Quantum-jump approach to dissipative processes: application to amplification without inversion

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Several recent studies have shown that the time evolution of an atom submitted to coherent laser fields and to dissipative processes, such as spontaneous emission of photons or excitation by a broadband incoherent field, can be considered to consist of a sequence of coherent evolution periods separated by quantum jumps occurring at random times. A general statistical analysis of this random sequence is presented for the case in which the number of relevant atomic states is finite and the delay functions giving the distribution of the time intervals between two successive jumps can easily be calculated. These general considerations are then applied to a simple model recently proposed for demonstrating the possibility of amplification without inversion of populations. We show how the quantum-jump approach allows one to calculate the respective contributions of the various physical processes responsible for the amplification or the attenuation of the probe field and to get new insights into the relevant physical mechanisms.

1. INTRODUCTION

During the past few years several experiments have shown that the time evolution of a single atomic system driven by coherent laser fields and subjected to dissipative processes such as spontaneous emission can exhibit discontinuous abrupt changes, also called quantum jumps.\(^1\)\(^-\)\(^3\) The analysis of these experiments has stimulated the development of new theoretical approaches to dissipative processes in which the time evolution of the atomic system is pictured as consisting of a series of coherent evolution periods separated by quantum jumps occurring at random times.\(^4\)\(^-\)\(^10\) The equivalence between these quantum-jump approaches and the usual description of dissipative processes in terms of master equations for the atomic-density operator has been demonstrated.\(^7\)\(^-\)\(^8\) Some connection seems also to exist between these analyses and general stochastic formulations of quantum mechanics.\(^9\)\(^-\)\(^11\)

When the number of relevant atomic states involved in dissipative processes is finite and when the Hamiltonian is time independent, the method of delay functions introduced in Refs. 4 and 5 is particularly convenient for performing Monte Carlo simulations of the sequence of quantum jumps.\(^5\)\(^6\) In this paper we show how this delay-function approach can be used to derive general statistical properties of the coherent evolution periods taking place between two successive quantum jumps. In order to keep the discussion as physical as possible, we introduce the method by dealing with a simple model that was recently proposed for demonstrating the possibility of an amplification without inversion.\(^12\)\(^-\)\(^13\) We show that the quantum-jump approach, based on the delay functions, gives the same results as the master-equation approach followed in Ref. 12 and provides in addition new insights into the underlying physical mechanisms. A brief account of these results, without any demonstration, was presented elsewhere.\(^14\)

The paper is organized as follows. We first give in Section 2 a general qualitative presentation of the method. Starting from the model of Ref. 12, we introduce the basic ideas that are used throughout the paper: quantum jumps associated with dissipative processes, coherent evolution periods between two successive quantum jumps, and physical processes associated with each coherent evolution period. We then introduce in Section 3 the various quantities that fully characterize the stochastic evolution of the atom, in particular the delay functions. Some results of Monte Carlo simulations are also presented. Section 4 is devoted to the derivation of general statistical properties of the sequence of quantum jumps, and we explain how it is possible to calculate various properties of the coherent evolution periods, such as their probabilities or their mean duration. This general method is applied in Section 5 to the model introduced in Section 2, and the probabilities of the various coherent evolution periods are calculated analytically. Finally, the results of this calculation are discussed in Section 6; these allow one to present a detailed analysis of the various competing physical mechanisms. Appendices A and B are devoted to the derivation of some properties of the coherent evolution periods.

2. PRINCIPLE OF THE METHOD

A. Model

As in Ref. 12, we consider a three-level atom with one excited state \(e\) and two lower states \(g_1\) and \(g_2\), forming a \(\Lambda\) configuration (Fig. 1). We denote by \(\omega_{g_1}\) and \(\omega_{g_2}\) the frequencies of the two allowed transitions \(g_1 \leftrightarrow e\) and \(g_2 \leftrightarrow e\). These two transitions are excited by two laser fields with
Fig. 1. Three-level atom forming a Λ configuration and subjected to dissipative processes, inducing transitions between the three levels e, g₁, g₂ with rates $g_1$, $g_2$, $R_1$, $R_2$ (arrows). The atom is also driven by two laser fields with frequencies $\omega_{L1}$ and $\omega_{L2}$ that are close, respectively, to the frequencies $\omega_{el}$ and $\omega_{e2}$ of the two transitions $g_1 \leftrightarrow e$ and $g_2 \leftrightarrow e$.

By spontaneous emission the atom can decay from e to $g_1$ or to $g_2$ with rates equal, respectively, to $R_1$ and $R_2$ (wavy oblique arrows of Fig. 1). The atom is also assumed to be subjected to broadband incoherent fields that induce both absorption and stimulated-emission processes between e and $g_1$, on the one hand, and $g_2$ on the other hand, with rates equal, respectively, to $R_1$ and $R_2$ (straight oblique arrows of Fig. 1).

Such a purely radiative and closed system has been introduced by the authors of Ref. 12 to show that the field $\omega_{L1}$, considered as a weak probe beam, can be amplified for certain values of the parameters, even if the lowest sublevel g1, contains more than one half of the total population. Such a result is derived in Ref. 12 from the solution of the optical Bloch equations, which describe the evolution of the density operator driven by the coherent fields $\omega_{L1}$ and $\omega_{L2}$ and subjected to the dissipative processes described by $\Gamma_1$, $\Gamma_2$, $R_1$, and $R_2$. Rather than solving the optical Bloch equations, we follow here the evolution of the state vector |$\psi$> of a single atom. The usual results provided by the optical Bloch equations are recovered by averaging over different realizations of the atomic stochastic evolution.

B. Manifolds of Atom + Laser Photon States

It is convenient to use here a quantum description of the two laser fields $\omega_{L1}$ and $\omega_{L2}$. If these fields are quasi-resonant ($|\delta_1| \ll \omega_{el}$, $|\delta_2| \ll \omega_{e2}$), the states of the total system atom + laser photons are grouped into manifolds $\mathcal{E}(N_1, N_2)$ of three quasi-degenerate states, which become degenerate if $\delta_1 = \delta_2 = 0$ (see Fig. 2):

$$\mathcal{E}(N_1, N_2) = \{|e, N_1, N_2>, |g_1, N_1 + 1, N_2>, |g_2, N_1, N_2 + 1>\}. \quad (2.2)$$

As a result of the above quasi-resonant assumption, the energy distance between two different manifolds is very large compared with the energy splittings within a given manifold.

The atom in $g_1$ ($g_2$) can absorb one $\omega_{L1}$ ($\omega_{L2}$) photon and be transferred to e, with the corresponding coupling being characterized by the Rabi frequency $\Omega_1$ ($\Omega_2$). More precisely, we have

$$\langle e, N_1, N_2|V_{L1}|g_1, N_1 + 1, N_2\rangle = \hbar \Omega_1/2, \quad (2.3a)$$
$$\langle e, N_1, N_2|V_{L2}|g_2, N_1, N_2 + 1\rangle = \hbar \Omega_2/2, \quad (2.3b)$$

where $V_{L1}$ is the atom–laser interaction Hamiltonian. These couplings are represented by the horizontal arrows of Fig. 2 and exist only within a given manifold. In the absence of dissipative processes, the system, initially in $\mathcal{E}(N_1, N_2)$, would remain forever in the same manifold, and its coherent evolution could be described entirely in terms of Rabi nutations between the three states of $\mathcal{E}(N_1, N_2)$.

C. Quantum Jumps Associated with Dissipative Processes

The system can leave $\mathcal{E}(N_1, N_2)$ only by a quantum jump that brings it into a neighboring manifold (oblique arrows starting from $\mathcal{E}(N_1, N_2)$ in Fig. 2). For example, it can jump from $|e, N_1, N_2\rangle$ into $|g_1, N_1, N_2\rangle$, which belongs to $\mathcal{E}(N_1 - 1, N_2)$, by a spontaneous-emission process or by an incoherent stimulated emission process with a total rate $\Gamma_1$ given by

$$\tilde{\Gamma}_1 = \Gamma_1 + R_1. \quad (2.4a)$$

Similar quantum jumps can take place between $|e, N_1, N_2\rangle$ and $|g_2, N_1, N_2\rangle$, which belongs to $\mathcal{E}(N_1, N_2 - 1)$, with a rate

$$\tilde{\Gamma}_2 = \Gamma_2 + R_2. \quad (2.4b)$$

The system can also jump from $|g_1, N_1 + 1, N_2\rangle$ ($|g_2, N_1, N_2 + 1\rangle$) into $|e, N_1 + 1, N_2\rangle$ ($|e, N_1, N_2 + 1\rangle$), which belongs to $\mathcal{E}(N_1 + 1, N_2)$ by an incoherent absorption process with a rate $R_1$ ($R_2$).

Figure 2 also represents the quantum jumps bringing the system into $\mathcal{E}(N_1, N_2)$ [oblique arrows arriving in $\mathcal{E}(N_1, N_2)$] from $|g_1, N_1, N_2\rangle$ ($|g_2, N_1, N_2\rangle$) into $|e, N_1, N_2\rangle$ with a rate $R_1$ ($R_2$), and from $|e, N_1 + 1, N_2\rangle$ ($|e, N_1, N_2 + 1\rangle$) into $|g_2, N_1 + 1, N_2\rangle$ ($|g_2, N_1, N_2 + 1\rangle$) with a rate $\tilde{\Gamma}_1$ ($\tilde{\Gamma}_2$).
D. Picturing the Time Evolution

Each coherent evolution period \((i, j)\) in a given manifold \(\mathcal{E}(N_1, N_2)\) may be characterized by the state \(i\) of \(\mathcal{E}(N_1, N_2)\) in which the system enters \(\mathcal{E}(N_1, N_2)\), after the quantum jump of entry, and by the state \(j\) of \(\mathcal{E}(N_1, N_2)\) from which the system leaves \(\mathcal{E}(N_1, N_2)\), through the quantum jump of exit. To keep the notation as simple as possible, we label the system leaves \(\mathcal{E}(N_1, N_2)\) and \(\mathcal{E}(N_1, N_2)\), and consider the reverse periods \((1, e)\), which corresponds to the absorption of one photon \(\omega_{12}\) without any change of \(N_e\),

\[
\text{period } (e, 1) \rightarrow \Delta N_1 = +1, \Delta N_2 = 0, \quad (2.5c)
\]

and consider the reverse periods \((1, e)\), which corresponds to the absorption of one photon \(\omega_{11}\) without any change of \(N_2\),

\[
\text{period } (1, e) \rightarrow \Delta N_1 = -1, \Delta N_2 = 0. \quad (2.5d)
\]

It may easily be checked that, for the five remaining periods \((i, j)\), \(\Delta N_i = 0\).

