ULTRACOLD COLLISIONS

2. Resonant scattering
Bound states and scattering resonances

Square well potential

A scattering resonance occurs each time a bound state enters the potential

Using the Bethe-Peierls prescription:

Look for bound states ($E<0$ solutions of Shrödinger’s Equation)

$$E = -\frac{\hbar^2 \kappa^2}{2\mu}$$

$$\psi(r) = A e^{-\kappa r}$$

A bound state exists only for $a>0$, and then $\kappa=a$: « halo dimer »
Control of the scattering length: Feshbach resonances

\[ E_b = \frac{\hbar^2}{ma^2} \]

\[ a \geq 0 \]

\[ a = \infty \]

\[ a \leq 0 \]
ULTRACOLD COLLISIONS

3. Quantum statistics
Quantum statistics

$Y_l^m$ has the parity of $l$

- Fermions must scatter in odd-$l$ wave
- Boson must scatter in even-$l$ wave

At low energy, scattering in $s$ ($l=0$)-wave only:

**no scattering between spin polarized fermions at low energy**

**For bosons**, competition kinetic energy/interaction energy.

Size of the single-particle ground state: $a_{ho} = \sqrt{\frac{\hbar}{m\omega}}$

\[\frac{E_{\text{int}}}{E_{\text{kin}}} \approx \frac{Na}{a_{ho}} \gtrsim 10\] for typical numbers.

Typical alkali BECs are dominated by the interaction energy.
Suppression of collisions in a spin polarized Fermi gas

THE WEAKLY INTERACTING BEC
THE WEAKLY INTERACTING BEC

1. The Mean-Field Approximation
The mean-field approximation

Quantum dynamics described by the many-body Hamiltonian

\[ H = \sum_{i=1}^{N} \left[ \frac{p_i^2}{2m} + U(r_i) \right] + \frac{1}{2} \sum_{i \neq j} V(r_i - r_j) \]

\( U \): trapping potential
\( V \): interaction potential

Variational procedure: Schrödinger’s Equation can be derived from a least-action principle associated with the Lagrangian

\[ L[\psi] = \langle \psi | H | \psi \rangle - i\hbar \langle \psi | \left( \frac{d}{dt} | \psi \rangle \right) \]

Variational Ansatz: place all particles in the same quantum state (a.k.a. mean-field approximation).

\[ \psi(r_1, r_2, ..., r_N, t) = \prod_{i=1}^{N} \Phi(r_i, t) \]
The Gross-Pitaevskii Equation

\[ L[\psi] = N \int d^3 r \Phi^*( -\frac{\hbar^2}{2m} \nabla^2 + U - i\hbar \partial_t ) \Phi + \frac{N(N-1)}{2} g \int d^3 r |\Phi|^4 \]

\[ g = 4\pi \hbar^2 a / m \] = coupling constant of the pseudo-potential

**Φ solution of the Gross-Pitaevski Equation**

\[ i\hbar \partial_t \Phi = -\frac{\hbar^2}{2m} \nabla^2 \Phi + U(r) \Phi + Ng |\Phi|^2 \Phi \]

Non-linear Schrödinger equation, due to interactions

Stationary solutions: \( \Phi(r, t) = \varphi(r)e^{-i\mu_0 t/\hbar} \)

\[ -\frac{\hbar^2}{2m} \nabla^2 \varphi + U(r) \varphi + Ng |\varphi|^2 \varphi = \mu_0 \varphi \]

\( \mu_0 \) is not the energy, but the chemical potential of the cloud \( \mu_0 = \frac{\partial E}{\partial N} \)
Scaling laws

\[ E[\psi] = \langle \psi | H | \psi \rangle = E_{\text{kin}} + E_{\text{trap}} + E_{\text{int}} \]

Isotropic harmonic trap of frequency \( \omega \)

### Scalings

<table>
<thead>
<tr>
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<th>( E_{\text{kin}} )</th>
<th>( E_{\text{trap}} )</th>
<th>( E_{\text{int}} )</th>
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<tbody>
<tr>
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<td>( N\hbar^2 / mL^2 )</td>
<td>( Nm\omega^2 L^2 )</td>
<td>( N^2 g / L^3 )</td>
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\( g > 0 \): \( E_{\text{kin}} \) and \( E_{\text{int}} \) expand the cloud; \( E_{\text{trap}} \) shrinks it

\( g < 0 \): \( E_{\text{kin}} \) expand the cloud; \( E_{\text{int}} \) and \( E_{\text{trap}} \) shrinks it
THE WEAKLY INTERACTING BEC

2. The repulsive BEC
Repulsive Bose-Einstein condensate scaling behavior

$N \sim a_{oh} (Na / a_{oh})^{1/5}$
For $a>0$ and $Na/a_{ho}>>1$, neglect the kinetic energy term in the EGP

**Thomas-Fermi profile** \( n(r) = N \left| \varphi(r) \right|^2 = \left( \mu_0 - U(r) \right) / g \) as long as $n>0$.

**Physical interpretation:**
This is the Local Density Approximation (LDA) for $\mu_{hom}(n) = gn$

For a harmonic trap of frequencies $\omega_x, \omega_y, \omega_z$:

\[
\int d^3r n(r) = N \implies \mu = \frac{\hbar \bar{\omega}}{2} \left( 15 \frac{Na}{a_{ho}} \right)^{2/5}
\]

with $\bar{\omega}^3 = \omega_x \omega_y \omega_z$, $a_{oh} = \sqrt{\hbar / m \bar{\omega}}$

Radius of the cloud in the direction $i=x,y,z$

\[
R_i^2 = \frac{\hbar \bar{\omega}}{m \omega_i^2} \left( 15 \frac{Na}{a_{ho}} \right)^{2/5} \propto a_{ho}^2 \left( \frac{Na}{a_{ho}} \right)^{2/5} \quad \text{(for an isotropic trap)}
\]
Comparison with experiment
THE WEAKLY INTERACTING BEC

3. The attractive BEC
Attractive Bose-Einstein condensate scaling behavior

The BEC is unstable above $N_{\text{crit}}$!

$E_{\text{pot}} \sim |E_{\text{int}}|$
$\alpha < 0$ in free space: collapse

Bose-Nova in $^{85}$Rb (Boulder)