal states modifies the external ground state of the gas. We perform exact diagonalization for $N = 4$ particles, in order to analyze the overlap between the exact ground state of the system and the Laughlin wave function, as a function of the strength of the atom–laser coupling. We identify a region of parameter space in which the ground state, despite having a small overlap with the exact Laughlin state, has an interaction energy close to zero, a large angular momentum, and a large entanglement entropy. We show that it can be represented as a Laughlin-like state with modified Jastrow factor.

We consider a small quasi two-dimensional ensemble of harmonically trapped bosonic atoms in the $x$–$y$ plane interacting with a single laser field. The single-particle Hamiltonian is given by

$$H_{sp} = \frac{p^2}{2M} + V(r) + \tilde{H}_{AL},$$  \hspace{1cm} (1)

where $M$ is the atomic mass and $V(r)$ is an external potential confining the atoms in the plane. $\tilde{H}_{AL}$ includes the atom–laser coupling as well as the internal energies. In order to minimize the technical aspects of the proposal, we consider here a very simple laser configuration to generate the geometrical gauge field. However, it is straightforward to generalize our method to more complex situations. The laser field is a plane wave with wave number $k$ and frequency $\omega_A$ propagating along the direction $y$ [see Fig. 1(a)]. It couples two internal atomic states $|g\rangle$ and $|e\rangle$ with a strength that is given by the Rabi frequency $\Omega_0$. We neglect the spontaneous emission rate of photons from the excited state $|e\rangle$, which is a realistic assumption if the intercombination line of alkali-earth or

\hspace{1cm} FIG. 1. (Color online) Schematic illustration of the considered setup. (a) Atoms are trapped in the $x$–$y$ plane and illuminated with a plane wave propagating along the $y$ direction. (b) The energy difference between the two internal states that are coupled by the laser field vary linearly along the $x$ direction. (c) Energy eigenvalues of the atom–laser coupling in the rotating wave approximation.
Ytterbium atoms is used. We further assume that the energies \( E_\text{e} = -\hbar \Omega x / (2w) \) and \( E_\text{c} = \hbar \omega_L + \hbar \Omega_\varphi x / (2w) \) of the uncoupled internal states vary linearly along \( x \), as sketched in Fig. 1(b). Here, \( w \) characterizes the length scale of the variation. This can experimentally be achieved by applying a real magnetic and/or electric field gradient. Using the rotating-wave approximation, \( H_{\text{AL}} \) can be written in the \( \{|e\}, |g\} \) basis [9]

\[
\hat{H}_{\text{AL}} = \frac{\hbar \Omega}{2} \left( \begin{array}{cc} \cos \theta & e^{i \phi} \sin \theta \\ e^{-i \phi} \sin \theta & -\cos \theta \end{array} \right),
\]

where \( \Omega = \Omega_0 \sqrt{1 + x^2/w^2} \), \( \tan \theta = w/x \), and \( \phi = ky \).

It is convenient to rewrite Eq. (1) in the basis of the local eigenvectors of \( \hat{H}_{\text{AL}} \), \( |\psi_1\rangle \) and \( |\psi_2\rangle \), associated to the eigenvalues \( \hbar \Omega/2 \) and \(-\hbar \Omega/2\), respectively. The atomic state can then be expressed as

\[
\chi(\mathbf{r},t) = a_1(\mathbf{r},t) \otimes |\psi_1(\mathbf{r})\rangle + a_2(\mathbf{r},t) \otimes |\psi_2(\mathbf{r})\rangle,
\]

where \( a_i \) captures the dynamics of the center of mass and \( |\psi_i\rangle \) of the internal degree of freedom. Projecting onto the basis \( \{|\psi_i\rangle\} \), the single-particle Hamiltonian is represented by the 2×2 matrix \( \hat{H}_{\text{sp}} = [H_{ij}] \) acting on the spinor \( [a_1(\mathbf{r},t), a_2(\mathbf{r},t)] \). We find in particular [10] [11]

\[
H_{jj} = \frac{\hbar^2 A_j^2}{2M} + U + \frac{\hbar \Omega}{2},
\]