If we are able to calculate the relative probabilities of the four periods \((2, 1), (1, 2), (1, e)\) and \((e, 1)\), we can thus determine whether the field \(\omega_{12}\) will be amplified or attenuated and identify the respective contributions of the various physical processes, stimulated Raman gain, stimulated Raman loss, induced emission, absorption, which are involved. This is the great advantage of the quantum-jump approach presented here as compared with the optical Bloch equations approach, which gives only the total gain (or loss).

3. Characterization of the Stochastic Evolution

In this section, we introduce the various quantities that are needed for characterizing completely the stochastic properties of the random sequence of Fig. 3.

A. Evolution within a Manifold

Consider first a coherent evolution period. We know that the system entered the manifold \(\mathcal{E}(N_1, N_2)\) in the state \(|j\rangle\) at time \(t\), and we want to study the subsequent time evolution of the projection of the state vector of the system onto \(\mathcal{E}(N_1, N_2)\). Since dissipative processes make the system leave the various states \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) with well-defined rates \(G_j\) given by (see Fig. 2)

\[
G_1 = R_1, \quad G_2 = R_2, \quad G_e = \tilde{R}_1 + \tilde{R}_2 = \tilde{R},
\]

one can show that the time evolution within \(\mathcal{E}(N_1, N_2)\) is governed by an effective non-Hermitian Hamiltonian \(\hat{H}_{eff}\) obtained by adding to the energies of the three states \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) an imaginary part \(-i\hbar G_j/2\). One possible method for demonstrating such a result is to study the projection of the resolvent operator onto \(\mathcal{E}(N_1, N_2)\) (see, for example, Chap. 3 of Ref. 15 and Sec. 1 of Ref. 16). Using Eqs. (2.1) and (2.3), we thus get

\[
\hat{H}_{eff} = h \begin{bmatrix}
-iG_1/2 + \delta_1 & 0 & \Omega_1/2 \\
0 & -iG_2/2 + \delta_2 & \Omega_2/2 \\
\Omega_1/2 & \Omega_2/2 & -iG_e/2
\end{bmatrix} \quad (3.2)
\]

From Eq. (3.2) one can calculate the probability amplitude

\[
c_{ij}(\tau) = \langle j|\exp(-i\hat{H}_{eff}\tau\hbar)|i\rangle
\]

for the system to be found in the state \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) at time \(t + \tau\), when it is known that it started from the state \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) at time \(t\). Multiplying \(|c_{ij}(\tau)|^2\) by \(G_j\,\mathrm{d}r\) then gives the conditional probability,

\[
W_{ij}(\tau)\,\mathrm{d}r = G_j|c_{ij}(\tau)|^2\,\mathrm{d}r,
\]

which corresponds to the stimulated emission of one photon \(\omega_{11}\) without any change of \(N_2\),

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\text{period } (e, 1) \rightarrow \Delta N_1 = +1, \Delta N_2 = 0, \quad (2.5c)
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one can show that the time evolution within \(\mathcal{E}(N_1, N_2)\) is governed by an effective non-Hermitian Hamiltonian \(\hat{H}_{eff}\) obtained by adding to the energies of the three states \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) an imaginary part \(-i\hbar G_j/2\). One possible method for demonstrating such a result is to study the projection of the resolvent operator onto \(\mathcal{E}(N_1, N_2)\) (see, for example, Chap. 3 of Ref. 15 and Sec. 1 of Ref. 16). Using Eqs. (2.1) and (2.3), we thus get

\[
\hat{H}_{eff} = h \begin{bmatrix}
-iG_1/2 + \delta_1 & 0 & \Omega_1/2 \\
0 & -iG_2/2 + \delta_2 & \Omega_2/2 \\
\Omega_1/2 & \Omega_2/2 & -iG_e/2
\end{bmatrix} \quad (3.2)
\]

From Eq. (3.2) one can calculate the probability amplitude

\[
c_{ij}(\tau) = \langle j|\exp(-i\hat{H}_{eff}\tau\hbar)|i\rangle
\]

for the system to be found in the state \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) at time \(t + \tau\), when it is known that it started from the state \(|j\rangle\) of \(\mathcal{E}(N_1, N_2)\) at time \(t\). Multiplying \(|c_{ij}(\tau)|^2\) by \(G_j\,\mathrm{d}r\) then gives the conditional probability,

\[
W_{ij}(\tau)\,\mathrm{d}r = G_j|c_{ij}(\tau)|^2\,\mathrm{d}r,
\]
that the system leaves $\mathcal{E}(N_1, N_2)$ by a quantum jump from the state $|j\rangle$ between times $t + \tau$ and $t + \tau + \tau$. The nine functions $W_{ij}(\tau)$, with $i, j = 1, 2, \ldots, e$, give the distribution of the time intervals. These delay functions are quite analogous to those introduced previously\textsuperscript{4,5} for analyzing the intermittent fluorescence that can be observed on a single trapped ion.\textsuperscript{4,3} The delay functions obey the normalization condition

$$\sum_j \int_0^\infty W_{ij}(\tau)d\tau = 1 \quad \text{for all } i,$$

proved in Appendix A, which results from the fact that the system has certainly left $\mathcal{E}(N_1, N_2)$ after an infinite time.

B. Characterization of a Jump

From Fig. 2 one can also find the probabilities $\pi_{kj}$, if a quantum jump starts from the state $|k\rangle$ of a manifold, that this quantum jump brings the system into the state $|j\rangle$ of a neighboring manifold:

$$\pi_{kj} = \delta_{kj}, \quad \pi_{kj} = \delta_{kj}, \quad \pi_{kj} = \delta_{kj} \Gamma_1 + \delta_{kj} \Gamma_2.$$  

These probabilities are obviously normalized:

$$\sum_j \pi_{kj} = 1 \quad \text{for all } k.$$  

Knowing $W_{ik}(\tau)$ and $\pi_{kj}$, one can decide randomly the time at which the system will leave $\mathcal{E}(N_1, N_2)$ and the state $|k\rangle$ from which the corresponding jump will occur, as well as the state $|j\rangle$ in which the system will arrive after such a jump, and so on. The stochastic properties of the random sequence of Fig. 2 are thus completely determined by the knowledge of the delay functions $W_{ik}(\tau)$, i.e., by solution (3.3) of the Schrödinger equation corresponding to Eq. (3.2), and by the probabilities $\pi_{kj}$. To complete our pictures, we must also give the expression of the state vector of the system between the jump of entry in $|j\rangle$ at time $t$ and the jump of exit from $|j\rangle$. If the jump of exit has not yet occurred at time $t + \tau$, the state vector certainly belongs to $\mathcal{E}(N_1, N_2)$ and is given by the normalized expression

$$|\psi(t + \tau)\rangle = \frac{\sum_k c_{ik}(\tau)|k\rangle}{\left(\sum_k |c_{ik}(\tau)|^2\right)^{1/2}}.$$  

C. Monte Carlo Simulations

In Section 4 we define from $W_{ij}(\tau)$ and $\pi_{kj}$ a certain number of probabilities for characterizing the mean statistical properties of the random sequence of Fig. 3. Analytical expressions are also derived for these probabilities in some limiting cases in Section 5. Beforehand we think it would be useful to give an example of Monte Carlo simulations of the time evolution of the system, because they provide nice pictorial views of the physical processes. For such simulations one can use the delay functions $W_{ij}(\tau)$ and the probabilities $\pi_{kj}$ introduced in this paper, which allow one to perform fast numerical calculations. One could also follow the Monte Carlo wave-function approach of Refs. 7 and 8, which requires more computing time but which, on the other hand, is simpler to program. In fact we have used a combination of both methods to generate the stochastic sequence reproduced in Fig. 4. In Fig. 4(a) we plot the number of probe photons as a function of time, each modification in the number of probe photons corresponding to the process indicated above by a small icon. More precisely, the arrows represent stimulated Raman processes, while $\uparrow$ and $\downarrow$ represent, respectively, absorption and stimulated emission processes of one photon. We notice from this figure that the total number of probe photons is increasing so that the total amplification is positive. A second observation is that this amplification is due mainly to stimulated Raman processes. In fact the stimulated emission of one photon is, for the parameters we have chosen here, a rare event. Since Fig. 4(a) is a little bit misleading because the photon number never has a well-defined value, except at the time of a quantum jump, it is better to plot the photon number as in Fig. 4(b), where, with an enlarged time scale, this value is shown only at the time of a quantum jump. Figure 4(c) shows a further magnification of the temporal sequence of the physical processes changing $N_1$. In the lower part of Fig. 4(c) we have also indicated the coherent evolution periods by means of oblique lines joining the entrance and

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Fig. 4. Stochastic evolution of the number of probe photons obtained by the Monte Carlo simulation explained in the text. The parameters are $\Gamma_1 = 0.25\Gamma$, $\Gamma_2 = 0.75\Gamma$, $R_1 = R_2 = (2.5 \times 10^{-3})\Gamma$, $\Omega_1 = (2.5 \times 10^{-5})\Gamma$, $\Omega_2 = 0.15\Gamma$, $\delta_1 = \delta_2 = 0$. Changes in probe photon number are tagged by the processes that produce them; $\uparrow$ indicates stimulated Raman gain; $\downarrow$ stimulated Raman absorption; $\perp$ one-photon loss. (a) Probe photon number versus normalized time $\tilde{t}$. The dashed line represents the mean rate of variation of $N_1$ computed from Eq. (4.16) below. (b) Enlarged part of the time evolution, with the number of probe photons being represented only at the time of the quantum jumps, where it has a well-defined value. One sees that between two successive changes of $N_1$ there are several quantum jumps during which $N_1$ does not change. (c) Further time enlargement allows one to distinguish the coherent evolution periods, which are represented by oblique lines joining their state in and their state out $|g_1\rangle, |g_2\rangle$, or $|\cdot\rangle$. 

