Lecture 7: BEC in double well potentials
One-dimensional optical lattices

Dipole potential felt by an atom with polarizability $\alpha > 0$ in an electric field $E$: 

$$V_{\text{dip}} = -\frac{1}{2} \alpha \chi_L |E|^2$$

A coherent superposition of waves with different wavevectors results in interferences.

Atoms are trapped near the interference maxima, by a potential varying very quickly on the scale of the optical wavelength.

Two counter-propagating laser beams create a standing wave with period $d = \frac{\pi}{k_L}$

Trapping potential of the form

$$V_{\text{lat}} = -2V_0 \left(1 + \cos(2k_Lx)\right) = -4V_0 \cos^2(k_Lx).$$

The lattice spacing can be varied by tuning the angle $\beta$ between the two beams
Atoms are trapped near the intensity maxima (red detuning) or minima (blue detuning)

One dimensional lattice: stacks of 2D gases

Additional (weaker) trapping potentials provide confinement in the plane perpendicular to the lattice

Three dimensional lattice: regular array of trapping sites in three dimensions

Cut of the potential in the x-y plane
Goal of the next three lectures
In the following lectures, we will examine the behavior of quantum gases periodic potentials.

Why is this interesting?
1) Natural connection with solid-state physics (band structure, etc ...)
2) Path to realize strongly interacting gases and new quantum phases (blackboard)

Outline
1) Minimal instance of Bose-Hubbard physics: many atoms interacting atoms in two wells
2) Non-interacting atoms and many wells: band structure and related physics
3) Many interacting atoms in many wells: superfluid-Mott insulator transition

In these lectures, we restrict ourselves to bosonic gases.

Exercise class
1) Bragg diffraction
2) Bloch oscillations in optical lattices
3) Josephson junction physics with BECs in double-well potential
Double well potential (Heidelberg experiment)

Experimental setup: standing wave + strong harmonic trap

Caricature of the actual potential: double square well

Height $V_0$

Characteristic energy $E_d = \frac{\hbar^2}{2md^2}$
Double square well: lowest-lying states

Energies of the two lowest states:

\[ E_0/E_d \approx \pi^2 \left( 1 - \frac{2}{\kappa d} - \frac{4}{\kappa d} e^{-\kappa d} \right) \]

\[ E_1/E_d \approx \pi^2 \left( 1 - \frac{2}{\kappa d} + \frac{4}{\kappa d} e^{-\kappa d} \right) \]
Double square well

Localized states:  \( \chi_{L/R}(x) = \frac{1}{\sqrt{2}}[\chi_0(x) \pm \chi_1(x)] \)

\[ H = E_0|\chi_0\rangle\langle\chi_0| + E_1|\chi_1\rangle\langle\chi_1| + \cdots, \]
\[ = \frac{E_0 + E_1}{2}(|\chi_L\rangle\langle\chi_L| + |\chi_R\rangle\langle\chi_R|) + \frac{E_0 - E_1}{2}(|\chi_L\rangle\langle\chi_R| + |\chi_R\rangle\langle\chi_L|) + \cdots \]

Splitting between the two lowest states = rate of quantum tunneling from left to right

Dealing with interactions is much easier using the basis of localized states due to the negligible overlap between them.
Double well potential (Heidelberg experiment)

Localized states: \( \chi_{L/R}(x) = \frac{1}{\sqrt{2}}[\chi_0(x) \pm \chi_1(x)] \)

Same qualitative conclusions as in the square well case
Interacting bosons in a double-well potential: 
Gross-Pitaevskii theory

Non-interacting atoms condense in the lowest single-particle state. The ground state many-body wavefunction is thus:

\[ |N : \chi_0 \rangle = \frac{1}{\sqrt{N!}} (\hat{a}_0^\dagger)^N |\varnothing \rangle, \]

where \( \hat{a}_0 \) annihilates a particle in the lowest state, \( \chi_0 \).

In Gross-Pitaevskii theory, one assumes that the many-body wavefunction retains the same global form: Only the spatial orbital the atoms condense into changes to minimize the energy. This corresponds to a many-body wavefunction of the form

\[ |N : \phi \rangle = \frac{1}{\sqrt{N!}} \left( \alpha_L \hat{a}_L^\dagger + \alpha_R \hat{a}_R^\dagger \right)^N |\varnothing \rangle, \]

where \( \hat{a}_L/R \) annihilates a particle in the localized states, \( \chi_L/R \).