with \( \epsilon_1 = 1 \) and \( \epsilon_2 = -1 \), where the vector \( \mathbf{A} \) and scalar \( U \) potentials read for the chosen gauge

\[
\mathbf{A}(\mathbf{r}) = \hbar k \left[ \begin{array}{c} y \\ x/4w \end{array} \right] - \frac{x}{2\sqrt{x^2 + w^2}}, \quad U(\mathbf{r}) = \frac{\hbar^2 A_j^2}{8M (x^2 + w^2)} \left( k^2 + \frac{1}{x^2 + w^2} \right).
\]

We consider atomic clouds extending over distances much smaller than \( w \). This allows us to expand the matrix elements \( H_{ij} \) up to second order in \( x \) and \( y \). In particular the artificial magnetic field can be considered as homogeneous over the cloud extent and it is equal to \( B_j = \epsilon_j \hbar k / (2w) \hat{\mathbf{e}}_z \) for an atom in \( |\psi_j\rangle \). We also choose the external potential \( V(\mathbf{r}) \) such that the total confinement for the state \( |\psi_2\rangle \) is isochoric with frequency \( \omega_\perp \):

\[
\frac{1}{2} m \omega_\perp^2 (x^2 + y^2) = U(\mathbf{r}) + V(r) - \frac{\hbar \Omega(r)}{2} + \frac{A^2(\mathbf{r})}{2M}.
\]

The Hamiltonian \( H_{\text{AL}} = p^2 / 2M + \mathbf{p} \cdot \mathbf{A} / M + M \omega_\perp^2 r^2 / 2 \) is thus circularly symmetric and its eigenfunctions are the Fock-Darwin (FD) functions \( \phi_{\ell,n} \), with \( \ell \) and \( n \) denoting the single-particle angular momentum and the Landau level, respectively. The interesting regime for addressing quantum Hall physics corresponds to quasi-flat Landau levels, which occurs when the magnetic field strength \( \eta \equiv \omega_c / 2 \omega_\perp \) is comparable to 1. Here \( \omega_c = |B_j| / M = \hbar k / (2Mw) \) is the ‘cyclotron frequency’. The energies of the states of the Lowest Landau Level (LLL) \( n = 0 \) are \( E_{\text{c},0} = \hbar \omega_\perp \left[ 1 + \ell(1 - \eta) + (k^2 \lambda_\perp^2 / 8) + \lambda_\perp^2 / (8w^2) \right] \), where \( \lambda_\perp = \sqrt{\hbar / M \omega_\perp} \).

Relevant energy scales of the single-particle problem are \( \hbar \Omega_0 \), which characterizes the internal atomic dynamics, and the recoil energy \( E_R = \hbar^2 k^2 / (2M) \), which gives the scale for the kinetic energy of the atomic center-of-mass motion when it absorbs or emits a single photon. For \( \hbar \Omega_0 \gg E_R \) the adiabatic approximation holds and the atoms initially prepared in the internal state \( |\psi_2\rangle \) will remain in this state in the course of their evolution.

The single-particle Hamiltonian \( H_{\text{LL}} \), in combination with repulsive contact interactions, then leads to quantum Hall-like physics which has already been extensively studied [3]. Our goal here is to consider corrections to the adiabatic approximation and to analyze in which respect these corrections still allow one to reach strongly correlated states. This aspect is particularly important from an experimental point of view, since the accessible range of \( \Omega_0 \) is limited if one wants to avoid spurious heating due to spontaneous emission of photons. Note that the strength of the atom–laser coupling, characterized by \( \Omega_0 \), is distinct from the strength of the magnetic field, characterized by \( \eta \). Because the magnetic field has a geometric origin, \( \eta \) is independent of the atom–laser coupling as long as the adiabatic approximation is meaningful.