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the exit states; these atomic states are represented in the
plot at three levels indicated by $|g_1\rangle$, $|g_2\rangle$, and $|e\rangle$. From
these lines we can notice that a stimulated Raman gain is
complete at the end of a period $(2,1)$ and a stimulated
Raman loss at the end of a period $(1,2)$. There are also
many transitions taking place between levels $g_2$ and $e$
without affecting the probe photon number. Finally, it
clearly appears that for most of the time the system is in
the period $(1,1)$. We show in Subsection 5.D that in such a
period the weight of the state $g$, is predominant in the
wave function, so that one concludes that the lowest level
is on the average the most populated, notwithstanding the
positive gain of the system. This very peculiar behavior
will be discussed in detail in the next sections.

4. GENERAL STATISTICAL PROPERTIES OF
THE SEQUENCE OF QUANTUM JUMPS

A. Probabilities of the Periods $(i, j)$

Our purpose in the present section is to define some useful
quantities relative to the sequence of quantum jumps and
to show how, in the particular case we are dealing with,
they are amenable to an analytical evaluation. A funda-
mental quantity here is the probability $P(i,j)$ of a period
$(i,j)$, i.e., the probability that a random choice among all
periods of the stochastic sequence gives as a result the pe-
riod $(i,j)$. Note that such a random choice is made inde-
pendently of the duration of the period, each period being
characterized only by the state of entry $|i\rangle$ and the state of
exit $|j\rangle$. Since each period corresponds to a well-defined
change in the photon numbers, the probabilities $P(i,j)$ are
directly connected to the absorption and the amplification
processes. Making use of Eqs. (2.5), we can for example
explicitly write the mean change of the number of probe
photons per period:

$$\langle \Delta N_i \rangle = P(2,1) + P(1,e) - P(1,2) - P(1,e),$$

where the roles of the amplification periods $(2,1)$ and $(e,1)$
and of the absorption period $(1,2)$ and $(1,e)$ clearly appear.
It has been also pointed out in Section 2 that during the
other periods the number of probe photons does not
change.

The probabilities $P(i,j)$ are linked to the probabilities
$P(i)$, that a randomly chosen period starts in the state $|i\rangle$, by
the relation

$$P(i,j) = P(i)P(j|i),$$

where $P(j|i)$ is the conditional probability that, given that the
period has started in $|i\rangle$, it ends in the state $|j\rangle$. Accord-
ing to Eqs. (3.4) we can write

$$P(j|i) = \int_0^\infty W_{ij}(\tau)d\tau = G_j \int_0^\infty d\tau |c_{ij}(\tau)|^2,$$

with

$$\sum_j P(j|i) = 1 \quad \text{for all } i,$$

making use of Eq. (3.5).

Equation (4.3) allows us to derive an interesting relation
between the conditional probabilities $P(j|i)$ and $P(i,j)$.
In Appendix B it is shown that for an effective non-
Hermitian Hamiltonian such as in Eq. (3.2), which sat-
sifies $(H_{\text{eff}}) = (H_{\text{eff}})^\dagger$, the following relation applies:
$c_{ij}(\tau) = c_{ji}(\tau)$. Thus in Eq. (4.3) the time integrals for
the two conditional probabilities are equal, and we get

$$P(j|i) = \frac{P(i) G_j}{G_i}.$$

The conditional probability of a given period is propor-
tional to the dissipative departure rate from the final
state of that period. Using Eqs. (4.2) and (4.5a), one can
also derive the exact equation

$$\frac{P(i,j)}{P(i,j)} = \frac{P(i) G_j}{P(i) G_i}.$$

The probability of a given period is thus proportional to
the probability of starting in the initial state and to the
dissipative departure rate from the final state.

We see from Eq. (4.2) that, once the coefficients $c_{ij}$ are
derived from Eq. (3.3), it is necessary to calculate the
probabilities $P(i)$ if we want to determine $P(i,j)$. In the
hypothesis of a stationary process, $P(i)$ is time inde-
dependent and is related to the conditional probability $Q(in:
|j\rangle in: |i\rangle)$ to start a period in the state $|j\rangle$, given that the
previous period has started in the state $|i\rangle$, by the equation

$$P(j) = \sum_i P(i)Q(in: |j\rangle in: i).$$

These conditional probabilities can be written as

$$Q(in: |j\rangle in: |i\rangle) = \sum_k P(k|i)\pi_{kj},$$

where the normalization condition (3.7) entails that

$$\sum_j Q(in: |j\rangle in: |i\rangle) = 1 \quad \text{for all } i.$$

This allows us to show that the homogeneous system (4.6)
has a nonzero solution, which can be normalized in such a
way that

$$\sum_i P(i) = 1.$$ 

To sum up, solving the Schrödinger equation associated
with Eq. (3.2) gives $c_{ij}$ [see Eq. (3.3)] and then $P(i,j)$, from
Eqs. (4.3) and (3.1). One then calculates $P(i)$, using
Eqs. (4.6), (4.7), (3.6), and (4.9), which gives according to
Eq. (4.2) the probabilities $P(i,j)$.

B. Average Statistical Quantities

Several average statistical quantities can now be calcu-
lated from the probabilities introduced above. Interesting
quantities are, for instance, the average duration, $T(i,j)$,
of a period $(i,j)$ and the average time, $T$, between two con-
secutive quantum jumps, given, respectively, by

$$T(i,j) = \frac{\int_0^\infty \tau W_{ij}(\tau)d\tau}{P(j|i)} = \frac{\int_0^\infty \tau |c_{ij}(\tau)|^2d\tau}{P(j|i)},$$

$$T = \sum_{i,j} T(i,j) P(i,j).$$
while the probability that, at a given randomly chosen time, the atom is found in the period $((i,j)$ is given by
\[
\Pi((i,j)) = \frac{\mathcal{P}(i,j)T(i,j)}{T}.
\]
Equation (4.12)

In a collection of atoms this quantity defines the fraction of them in the period $(i,j)$.

We also derive here an expression for the populations of the atomic levels, with it understood that a similar procedure can be followed for other physical observables as well. Before starting this derivation, we write Eq. (3.8) (with $t = 0$) in a more physically meaningful way, i.e.,
\[
\frac{\sum \langle c_{ik}(t)\rangle |k\rangle}{[N_i(t)]^{1/2}},
\]
where we have introduced the probability $N_i(t)$ that no quantum jump has occurred in the time interval $[0,\tau]$ for a coherent evolution period starting at time $\tau = 0$ in $|i\rangle$:
\[
N_i(t) = \sum_k |c_{ik}(t)|^2 = 1 - \int_0^\tau \sum_k W_{ik}(\tau')d\tau'.
\]
In deriving expression (4.14) for $N_i(t)$, use has been made of relation (A6) below. The average population of the state $|k\rangle$ is calculated by integration of its instantaneous value $\langle k | \psi_i(\tau) \rangle^2$ over the time $T_a = nT$ corresponding to a very large number $(n \rightarrow \infty)$ of coherent evolution periods. The number $dn_i(t)$ of periods starting in $|i\rangle$ and lasting a time between $\tau$ and $\tau + d\tau$ is $dn_i(t) = n\mathcal{P}(i)\sum W_{ik}(\tau)d\tau$. The contribution to the average population of level $|k\rangle$ coming from a coherent evolution period that started in $|i\rangle$ is $\int_0^\tau \langle k | \psi_i(\tau') \rangle^2 d\tau'$. Then we obtain for the average population
\[
\Pi_i = \sum_{i,j} \frac{\mathcal{P}(i)}{T} \int_0^\tau W_{ij}(\tau) \left[ \int_0^\tau \langle k | \psi_i(\tau') \rangle^2 d\tau' \right] d\tau
\]
\[
= \sum_{i,j} \frac{\mathcal{P}(i)}{T} \int_0^\tau |c_{ik}(\tau)|^2 d\tau
\]
(4.15)
after having performed an integration by parts and having used relation (4.14) for $N_i(t)$.

One final quantity to be calculated is the average rate of increase of the number of probe photons, $N_i$. First, it is clear from Fig. 4(a) that the increase (or decrease) of $N_i$ takes place only through random quantum jumps. Thus, for the time evolution of a single atom, such as that pictured in Fig. 4(a), $dN_i/dt$ cannot be defined. We may define only a coarse-grained rate of variation, averaged over a very large number of coherent evolution periods, $n$, lasting a time $T_a = nT$. Because of the random occurrence of the quantum jumps, such a coarse-grained rate of variation of $N_i$ may be approximated as the ratio between the variation in the probe photon number and the time. The average change in the photon number $N_i$ during each coherent evolution period is given by Eq. (4.1), so that the coarse-grained rate of variation of $N_i$ is equal to
\[
\langle \frac{dN_i}{dt} \rangle = \frac{n\langle \Delta N_i \rangle}{T_a} = \frac{\langle \Delta N_i \rangle}{T}
\]
\[
= \frac{\mathcal{P}(2,1) + \mathcal{P}(e,1) - \mathcal{P}(1,2) - \mathcal{P}(1,e)}{\sum_{i,j} T(i,j)\mathcal{P}(i,j)}. \tag{4.16}
\]
In Fig. 4 the value of $\langle dN_i/dt \rangle$ given by this expression is represented by the slope of the dashed line passing through the photon-number values.

5. APPLICATION TO AMPLIFICATION

The determination of the delay functions introduced in Section 3 and the use of these functions in the equations for the probabilities and conditional probabilities derived in Section 4 allow one to calculate the average variation in the photon number, $\Delta N_i$, the average time between quantum jumps, and through Eq. (4.16) the amplification coefficient corresponding to the scheme of Fig. 1. In this section such a determination will be performed analytically in the limiting case of a weak probe field and a not-too-strong pump field.