Following the steps leading to the Gross-Pitaevskii equation, this corresponds to a condensate wavefunction

\[ \psi(\mathbf{r}) = \alpha_L \chi_L(\mathbf{r}) + \alpha_R \chi_R(\mathbf{r}) \]

The complex amplitudes \( \alpha_{L/R} = \sqrt{\frac{N_{L/R}}{N}} e^{i\phi_{L/R}} \) define the mean populations and relative phase.
Two-mode Bose-Hubbard Hamiltonian

Generic Hamiltonian for contact interactions

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]

\[ \hat{H}_0 = \int d\mathbf{r} \, \hat{\Psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2M} \Delta + V(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) \]

\[ \hat{H}_{\text{int}} = \frac{g}{2} \int d^{(3)}r \, \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \]

Projection to lowest doublet:

\[ \hat{\Psi}(\mathbf{r}) = \chi_L(\mathbf{r}) \hat{a}_L + \chi_R(\mathbf{r}) \hat{a}_R + \cdots \]

\[ \hat{H}_0 = -\frac{E_1 - E_0}{2} (\hat{a}_R^\dagger \hat{a}_L + \text{h.c.}) \]

\[ \hat{H}_{\text{int}} \approx \frac{U}{2} \times \left( \hat{N}_L(\hat{N}_L - 1) + \hat{N}_R(\hat{N}_R - 1) \right) \]

Number operators: \( \hat{N}_{L/R} = \hat{a}_{L/R}^\dagger \hat{a}_{L/R} \)

Final result is the (two-mode) Bose-Hubbard Hamiltonian:

\[ \mathcal{H}_{\text{BH}} = -J(\hat{a}_L^\dagger \hat{a}_R + \text{h.c.}) + U \Delta \hat{n}^2 \]

where \( \Delta \hat{n} = \frac{\hat{N}_L - \hat{N}_R}{2} \) quantifies the population imbalance between the two wells.
Measurement of population imbalance and relative phase

Population imbalance:

direct counting using imaging with high spatial resolution


Relative phase:

Time of flight expansion from the double well potential produces interference fringes

\[ n_{\text{t.o.f.}}(r) = \left( \frac{m}{\hbar t} \right)^3 N W \left( k = \frac{m r}{\hbar t} \right)^2 \left[ 1 + \mathcal{C} \cos \left( \frac{mdx}{2\hbar t} \right) + S \sin \left( \frac{mdx}{2\hbar t} \right) \right] \]

\[ \mathcal{C} = \frac{\langle \hat{a}^\dagger_L \hat{a}_R + \hat{a}_R \hat{a}_L \rangle}{N}, \quad S = i \frac{\langle \hat{a}_L \hat{a}_R - \hat{a}^\dagger_R \hat{a}_L \rangle}{N}, \]

For a GP state released from the double-well potential, the fringe position reveals the initial relative phase

\[ n_{\text{t.o.f.}}(r) = \left( \frac{m}{\hbar t} \right)^3 N \left| W \left( k = \frac{m r}{\hbar t} \right) \right|^2 \left[ 1 + 2 \sqrt{\frac{N_L N_R}{N^2}} \cos \left( \phi + \frac{mdx}{\hbar t} \right) \right] \]

Interference and number-squeezing

Series of t.o.f. images
Relative phase is stable
Average t.o.f. image

Population imbalance histograms

Binomial distribution


Conclusion:
GP theory not always valid!

# of shots

Conclusion:
GP theory not always valid!
Coherence factor

\[ \alpha \]

Variance of population imbalance

\[ \Delta n^2 \]

Distribution in Fock space

\[ \rho(n) \]

T.o.f. interference pattern

\[ \rho(k) \]
Number squeezing and entanglement

Number squeezing
\[ \zeta_N = \frac{4\langle \Delta n^2 \rangle}{N} \]

Separability criterion
\[ \zeta_S = \frac{4\langle \Delta n^2 \rangle}{NC} \]

See also: Maussang et al., PRL 105, 080403 (2010).