In the following we consider the situation where \( \hbar \Omega_0 \) is still relatively large compared to \( E_R \), so that we can treat the coupling between the internal subspaces related to \( |\psi_{1,2}\rangle \) in a perturbative manner. In a systematic expansion in powers of \( \Omega_0^{-1} \), the first correction to the adiabatic approximation consists of the spinor component \( a_2 \) in replacing \( H_{\text{LL}} \) by the effective Hamiltonian \( H_{\text{eff}}^{(2)} = H_{\text{LL}} - H_{\text{LL}} H_{\text{LL}} / (\hbar \Omega_0) \). The additional term \( H_{21} H_{12} / (\hbar \Omega_0) \) is somewhat reminiscent of the anisotropic potential that is applied to set an atomic cloud in rotation [12] [13]. It is however mathematically more involved and physically richer, as it includes not only powers of \( x \) and \( y \), but also spatial derivatives with respect to these variables (see Supplementary Material).

The quasi-degeneracy in the LLL can lead to strong correlations as the interaction picks a many-body ground state for the system. The interaction between the atoms is well described by a contact interaction with a coupling constant \( g = 8\pi \sqrt{\alpha^{-1} \ell} \) for the quasi two-dimensional confinement. Here \( \alpha_s \) is the s-wave scattering length and \( \ell \) the thickness of the gas in the strongly-confined \( z \) direction. The many-body Hamiltonian then reads

\[
H = \sum_{i=1}^{N} H_{\text{eff}}^{(2)}(i) + \frac{\hbar^2 g}{M} \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j).
\]

Using an algorithm for exact diagonalization within the LLL of \( H_{\text{LL}} \), we have determined the many-body ground state (GS) of a system for various atom numbers \( N \), laser couplings \( \Omega_0 \) and magnetic field strengths \( \eta \). To ensure
the validity of the LLL assumption, we demand that the difference in energy between different Landau levels is larger than the kinetic energy of any particle in a FD state inside a Landau level. In addition, in the full many body problem the interaction energy per particle is always much smaller than the energy difference between adjacent landau levels. The main results are summarized in Fig. 2 for $N = 4$ atoms, $k = 10/\Lambda_\perp$, and $gN = 6$. For the whole range of parameters considered our algorithm can run to $N = 6$, but to easily locate the various regimes on a single graph, we restrict ourselves to four particles.

In Fig. 2(a) we show the expectation value of the total angular momentum of the GS as a function of $\eta$ and $\hbar \Omega_0/E_R$. For large $\Omega_0$ we recover the step-like structure that is well known for rotating bosonic gases, with plateaus at $L = 0, 4, 8,$ and $12$ [3, 14]. For an axisymmetric potential containing $N$ bosons, the value $L = N$ (here $L = 4$) corresponds to a single, well centered vortex, and the value $L = N(N - 1)$ (here $L = 12$) corresponds to the Laughlin state, with a filling factor $1/2$. For decreasing values of $\Omega_0$ the transitions between the plateaus become broader and are displaced towards smaller values of $\eta$.

An interesting measure for the correlations in the GS is the entanglement entropy $S = -\text{Tr} \left[ \rho^{(1)} \ln \rho^{(1)} \right]$, where $\rho^{(1)}$ is the one-body density matrix associated to the GS wave-function [15]. This quantity is zero for a true Bose–Einstein condensate, since all particles occupy the same mode and $\rho^{(1)}$ represents a pure state. For the Laughlin wave-function with $N$ bosons, $2N - 1$ single-particle states are approximately equally populated, and the entropy is $\sim \ln(2N - 1)$. The entropy $S$ is plotted in Fig. 2(b) and it presents features that are similar to that of Fig. 2(a). The region of $\langle L \rangle = 0$ corresponds to a fairly condensed region with $S \sim 0$. The entropy decreases with $\Omega_0$ but the dependence on $\eta$ exhibits steps, similarly to that of $\langle L \rangle$. In the one vortex region, corresponding to $\langle L \rangle = N$, the condensation is already not complete, and the entropy approaches 1. Finally, it gradually increases as we increase $\eta$, and reaches its maximum value in the Laughlin-like region, $\eta > 0.95$.