A. Assumptions

The following relations are assumed to be valid for the analytical calculations:
\[
\Omega_1, \Omega_2 \ll \Gamma, \tag{5.1a}
\]
\[
R_1, R_2 \ll \Gamma' = \Omega^2/\Gamma, \tag{5.1b}
\]
\[
\Omega_1 \ll \Omega_2, R_1, R_2, \Gamma'. \tag{5.1c}
\]
Combining inequalities (5.1a) and (5.1b) leads to
\[
R_1, R_2 \ll \Gamma' \ll \Gamma. \tag{5.1d}
\]

These assumptions correspond to well-defined conditions to be realized in experimental configurations. Inequality (5.1a) for the Rabi frequencies expresses a weak-excitation condition, meaning that the two transitions are not saturated. Inequality (5.1b) expresses that the absorption rate $\Gamma'$ of $\omega_{12}$ photons from the $g_2$ level is larger than the absorption rate $R_1$ and $R_2$ of incoherent radiation from levels $g_1$ and $g_2$, respectively. Finally, according to inequality (5.1c), the field at frequency $\omega_{12}$ is considered a weak probe field, as is usually done near the threshold for lasing. From these assumptions it follows that the populations of the three levels are not modified by the application of the $\omega_{12}$ field. In contrast, owing to inequality (5.1b), the populations of levels $g_2$ and $e$ are modified appreciably by the application of the $\omega_{12}$ field. Note that while the population level $\Gamma_1$ is not affected by the probe field $\omega_{12}$, owing to the depletion of level $g_2$ by the pump field $\omega_{11}$, the population of level $g_1$ can become larger than the population of level $g_2$, so that we can realize a condition of no inversion between two levels connected by a Raman transition. In summary, the threshold conditions for the amplification of the $\omega_{12}$ field are investigated with the assumption that we apply a field $\omega_{12}$ strong enough to modify the population of $g_2$ and at the same time weak enough not to saturate the transition $g_2 \leftrightarrow e$.

Two more assumptions will be used in this section. The first one,
\[
\delta_2 = 0, \tag{5.2}
\]
is not essential, but allows one to write simpler analytical expressions. The second one,
\[
0 \leq |\delta_1| \ll \Gamma, \tag{5.3}
\]
expresses that the frequency $\omega_{11}$ can be modified for scanning the Raman resonance, with the frequency $\omega_{12}$ remaining in resonance with the transition $g_2 \leftrightarrow e$. No hypothesis is introduced concerning the relative magnitudes of the pumping rate $\Gamma'$ and the frequency detuning $\delta_1$.

B. Calculation of the Relevant Probabilities
The Schrödinger equation associated with the effective Hamiltonian (3.2) leads to the following differential equations for the amplitudes $c_i(\tau)$ introduced in Eq. (3.3) and describing the coherent evolution within the manifold $\mathcal{E}(N_1, N_2)$ for a system starting from the state $|i\rangle$ at time $\tau = 0$:

$$|\psi_i(\tau)\rangle = c_{i}(\tau)|e, N_1, N_2\rangle + c_{1i}(\tau)|g_1, N_1 + 1, N_2\rangle + c_{12}(\tau)|g_2, N_1, N_2 + 1\rangle,$$  \hspace{1cm} (5.4)

$$\dot{c}_i(\tau) = -i\frac{\Gamma}{2}c_i(\tau) + \frac{\Omega_1}{2}c_{1i}(\tau) - i\frac{\Omega_2}{2}c_{12}(\tau),$$  \hspace{1cm} (5.5a)

$$\dot{c}_{1i}(\tau) = -i\frac{\Omega_1}{2}c_i(\tau) + i\frac{\Omega_2}{2}c_{12}(\tau),$$  \hspace{1cm} (5.5b)

$$\dot{c}_{12}(\tau) = -i\frac{\Omega_2}{2}c_i(\tau) - \frac{R_2}{2}c_{12}(\tau).$$  \hspace{1cm} (5.5c)

$$c_i(\tau = 0) = \delta_i.$$  \hspace{1cm} (5.5d)

In these equations, use has been made of the resonant condition $\delta = 0$ for the pump field at frequency $\omega_{12}$. Since by assumption the rate $\Gamma$ defines the fastest time constant, we are allowed to perform, at times $t > 1/\Gamma$, an adiabatic elimination of the variable $c_i$ in terms of the slow variables $c_{1i}$ and $c_{12}$. In the resulting equations, owing to inequalities (5.1), $\Omega_i^2/\Gamma$ can be neglected with respect to $R_1$, and $R_2$ can be neglected with respect to $\Gamma'$, so that the following equations are obtained (for times $t > 1/\Gamma$):

$$\dot{c}_{1i}(\tau) = -i\frac{\Omega_1}{\Gamma}c_{1i}(\tau) + i\frac{\Omega_2}{2\Gamma}c_{12}(\tau),$$  \hspace{1cm} (5.6a)

$$\dot{c}_{12}(\tau) = -i\frac{\Omega_2}{2\Gamma}c_{1i}(\tau) - \frac{R_2}{2\Gamma}c_{12}(\tau).$$  \hspace{1cm} (5.6b)

On the other hand, the initial conditions of Eqs. (5.5) are modified in a negligible way under the assumptions specified above. Solving Eqs. (5.6), with the proper initial conditions, and substituting the corresponding results into the relation (4.3) allow one to derive the conditional probabilities $\mathcal{P}(j/i)$. In what follows, the conditional probabilities for the four processes contributing in Eq. (4.1) to the variation of the probe photon number will be explicitly derived.

1. $g_2 \rightarrow g_1$: Stimulated Raman Gain
According to Eq. (2.5a) the stimulated Raman gain takes place during the periods (2,1), whose probability $\mathcal{P}(2/1)$ depends, according to Eq. (4.2), on the conditional probability $\mathcal{P}(1/2)$. For one to calculate

$$\mathcal{P}(1/2) = R_1 \int_0^\infty d\tau |c_{21}(\tau)|^2,$$  \hspace{1cm} (5.7)

Eqs. (5.6) have to be solved with $i = 2$. Using inequalities (5.1) and neglecting terms containing $1/\Gamma'$ with respect to those containing $1/R_1$, we obtain the following expression:

$$\mathcal{P}(1/2) = \Omega_1^2 \frac{1}{\Omega_2^2 + 1} \left( \frac{\Gamma'}{\delta_1 + i(\Gamma'_2/2)} \right) \exp \left( -\frac{\Gamma'_2}{2\tau} \right).$$  \hspace{1cm} (5.8)

The frequency dependence of this probability is equal to that discussed for the conditional probability of the Raman gain process.

2. $g_1 \rightarrow g_2$: Stimulated-Emission Gain
Following Eq. (2.5c), we have to study the periods (1,2), with conditional probability $\mathcal{P}(2/1)$, to be calculated through

$$\mathcal{P}(2/1) = R_2 \int_0^\infty d\tau |c_{12}(\tau)|^2.$$  \hspace{1cm} (5.9)

The calculation of $c_{1i}(\tau)$ or $c_{12}(\tau)$ will be presented in the following paragraph. We just note here that, owing to Eqs. (4.5a) and (3.1), the following relation applies:

$$\mathcal{P}(1/2) = R_2 \frac{\Omega_1^2}{\Omega_2^2 + 1} \frac{1}{(\delta_1 + i(\Gamma'_2/2))^2}.$$  \hspace{1cm} (5.10)

We can thus neglect stimulated-emission gain $e \rightarrow g_1$ in comparison with absorption loss $g_1 \rightarrow e$.

3. $e \rightarrow g_1$: Absorption Loss
Following Eq. (2.5d), we have to study the periods (1, e) and calculate

$$\mathcal{P}(e/1) = R_1 \int_0^\infty d\tau |c_{e1}(\tau)|^2.$$  \hspace{1cm} (5.11)

The calculation of $c_{e1}(\tau)$ or $c_{12}(\tau)$ will be presented in the following paragraph. We just note here that, owing to Eqs. (4.5a) and (3.1), the following relation applies:

$$\mathcal{P}(e/1) = (R_1/\Gamma) \mathcal{P}(e/1),$$  \hspace{1cm} (5.12)

so that it results from inequality (5.1d) that

$$\mathcal{P}(e/1) \ll \mathcal{P}(e/1).$$  \hspace{1cm} (5.13)

4. $g_1 \rightarrow e$: Absorption Loss
Following Eq. (2.5d), we have to study the periods (1, e) and calculate

$$\mathcal{P}(e/1) = \Gamma_1 \int_0^\infty d\tau |c_{1e}(\tau)|^2.$$  \hspace{1cm} (5.14)

Equations (5.5), with $i = 1$, determine $c_{1e}(\tau)$. At times $t > 1/\Gamma$, where an adiabatic elimination of the variable $c_{1e}$ is performed, Eqs. (5.6b) and (5.6c) for $c_{1i}$ and $c_{12}$ are solved, and the solution is substituted into Eq. (5.6a). The final result for $c_{1e}$, with the usual approximations, is

$$c_{1e}(\tau) = \Omega_1 \frac{1}{\Gamma} \left[ \frac{\Gamma'}{\delta_1 + i(\Gamma'_2/2)} \right] \left[ \frac{\Gamma'_2}{\Gamma} \right] \exp \left( -\frac{\Gamma'_2}{2\tau} \right) - i\delta_1 \exp \left[ -\left( \frac{R_1}{2} + i\delta_1 \right) \tau \right].$$  \hspace{1cm} (5.15)
In Eq. (5.15) two separate terms contribute to $c_v$ and to $\mathcal{P}(e/i)$, one of them being zero at resonance ($\delta_1 = 0$). The general expression for $\mathcal{P}(e/i)$ results:

$$\mathcal{P}(e/i) = \frac{\Omega^2}{\Omega_e^2} \left( \frac{\Gamma'_v/2}{\Gamma'_v/2} + \delta_i^2 \right) \left[ 1 + \frac{4\delta_i^2}{R_1\Gamma'_v} - \frac{4\delta_i^2}{(\Gamma'_v/2)^2 + \delta_i^2} \right],$$

(5.16a)

with the limiting cases at resonance ($\delta_1 = 0$),

$$\mathcal{P}(e/i) = \frac{\Omega^2}{\Omega_e^2},$$

(5.16b)

and far from resonance ($|\delta_1| \gg \Gamma'_v$),

$$\mathcal{P}(e/i) = \frac{\Omega^2}{\Gamma'_v} = \frac{\Omega^2}{\Gamma'_v},$$

(5.16c)

Because of inequality (5.1b), the conditional probability for absorption $\mathcal{P}(e/i)$ takes its minimum value at resonance.

5. Probabilities $\mathcal{P}(i)$ and $\mathcal{P}(i, j)$

The next step before deriving the probabilities $\mathcal{P}(i, j)$ is to calculate the probabilities $\mathcal{P}(i)$ that a randomly chosen period starts in the state $|i\rangle$. Since all the probabilities $\mathcal{P}(j/i)$ just determined and contributing to the modification of the probe photon number $\Delta N_1$ are proportional to $\Omega_i^2$, it is sufficient here to calculate $\mathcal{P}(i)$ to order 0 in $\Omega_1$.

It is clear from Fig. 2 that the system enters into levels $|g_1\rangle$ and $|g_2\rangle$ only through quantum jumps from $|e\rangle$, either in spontaneous-emission processes or in emission processes stimulated by the incoherent radiation. The relative probabilities of entering into $|g_1\rangle$ and $|g_2\rangle$ are proportional, respectively, to $\Gamma'_v/\Gamma$ and $\Gamma_2/\Gamma$. Thus the ratio between the probabilities $\mathcal{P}(1)$ and $\mathcal{P}(2)$ of starting a period in states $|1\rangle$ and $|2\rangle$ is given exactly by

$$\mathcal{P}(1)/\mathcal{P}(2) = \frac{R_1}{\Omega_2^2}. \frac{\Gamma_2}{\Gamma},$$

(5.17)

An alternative method for determining the probabilities $\mathcal{P}(i)$ is to make use of Eqs. (4.6) and (4.9), the conditional probabilities $Q(i/:e/i)$ being derived from Eq. (4.7). This derivation will be performed here within the assumptions specified above. In this limit we notice from Fig. 2 that, if a period starts in $|g_1\rangle$, the following period always starts from $|e\rangle$. This implies that

$$Q(i/:e/i) = \delta_{ij}. \quad (5.18a)$$

If instead a period starts in $|e\rangle$, at the lowest order in $\Gamma_2/\Gamma$, the probabilities of quantum jumps to levels $g_1$ and $g_2$ are larger than the probability for a coherent evolution toward $|2\rangle$ within the manifold $\mathcal{E}(N_1, N_2)$ followed by a jump from this level. Thus we may write

$$Q(i/:e/i) = \frac{R_1}{\Gamma_2} \delta_{ij} + \frac{R_2}{\Gamma} \delta_{ij}. \quad (5.18b)$$

Finally, for a period starting in $|g_2\rangle$, the most probable process is a coherent evolution toward the state $|e\rangle$ of the same manifold $\mathcal{E}(N_1, N_2)$ followed by a quantum jump toward $g_1$ or $g_2$. Thus we may write

$$Q(i/:g_2\rangle) = \frac{R_1}{\Gamma_2} \delta_{ij} + \frac{R_2}{\Gamma} \delta_{ij}. \quad (5.18c)$$

From relations (5.18) it follows that

$$Q(i/:e/i) = \delta_{1i}. \quad (5.19)$$

Applying Eq. (4.6) to the state $|j\rangle = |e\rangle$ and using relation (5.19), we derive

$$\mathcal{P}(e) = \mathcal{P}(1). \quad (5.20)$$

Finally, from Eq. (5.17) and relation (5.20), and from the normalization condition given by Eq. (4.9), we get

$$\frac{\mathcal{P}(1)}{\Gamma_1} = \frac{\mathcal{P}(2)}{\Gamma_2} = \frac{\mathcal{P}(e)}{\Gamma}, \frac{1}{2\Gamma_1 + \Gamma_2}, \quad (5.21)$$

which leads to

$$\mathcal{P}(1) = \frac{R_1}{2\Gamma_1 + \Gamma_2} \Gamma_2, \quad \mathcal{P}(2) = \frac{R_2}{2\Gamma_1 + \Gamma_2} \Gamma_2, \quad \mathcal{P}(e) = \frac{\Gamma_1}{2\Gamma_1 + \Gamma_2}. \quad (5.22)$$

At this point Eqs. (5.8), (5.10), (5.12), (5.16), and (5.22) represent the elements required for calculation of the probabilities of the periods contributing to a variation in the probe photon number. For the stimulated Raman processes we get

$$\mathcal{P}(2,1) = \mathcal{P}(2)/\mathcal{P}(1) = \frac{R_1}{\Omega_2^2} \frac{\Gamma_2}{1 + (2\delta_1/\Gamma_1')^2} \frac{\Gamma_1}{2\Gamma_1 + \Gamma_2}, \quad (5.23)$$

$$\mathcal{P}(1,2) = \mathcal{P}(1)/\mathcal{P}(2) = \frac{R_2}{\Omega_2^2} \frac{\Gamma_1}{1 + (2\delta_1/\Gamma_1')^2} \frac{\Gamma_2}{2\Gamma_1 + \Gamma_2}, \quad (5.24)$$

while for the one-photon processes at resonance ($\delta_1 = 0$) we have

$$\mathcal{P}(e,1) = \mathcal{P}(e)/\mathcal{P}(1/e) = \frac{R_1}{\Gamma_2} \frac{\Gamma_2}{\Omega_2^2} \frac{2\Gamma_1 + \Gamma_2}{\Gamma_2}, \quad (5.25a)$$

$$\mathcal{P}(1/e) = \mathcal{P}(1)/\mathcal{P}(e/1) = \frac{\Gamma_1}{\Omega_2^2} \frac{\Gamma_2}{\Gamma_2} \frac{2\Gamma_1 + \Gamma_2}{\Gamma_2}, \quad (5.25b)$$

and far from resonance ($|\delta_1| \gg \Gamma'_v$)

$$\mathcal{P}(e,1) = \mathcal{P}(e)/\mathcal{P}(1/e) = \frac{\Gamma_1}{\Omega_2^2} \frac{2\Gamma_1 + \Gamma_2}{\Gamma_2}, \quad (5.25b)$$

$$\mathcal{P}(1/e) = \mathcal{P}(1)/\mathcal{P}(e/1) = \frac{\Gamma_1}{\Gamma_2} \frac{2\Gamma_1 + \Gamma_2}{\Gamma_2}. \quad (5.26b)$$

Comparison of Eqs. (5.23) and (5.24) shows that

$$\frac{\mathcal{P}(1,2)}{\Gamma_1 R_2} = \frac{\mathcal{P}(2,1)}{\Gamma_2 R_1}. \quad (5.27)$$

In fact, such a relation could have been derived directly from the exact equations (4.2), (4.5b), and (5.17), which shows that Eq. (5.27) remains valid even if the assumptions made in Subsection 5A were not valid. We return to the physical meaning of Eq. (5.27) in Subsection 6A.

C. Condition for Amplification

By substitution of the probabilities for the four coherent evolution periods modifying the probe photon number into
Eq. (4.16), or more simply into Eq. (4.1), the conditions required for realizing amplification may be determined. We examine the condition \( (\Delta N_1) > 0 \) for the case \( \delta_1 = 0 \), where, according to Eq. (5.23), the probability \( P(2,1) \) of the Raman gain process is maximum, whereas, according to Eqs. (5.26), the probability of the one-photon loss process is minimum. Making use of Eqs. (5.23)–(5.27) with \( \delta_1 = 0 \), we obtain

\[
(\Delta N_1) = \frac{1}{2R_1 + \frac{\Omega_1^2}{\Omega_2^2} R_1} \left( \frac{\tilde{g}_2 R_1 + \tilde{g}_1 R_1 - \tilde{g}_2 R_1 - \tilde{g}_1 R_1}{\tilde{g}_1 + \tilde{g}_2} \right). \tag{5.28}
\]

The second term in parentheses on the right-hand side of this equation, i.e., the one-photon process gain \( P(\epsilon,1) \), can be neglected because it is much smaller than the other terms. Thus the amplification condition \( (\Delta N_1) > 0 \) may be written as

\[
\frac{\tilde{g}_2}{\tilde{g}_1} > 1 + \frac{R_2}{R_1}. \tag{5.29}
\]

We may also consider separately the condition for having the Raman gain larger than the Raman losses, \( P(2,1) > P(1,2) \):

\[
\frac{\tilde{g}_2}{\tilde{g}_1} > \frac{R_2}{R_1}. \tag{5.30}
\]

Comparison, in inequalities (5.30) and (5.29), of the two terms appearing on the right-hand side identifies the contributions to the losses that are due to the one-photon and the Raman processes.

D. Calculation of Average Populations

The occurrence of amplification without inversion also entails an average population in level \( g_2 \) larger than the sum of populations in levels \( g_1, g_2, e \). Because we are interested in the conditions for reaching amplification without inversion in threshold conditions, we need the populations \( \Pi \) of levels \( |k\rangle \) in the absence of the field \( \omega_{k\ell} \), i.e., to order 0 in \( \Omega_1 \). To perform a complete analysis, we calculate the populations for both \( \Omega_2 = 0 \) and \( \Omega_2 \neq 0 \). Equations (4.15) represent the exact approach to the determination of the populations. To order 0 in \( \Omega_1 \) and \( \Omega_2 \) we have \( |c_{ik}(\tau)|^2 = \delta_{ik} \exp(-G_{ik}) \), and from Eqs. (4.15) we deduce

\[
\Pi_1 = \frac{P(1)}{TR_1}; \quad \Pi_2 = \frac{P(2)}{TR_2}; \quad \Pi_\epsilon = \frac{P(\epsilon)}{TR}. \tag{5.31}
\]

Making use of Eq. (5.21) and of the normalization condition for the populations,

\[
\sum_k \Pi_k = 1, \tag{5.32}
\]

we obtain

\[
\frac{\Pi_1}{(\tilde{g}_1/R_1)} = \frac{\Pi_2}{(\tilde{g}_2/R_2)} = \frac{\Pi_\epsilon}{(\tilde{g}_1/\tilde{g}_2)} = \frac{1}{(\tilde{g}_1/R_1) + (\tilde{g}_2/R_2) + (\tilde{g}_1/\tilde{g}_2)}. \tag{5.33}
\]

Owing to the hypothesis \( \tilde{g}_1 >> R_1, R_2 \), the population \( \Pi_\epsilon \) remains negligible in comparison with \( \Pi_1 \) and \( \Pi_2 \).