In Fig. 2(c) we depict the average interaction energy as a function of $\eta$ and $\hbar \Omega_0/E_R$. In the inset we also plot for $L = 0$ and $L = N = 4$ the analytical result expected in an axisymmetric potential, $E_{\text{int}} = gN(2N - L - 2)/(8\pi)$, valid for $L = 0$ and $2 \leq L \leq N$ [10]. The interaction energy approaches zero as we increase $\eta$, indicating the Laughlin-like nature of the states in the region $\eta \geq 0.95$.

The standard bosonic Laughlin state (at half filling) has the analytical form [17, 18]

$$\Psi_L(z_1, \ldots, z_N) = N \prod_{i<j} |z_i - z_j|^2 e^{-\sum |z_i|^2/2\lambda^2_{\perp}},$$

where $N$ is a normalization constant and $z = x + iy$. We calculated the dependence of the squared overlap $|\langle \Psi_L | \Psi_{\text{GS}} \rangle|^2$ of the Laughlin state with the exact GS as a function of the magnetic field strength $\eta$ and the atom–laser coupling $\Omega_0$. The result is plotted in Fig. 3 for

![Figure 2](image_url)

**FIG. 2.** (Color online) Characterization of the ground state for $N = 4$ atoms as a function of $\eta$ and $\hbar \Omega_0/E_R$. The insets concentrate on two different values of $\hbar \Omega_0/E_R = 40$, and 100, respectively. (a) Average value of the total angular momentum, in units of $\hbar$. (b) Entanglement entropy. (c) Interaction energy, in units of $\hbar \omega_{\perp}$.

![Figure 3](image_url)

**FIG. 3.** (Color online) Squared overlap $|\langle \Psi_{\text{LS}} | \Psi_{\text{GS}} \rangle|^2$ as a function of $\eta$ and $\hbar \Omega_0/E_R$ for $N = 4$. The dashed line marks the region of squared overlap larger than 0.8. The inset depicts the squared overlap for $\hbar \Omega_0/E_R = 100$ (solid) and 40 (dashed) as a function of $\eta$. 

In conclusion we have performed exact diagonalization to analyze the ground state of a small cloud of bosonic atoms subjected to an artificial gauge field. Our approach allowed us to explore both the regime of very large atom–laser coupling, where the adiabatic approximation is valid, and the case of intermediate coupling strengths. In the first case we recovered the known results for a single component gas in an axisymmetric potential. The second case is crucial for practical implementations because it requires less light intensity on the atoms, which decreases the residual heating due to photon scattering. In this case we have identified a regime where a strongly correlated ground state emerges, which shares many similarities with the Laughlin state in terms of angular momentum and energy, although the overlap between the two remains small. We have also proposed an ansatz that represents this state quite accurately for a region of the parameter space.

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Note that the Laughlin-like region decreases notably in size as $N$ is increased.
For completeness we provide here the explicit expression for the term $H_{21}H_{12}$ appearing in the perturbatively derived Hamiltonian $H_{22}^{\text{eff}}$. As explained in the text we consider up to quadratic terms in $x$ and $y$. The explicit expression then reads

$$H_{21}H_{12} = \left( \frac{\hbar^4}{4M^2w^3} - \frac{2x^2\hbar^4}{M^2w^6} + \frac{k^2x^2\hbar^4}{16M^2w^4} + \frac{k^4x^2\hbar^4}{64M^2w^2} \right)$$

$$+ \left( \frac{ikxy\hbar^4}{4M^2w^5} - \frac{k^2y^2\hbar^4}{64M^2w^2} \right) \partial_y$$

$$+ \left( \frac{i\bar{h}x\hbar^4}{4M^2w^3} - \frac{ik^3x\hbar^4}{8M^2w} \right) \partial_y$$

$$+ \left( -\frac{i\bar{h}y\hbar^4}{4M^2w^3} + \frac{k^2y^2\hbar^4}{4M^2w^2} \right) \partial_x$$

$$+ \left( -\frac{\hbar^4}{4M^2w^2} + \frac{x^2\hbar^4}{2M^2w^4} \right) \partial_x^2. \quad (11)$$