For \( \Omega_2 \) different from zero and satisfying inequalities (5.1), with \( \Omega_1 \) still equal to zero, the result of the first of Eqs. (5.31), for \( \Pi_1 \), remains the same. For the other populations we note that the eigensolutions of Eqs. (5.5) are some states \( \{|2\rangle \} \) containing mainly states \( |2\rangle \) and \( |\epsilon\rangle \), with decay rates respectively equal to \( \Gamma_2 \) and \( \Gamma_\epsilon \). If a period starts in state \( |2\rangle \), then state \( |2\rangle \) is mainly occupied, and the integral appearing in Eq. (4.15) for \( \Pi_2 \) gives \( 1/\Gamma_2 \). On the other hand, if a period starts in state \( |\epsilon\rangle \), then state \( |\epsilon\rangle \) is mainly occupied, and the integral appearing in Eq. (4.15) for \( \Pi_\epsilon \) gives \( 1/\Gamma_\epsilon \). Thus we deduce

\[
\Pi_1 = \frac{P(1)}{TR_1}; \quad \Pi_2 = \frac{P(2)}{TR_2}; \quad \Pi_\epsilon = \frac{P(\epsilon)}{TR}. \tag{5.34}
\]

Making use of Eq. (5.21) and of the normalization condition (5.32), we obtain for the populations at the lowest order in \( \Omega_2 \)

\[
\frac{\Pi_1}{(\tilde{g}_1/R_1)} = \frac{\Pi_2}{(\tilde{g}_2/R_2)} = \frac{\Pi_\epsilon}{(\tilde{g}_1/\tilde{g}_2)} = \frac{1}{(\tilde{g}_1/R_1) + (\tilde{g}_2/R_2) + (\tilde{g}_1/\tilde{g}_2)}. \tag{5.35}
\]

For this case the population in \( |\epsilon\rangle \) is also negligible. Furthermore, if \( \tilde{g}_1 \) and \( \tilde{g}_2 \) are comparable, the condition \( R_1 \ll \Gamma_2 \) [see inequality 5.1(b)] entails that \( \Pi_1 \) be larger than \( \Pi_2 \), so that the population of state \( g_1 \) is the largest one, in agreement with the results of the Monte Carlo simulations of Fig. 4(c).

Note that when these results for the populations are combined with Eqs. (5.34) and (5.22) the following expression for the average time between two consecutive quantum jumps, \( T \), is obtained:

\[
T = \frac{P(1)}{\Pi_1 R_1} + \frac{P(2)}{\Pi_2 R_2} - \frac{P(\epsilon)}{\Pi_\epsilon TR} = 2\frac{1}{\tilde{g}_1 + \tilde{g}_2} \left( \frac{1}{R_1} + \frac{1}{\tilde{g}_1} \right)
+ \frac{1}{2\tilde{g}_1 + \tilde{g}_2}. \tag{5.36}
\]

E. Condition for Noninversion

Using the expressions derived in the previous subsection for the populations of the different states, we may now determine the conditions to be satisfied in order to realize noninversion between states \( g_1 \) and \( g_2 \), i.e., \( \Pi_1 > \Pi_2 \). Making use of Eqs. (5.35), for the case in which \( \Omega_1 = 0 \) and \( \Omega_2 \neq 0 \) and at the lowest order in this parameter, we obtain for the noninversion condition

\[
\frac{\Gamma_2'}{R_1} \geq \frac{\Gamma_2}{\tilde{g}_1}. \tag{5.37}
\]

For the case \( \Omega_2 = 0 \) and \( \Omega_1 = 0 \), the noninversion condition results from Eq. (5.33):

\[
\frac{R_2}{R_1} \geq \frac{\Gamma_2}{\tilde{g}_1}. \tag{5.38}
\]

One must compare these conditions for noninversion with that for amplification, inequality (5.29), for one to see whether they are compatible. It is obvious from inspec-
We have confirmed that complete agreement exists among the expressions we have derived, through the quantum jump approach, for the gain condition, the noninversion condition, the gain per unit time, and the results obtained from the solution of optical Bloch equations for the density matrix of the three-level system. We have performed this comparison by solving the optical Bloch equations numerically for different sets of atom and laser parameters and by testing the equality between the gain per unit time derived from such a numerical solution with that obtained with Eq. (4.16). Furthermore, it has been verified that the conditions for gain and noninversion derived above coincide with those derived in Ref. 12 by a solution of the density-matrix equations within the assumptions specified in Subsection 5.A.

6. PHYSICAL DISCUSSION

In this section we discuss the physical content of the results derived in the previous section, and we show how the quantum-jump approach provides interesting physical insights into the physical mechanisms at the origin of amplification without population inversion.

A. Amplification Mechanism: Raman Gain versus Raman Loss

The analysis of Subsection 2.E shows that there are two physical processes permitting the probe field to be amplified: two-photon stimulated Raman processes \( g_2 \rightarrow g_1 \) and one-photon stimulated-emission processes \( e \rightarrow g_1 \).

We have seen in Section 5 that the one-photon processes \( e \rightarrow g_1 \) make a negligible contribution if the assumptions of Subsection 5.A are fulfilled. This shows that stimulated Raman processes \( g_2 \rightarrow g_1 \) play a key role in determination of the amplification of the three-level system.

The balance between stimulated Raman gain and stimulated Raman loss is determined by the ratio between \( \mathcal{P}(2,1) \) and \( \mathcal{P}(1,2) \), which can be calculated from the exact equation (5.27). One obtains

\[
\frac{\mathcal{P}(2,1)}{\mathcal{P}(1,2)} = \frac{R_2 \bar{\Gamma}_1}{R_1 \bar{\Gamma}_2} = \frac{(\text{rate out of 2}) \times (\text{rate in 1})}{(\text{rate out of 1}) \times (\text{rate in 2})}.
\]

It thus appears that the dissymmetry between the two stimulated inverse Raman processes \( g_2 \rightarrow g_1 \) and \( g_1 \rightarrow g_2 \) can come only from the dissymmetry between the rates of the quantum jumps through which the system enters and leaves these two states. The dissymmetry is completely independent of the amplitudes of the two laser fields, i.e., for \( \Omega_1 \) and \( \Omega_2 \). This problem of the dissymmetry between the transition probabilities of two inverse processes has been discussed largely in the context of amplification without inversion." In fact, if the incoherent fields responsible for the rates \( R_1 \) and \( R_2 \) are thermal fields with temperature \( \Theta_1 \) and \( \Theta_2 \), we have

\[
R_i = \frac{\Gamma_i}{\exp(h\omega_i/k_B\Theta_i) - 1}.
\]

with \( i = 1, 2 \), so that condition \( R_1 \bar{\Gamma}_1 > R_2 \bar{\Gamma}_2 \) implies that \( \Theta_1 > \Theta_2 \), which is just the inversion condition for the incoherent fields introduced in Ref. 12.

B. QUENCHING OF ABSORPTION

To have the probe field amplified, it is not sufficient to have stimulated Raman gain larger than stimulated Raman loss. The one-photon absorption loss \( g_1 \rightarrow e \), described by \( \mathcal{P}(1,e) \), must be weak enough that \( (\Delta N_1) \), given by Eq. (4.1), remains positive.

In this respect it is important to note the small value of \( \mathcal{P}(1,e) \), which implies a quenching of the absorption process, and to understand the physical origin of this quenching. Consider for example the value of \( \mathcal{P}(e/1) \), which, according to Eq. (5.16b), is equal to \( \Omega_1^2/\Omega_2^2 \) at resonance (\( \delta_1 = 0 \)). A naïve argument would give, at resonance, an absorption rate from \( g_1 \) to \( e \) equal to \( \Gamma' = \Omega_1^2/\Gamma \), equivalent to the rate \( \Gamma' = \Omega_2^2/\Gamma \) given in relation (6.1b). Multiplying this rate \( \Gamma' \) by the mean lifetime \( T_i = 1/R_i \) of \( g_1 \) would then lead to a value of \( \mathcal{P}(e/1) \) equal to

\[
[\mathcal{P}(e/1)]_{\text{naïve}} = \frac{\Omega_1^2}{\Gamma R_1} = \frac{\Omega_2^2 \Gamma'}{\Omega_2^2 R_1},
\]

which, according to inequality (5.1d), is much larger than the true value \( \Omega_1^2/\Omega_2^2 \). If Eq. (6.3) were true, the one-photon absorption loss would be too large to be overcome by stimulated Raman gain.
ence between two absorption amplitudes, which represents the basis of the phenomenon of coherent population trapping,\textsuperscript{19} to be discussed in what follows, is essential for realizing amplification without inversion. More precisely, let us introduce the following linear combinations of the two states $|g_1, N_1 + 1, N_2\rangle$ and $|g_2, N_1, N_2 + 1\rangle$ belonging to the manifold $\mathcal{E} (N_1, N_2)$:

$$
|\psi_{\text{NC}}\rangle = (\Omega_2/\Omega)|g_1, N_1 + 1, N_2\rangle - (\Omega_1/\Omega)|g_2, N_1, N_2 + 1\rangle,
$$

(6.4a)

$$
|\psi_{\text{C}}\rangle = (\Omega_1/\Omega)|g_1, N_1 + 1, N_2\rangle + (\Omega_2/\Omega)|g_2, N_1, N_2 + 1\rangle,
$$

(6.4b)

where

$$
\Omega = (\Omega_1^2 + \Omega_2^2)^{1/2} = \Omega_3,
$$

(6.5)

with the last relation resulting from inequality (5.1c). In the state $|\psi_{\text{NC}}\rangle$, the two absorption amplitudes from $|g_1, N_1 + 1, N_2\rangle$ and $|g_2, N_1, N_2 + 1\rangle$ toward $|e, N_1, N_2\rangle$ interfere destructively.\textsuperscript{19} Using Eqs. (2.3), we get

$$
\langle e, N_1, N_2 | V_{\text{NL}} | \psi_{\text{NC}}\rangle = 0.
$$

(6.6)

In contrast, these absorption amplitudes interfere constructively in the coupled state $|\psi_{\text{C}}\rangle$, since

$$
\langle e, N_1, N_2 | V_{\text{NL}} | \psi_{\text{C}}\rangle = \hbar \Omega / 2 = \hbar \Omega_3 / 2.
$$

(6.7)

Inverting Eqs. (6.4), one gets the following relation between the state $|g_1, N_1 + 1, N_2\rangle$ and the states $|\psi_{\text{NC}}\rangle$ and $|\psi_{\text{C}}\rangle$:

$$
|g_1, N_1 + 1, N_2\rangle = (\Omega_2/\Omega)|\psi_{\text{NC}}\rangle + (\Omega_1/\Omega)|\psi_{\text{C}}\rangle.
$$

(6.8)

Since $\Omega_1 \ll \Omega_2 = \Omega$, Eq. (6.8) shows that $|g_1, N_1 + 1, N_2\rangle$ nearly coincides with $|\psi_{\text{NC}}\rangle$. Thus it is only because $|g_1, N_1 + 1, N_2\rangle$ is slightly contaminated by $|\psi_{\text{C}}\rangle$ (with a weight $\Omega_1^2/\Omega_2^2$) that a small absorption can take place from state $g_1$. One understands in this way why the absorption of one photon $\omega_{21}$ from $g_1$ is quenched.

In the previous discussion we have supposed the two detunings $\delta_1$ and $\delta_2$ to be equal to zero. In such a case the two states $|g_1, N_1 + 1, N_2\rangle$ and $|g_2, N_1, N_2 + 1\rangle$ are degenerate in the manifold $\mathcal{E} (N_1, N_2)$. It follows that the two linear combinations $|\psi_{\text{NC}}\rangle$ and $|\psi_{\text{C}}\rangle$ are, as $|g_1, N_1 + 1, N_2\rangle$ and $|g_2, N_1, N_2 + 1\rangle$, eigenstates of the unperturbed atom-laser photon Hamiltonian $H_0$ (i.e., without the interaction Hamiltonian $V_{\text{NL}}$). Consequently, the state $|\psi_{\text{NC}}\rangle$ is, at resonance, a stationary state with respect to $H_0$. Such a state thus is not only insensitive to $V_{\text{NL}}$ [see Eq. (6.6)] but also does not evolve under the effect of $H_0$. If $\delta_1$ is no longer equal to zero, the two states $|g_1, N_1 + 1, N_2\rangle$ and $|g_2, N_1, N_2 + 1\rangle$, which are still eigenstates of $H_0$, are no longer degenerate, so that $|\psi_{\text{NC}}\rangle$ is no longer stationary with respect to $H_0$. There is a nonzero off-diagonal element of $H_0$ between $|\psi_{\text{NC}}\rangle$ and $|\psi_{\text{C}}\rangle$, which is of the order of $\hbar \delta_1$. As a consequence of this coupling, a system initially in $|\psi_{\text{NC}}\rangle$ can be transferred by $H_0$ into $|\psi_{\text{C}}\rangle$, from where it can be excited to $|e, N_1, N_2\rangle$ by $V_{\text{NL}}$ [see relation (6.7)]. Out of resonance ($\delta_1 \neq 0$), an atom in the state $|\psi_{\text{NC}}\rangle$ can thus absorb the probe field $\omega_{21}$. The critical value of $\delta_1$, characterizing the breaking of the quenching of absorption, is such that the off-diagonal coupling $\hbar \delta_1$ induced by $H_0$ between $|\psi_{\text{NC}}\rangle$ and $|\psi_{\text{C}}\rangle$ is of the order of the radiative width $\hbar \Gamma'_e$ of $|\psi_{\text{C}}\rangle$. One understands in this way why, when $|\delta_1| >> \Gamma'_e$, the correct value of $\mathcal{P}(e/1)$, given by Eq. (5.16c), coincides with the value deduced from the naive argument neglecting interference effects and leading to Eq. (6.3).

These general considerations are confirmed by the results of numerical calculations of $\mathcal{P}(1, e)$ and $\mathcal{P}(2, 1)$ versus $\delta_1$, represented in Fig. 5. In Fig. 5(a), $\delta_2$ is supposed to be equal to zero. Near $\delta_1 = 0$, in a very narrow interval with a width of the order of $\Gamma'_e$, $\mathcal{P}(1, e)$ is strongly quenched, whereas $\mathcal{P}(2, 1)$ is enhanced. If $\delta_2$ is no longer equal to zero [Fig. 5(b)], the same effect (enhancement of the Raman gain and quenching of the absorption loss) still exists, but it now occurs near $\delta_1 = \delta_2$, i.e., near the resonance condition for the Raman processes.

We conclude this subsection with two remarks:

(i) A diagrammatic approach for understanding the quenching of absorption in the $\Lambda$ configuration of Fig. 1 was recently introduced.\textsuperscript{20} At the lowest order in $\Omega_1$, there are two interfering diagrams, allowing the atom to reach $e$ from $g_1$ (see Fig. 5 of Ref. 20): the first diagram corresponds to the direct one-photon absorption process $g_1 \rightarrow e$; the second corresponds to a three-photon process that consists of a two-photon stimulated Raman process $g_1 \rightarrow g_2$ followed by a one-photon absorption process $g_2 \rightarrow e$. These two ways of reaching $e$ in fact correspond to the two terms on the right-hand sides of Eqs. (5.6a) and (5.15). If the broad level $e$ is considered a continuum, the first and second interfering diagrams correspond, respectively, to a direct transition to the continuum and to an indirect transition through the narrow discrete level $g_2$ with a width $\Gamma'_e$. In this way one can interpret the narrow structures appearing for $\mathcal{P}(1, e)$ in Fig. 5 as being characteristic of a Fano profile.

Another example of quenching of absorption by destructive interference between two physical processes was recently proposed for explaining the physical origin of amplification without inversion near the central resonance of the Mollow absorption spectrum.\textsuperscript{31}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{Fig. 5. Probabilities $\mathcal{P}(2, 1)$ and $\mathcal{P}(1, e)$ for the coherent evolution periods (2, 1) and (1, e) contributing to the gain or loss of the probe photon number versus the detuning $\delta_1$ of the probe laser at frequency $\omega_{21}$. The parameters are the same as in Fig. 4, with (a) $\delta_1 = 0$ and (b) $\delta_1 = -0.5\Gamma'$.}
\end{figure}
(ii) The quantum interference effects discussed in this subsection play an equivalent role in the absorption and stimulated-emission processes between e and $g_1$. In fact the stimulated-emission process from $|e, N_1, N_2\rangle$ to $|g_2, N_1 + 1, N_2\rangle$ depends on the contamination of $|\psi_c\rangle$ in the linear combination of Eqs. (6.4), exactly as in the absorption process. Another simple way of getting this result is to use Eq. (5.12), which shows that $\mathcal{P}(1/e)$ and $\mathcal{P}(e/1)$ have the same $\delta_1$ dependence, reflecting the presence of quantum interference effects. Note also that $\mathcal{P}(1,e)$ and $\mathcal{P}(e,1)$ are not connected by an equation similar to Eq. (6.1), because $\mathcal{P}(1)$ and $\mathcal{P}(e)$ are not related by an exact equation analogous to Eq. (5.17).

C. Dissymmetry between the One-Photon Absorption Processes from $g_1$ and $g_2$

In the previous subsection we have seen that the absorption of the $\omega_{12}$ field is quenched by the presence of the $\omega_{12}$ field. There is no similar effect for the absorption of the $\omega_{12}$ field, which is not quenched by the presence of the $\omega_{12}$ field. Such a dissymmetry arises because we have supposed that $\Omega_1 \ll \Omega_2$, so that the equation equivalent to Eq. (6.8),

$$|g_2, N_1, N_2 + 1\rangle = (\Omega_2/\Omega)|\psi_c\rangle - (\Omega_1/\Omega)|\psi_{nc}\rangle,$$

(6.9)

shows that $|g_2, N_1, N_2 + 1\rangle$ nearly coincides with the coupled state $|\psi_c\rangle$. There is therefore a one-photon absorption rate from $g_2$ that is correctly described by the parameter $\Gamma_1'$ introduced in inequality (5.1b).

The dissymmetry between $\mathcal{P}(1,e)$ and $\mathcal{P}(2,e)$ is important for achieving amplification without inversion. We have seen in Subsection 6.A that the balance between stimulated Raman gain and stimulated Raman loss is independent of the amplitude of the two laser fields, i.e., of $\Omega_1$ and $\Omega_2$. When $\Omega_1$ and $\Omega_2$ are different from zero, with $\Omega_1 \ll \Omega_2$, the population of $g_1$ is not modified appreciably because of the quenching of absorption from $g_1$, discussed in Subsection 6.B. In contrast, if $\Gamma_1' \gg \Gamma_2$, as we have supposed in inequality (5.1b), the absence of quenching of absorption from $g_2$ results in the fact that the population of $g_2$ is considerably reduced from its value for $\Omega_2 = 0$, so that $g_2$ becomes less populated than $g_1$.

7. CONCLUSION

In this section we summarize the main results, which have been obtained by applying a quantum-jump approach to the model of reference.12 The respective contributions of the various physical processes responsible for the absorption or the amplification of the probe field $\omega_{11}$ have been identified. Analytical expressions have been obtained for the probabilities of these processes in threshold conditions for the field $\omega_{11}$ and in the limit where the field $\omega_{12}$ is strong enough for modifying the population of level $g_2$ but also weak enough for not saturating the transition $g_2 \leftrightarrow e$.

In such conditions amplification is due to the two-photon stimulated Raman processes $g_2 \rightarrow g_1$, which predominate over the inverse processes $g_1 \rightarrow g_2$ if there is a proper dissymmetry between the rates in and out of $g_1$ and $g_2$ [see Eq. (6.1)]. Quantum interference effects play an essential role in the limit $\Omega_1 \ll \Omega_2$ by quenching the one-photon absorption processes $g_1 \rightarrow e$. Within the same limit they do not change the one-photon absorption processes $g_2 \rightarrow e$, which can thus deplete $g_2$, since we have assumed that the corresponding coherent absorption rate $\Gamma_1'$ is larger than the pumping rate $R_2$ of the incoherent fields. One can thus understand in this way how the atom can spend most of its time in $g_1$, which then becomes more populated than $g_2$, without the introduction of too-large absorption losses that otherwise would prevent the amplification of the field $\omega_{11}$.

In this problem interference effects do not introduce a dissymmetry between the one-photon absorption processes $g_1 \rightarrow e$ and the reverse stimulated processes $e \rightarrow g_1$. They do not modify $\mathcal{P}(e,1)/\mathcal{P}(1,e)$. In contrast, they do introduce a dissymmetry between the one-photon absorption processes $g_1 \rightarrow e$ and $g_2 \rightarrow e$ starting, respectively, from $g_1$ and $g_2$. They modify $\mathcal{P}(1,e)/\mathcal{P}(2,e)$.

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APPENDIX A: NORMALIZATION OF THE DELAY FUNCTIONS

In this appendix we prove Eq. (3.5), which is the normalization condition for the delay functions. By use of Eq. (3.4), that condition may be written as

$$\sum_j \int_0^\infty W_{ij}(\tau) d\tau = \sum_j G_j \int_0^\infty |c_{ij}(\tau)|^2 d\tau = 1 \quad \text{for all } i. \quad (A1)$$

The coefficients $c_{ij}$ obey differential equations derived from the Schrödinger equation corresponding to the effective Hamiltonian (3.2). For the off-diagonal elements of that Hamiltonian, which are the Rabi frequencies associated with the Hermitian atom–laser interaction operator, the following notation is introduced:

$$\left\langle j \left| \frac{\hat{H}_{eff}}{\hbar} \right| k \right\rangle = \frac{\Omega_{jk}}{2} = \left\langle k \left| \frac{\hat{H}_{eff}}{\hbar} \right| j \right\rangle^* = \frac{\Omega_{kj}^*}{2}. \quad (A2)$$

The equations for the coefficients $c_{ij}$ may be written as

$$\dot{c}_{ij}(\tau) = -\left( \frac{G_j}{2} + i\delta_j \right) c_{ij} - i \left( \sum_k \frac{\Omega_{jk}}{2} c_{ik} \right) \quad \text{with } j = 1,2,e, \quad (A3)$$

where we have introduced $\delta_e = 0$. The initial conditions $c_{ij}(0)$ depend on the coherent evolution period, but they satisfy the normalization condition

$$\sum_j |c_{ij}(0)|^2 = 1. \quad (A4)$$
Using Eq. (A3) and its complex conjugate and summing over the index \( j \), we obtain

\[
\frac{d}{d\tau} \sum_j |c_{ij}(\tau)|^2 = -\sum_j G_j |c_{ij}(\tau)|^2 - i \sum_{j,k} \left[ c_{ij}^*(\tau) \frac{\Omega_{jk}}{2} c_{ik}(\tau) - c_{ik}^*(\tau) \frac{\Omega_{jk}^*}{2} c_{ij}(\tau) \right]
\]

\[
= -\sum_j G_j |c_{ij}(\tau)|^2.
\]

(A5)

We have used Eq. (A2) to replace \( \Omega_{jk}^* \) and exchanged the \( j \) and \( k \) indices to find that the second term on the right-hand side vanishes. Integration of Eq. (A5) on the \((0,\tau)\) interval then leads to

\[
\sum_j |c_{ij}(\tau)|^2 = \sum_j |c_{ij}(0)|^2 - \sum_j G_j \int_0^\tau |c_{ij}(\tau')|^2 d\tau'
\]

\[
= 1 - \sum_j G_j \int_0^\tau |c_{ij}(\tau')|^2 d\tau',
\]

(A6)

where use has been made of Eq. (A4). If we let \( \tau \) tend to \( \infty \) and if we use the fact that \( c_{ij}(+\infty) = 0 \) for all \( j \), because at \( \tau = +\infty \) the system has certainly left the manifold \( \mathcal{G}(N_1, N_2) \), we get Eq. (A1).

**APPENDIX B: SYMMETRY OF THE \( c_{ij} \) COEFFICIENTS**

In this appendix we prove that, if the expansion of \( |\psi(t)\rangle \) on an orthonormal basis \( |i\rangle \) is used,

\[
|\psi(t)\rangle = \sum_j c_{ij}(t) |j\rangle,
\]

(B1)

with the initial conditions

\[
c_{ij}(t = 0) = \delta_{ij}
\]

(B2)

in order to describe the time evolution under a Hamiltonian represented by a matrix \( H \) such that

\[
(H)^* = (H)^T,
\]

then we have the following symmetry relation:

\[
c_{ij}(t) = c_{ji}(t).
\]

(B4)

Owing to Eq. (3.3), this symmetry relation may also be written as

\[
\langle j | \exp(-iHt/\hbar) | i \rangle = \langle i | \exp(-iHt/\hbar) | j \rangle.
\]

(B5)

In our analysis the effective Hamiltonian \( H_{\text{eff}} \), given by Eq. (3.2), has for off-diagonal elements the Rabi frequencies associated with the Hermitian atom-laser interaction operator, as in Eq. (A2). The diagonal elements of the effective Hamiltonian

\[
\langle i | \frac{H_{\text{eff}}}{\hbar} | i \rangle = \delta_i - i\hbar \frac{G_i}{2}
\]

(B6)

contain an imaginary part describing the departure rate \( G_i \) from state \( |i\rangle \) that is due to the dissipative processes. Thus the Hamiltonian \( H_{\text{eff}} \) satisfies the conditions expressed by Eq. (B3).

Let \( |\phi_\alpha\rangle \) be the eigenvector of \( H \) with eigenvalue \( E_\alpha \):

\[
H |\phi_\alpha\rangle = E_\alpha |\phi_\alpha\rangle.
\]

(B7)

Since \( H \) is not Hermitian, its eigenvalues \( E_\alpha \) are not real and its eigenvectors \( |\phi_\alpha\rangle \) are not orthogonal.

We may use an expansion of \( |\phi_\alpha\rangle \) on the orthonormal basis \( |i\rangle \) with coefficients \( c_\alpha^i \):

\[
|\phi_\alpha\rangle = \sum_i c_\alpha^i |i\rangle.
\]

(B8)

From this expansion we may introduce a new vector \( |\bar{\phi}_\alpha\rangle \), defined as

\[
|\bar{\phi}_\alpha\rangle = \sum_i (c_\alpha^i)^* |i\rangle.
\]

(B9)

This vector has several interesting properties. Inserting expansion (B8) into Eq. (B7) and taking the complex conjugate, we get

\[
H^* \langle \bar{\phi}_\alpha | = E_\alpha \langle \bar{\phi}_\alpha |
\]

(B10)

From the adjoint of this relation and from Eq. (B2), we derive

\[
\langle \bar{\phi}_\alpha | H | \phi_\alpha \rangle = E_\alpha \langle \bar{\phi}_\alpha | \phi_\alpha \rangle = E_\beta \langle \bar{\phi}_\beta | \phi_\alpha \rangle.
\]

(B14)

We now demonstrate that, for nondegenerate eigenvalues \( E_\alpha \) and \( E_\beta \), we have

\[
\langle \bar{\phi}_\alpha | \phi_\alpha \rangle = 1.
\]

(B12)

Consider \( \langle \bar{\phi}_\beta | H | \phi_\alpha \rangle \). Applying Eqs. (B7) and (B11), we get

\[
\langle \bar{\phi}_\beta | H | \phi_\alpha \rangle = E_\alpha \langle \bar{\phi}_\beta | \phi_\alpha \rangle = E_\beta \langle \bar{\phi}_\beta | \phi_\alpha \rangle.
\]

We show now that the operator \( \sum_\beta \langle \bar{\phi}_\beta | \phi_\alpha \rangle \) is the identity operator \( I \). Applying this operator to any eigenvector \( |\phi_\alpha\rangle \), we obtain

\[
\left( \sum_\beta \langle \bar{\phi}_\beta | \phi_\alpha \rangle \right) |\phi_\alpha\rangle = \sum_\beta \langle \bar{\phi}_\beta | \phi_\alpha \rangle |\phi_\beta\rangle = |\phi_\alpha\rangle.
\]

(B15)

If the \( |\phi_\alpha\rangle \)'s form a complete set, Eq. (B15) implies that

\[
\sum_\beta |\phi_\beta\rangle \langle \bar{\phi}_\beta | = I.
\]

(B16)

Making use of Eqs. (B7) and (B16), we get

\[
H = \sum_\alpha E_\alpha |\phi_\alpha\rangle \langle \bar{\phi}_\alpha |,
\]

(B17)

exp\((-iHt/\hbar)\) = \sum_\alpha \exp\((-iE_\alpha t/\hbar)\)|\phi_\alpha\rangle \langle \bar{\phi}_\alpha |.  \]

(B18)
ing Eqs. (B18), (B8), and (B9), we derive

\[
\langle j | \exp(-iHt/\hbar) | i \rangle = \sum_{\alpha} \exp(-iE_{\alpha}/\hbar) \langle j | \phi_\alpha \rangle \langle \phi_\alpha | i \rangle = \sum_{\alpha} \exp(-iE_{\alpha}/\hbar) c^\dagger \alpha c_\alpha \langle j | \exp(-iHt/\hbar) | i \rangle,
\]

(B19)

since \( i \) and \( j \) play a symmetric role in the second line. This proves Eqs. (B5) and (B4).

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