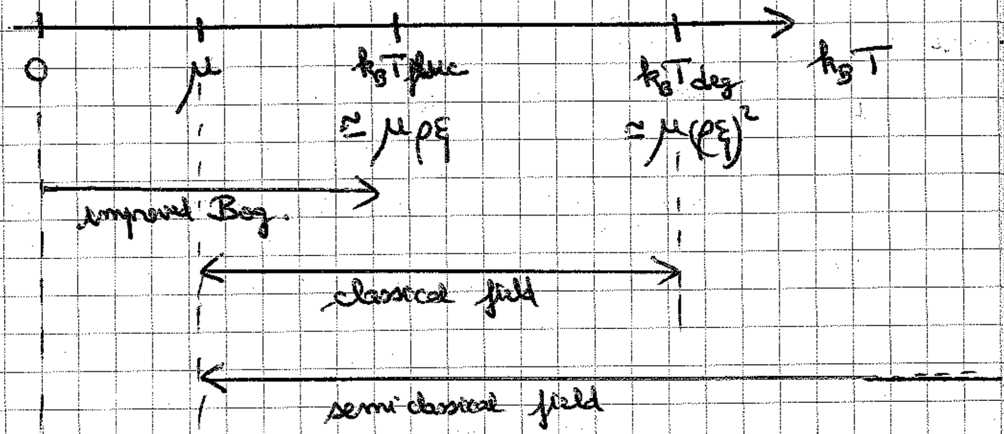
5. Classical field models for weakly interacting bosons

5.1 Motivations

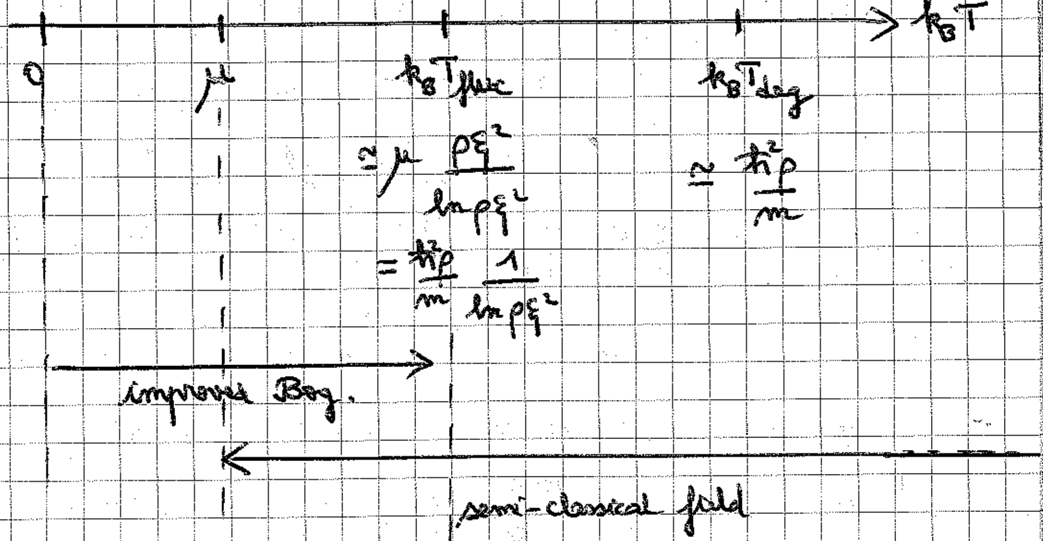
* study degenerate gas in 1D, 2D in "high" temperature regime where density fluctuations are large : $k_B T > k_B T_{fluc}$

1D: [here μ is $T=0$ chemical potential]



[T_{deg} is degeneracy temperature such that $\rho \lambda^d = 1$]

2D : [here μ is $T = 0$ chemical potential]



* classical field methods are widely used
  - → to study dynamics of formation of 3D BEC : Kagan, Svistunov, Shlyapnikov ; Sachdev
  - → equilibrium properties in 1D, 2D, 3D : Burnett, Rzążewski, Castin, Simula
  - → calculation of T_c in 2D (↙ BKT) and 3D (↘ BEC) : Svistunov, Prokofiev

5.2 Classical field model in 1D

* take contact interaction $g\,\delta(z)$, replace

$\hat{\psi}(z)$ by $\psi(z) \in \mathbb{C}$

$\hat{\psi}^\dagger(z)$ by $\psi^*(z) \in \mathbb{C}$

$$E[\psi] = \int_0^L dz \left[\frac{\hbar^2}{2m} \left|\frac{d\psi}{dz}\right|^2 + \frac{g}{2}|\psi|^4 - \mu|\psi|^2 \right]$$

thermal distribution $P[\psi] \propto e^{-\beta E[\psi]}$

* quite well behaved (in absence of cut-off)

$$g=0: \quad \rho = \int \frac{dk}{2\pi} \frac{k_B T}{E_k - \mu} < +\infty \qquad E_k = \frac{\hbar^2 k^2}{2m}$$

from equipartition theorem

but $\langle E \rangle = \infty$ of course

* exactly solvable : Scalapino, Sears, Ferrel (1972)

$$\int \mathcal{D}\psi \; e^{-\beta E} = \int d\psi_0 \underbrace{\int_{\psi(0)=\psi_0}^{\psi(L)=\psi_0} \mathcal{D}\psi \; e^{-\beta E[\psi]}}$$

Feynman path integral for imaginary time propagator of a quantum particle moving in 2D with Hamiltonian

$$\mathcal{H} = \frac{m k_B T}{\hbar^3} \hat{\vec{p}}^2 + \frac{\beta g}{2} (\hat{x}^2+\hat{y}^2)^2 - \beta\mu (\hat{x}^2+\hat{y}^2)$$

$$\int \mathcal{D}\psi \; e^{-\beta E[\psi]} = \mathrm{Tr}\left[e^{-\hat{\mathcal{H}} L/\hbar}\right]$$

therm. limit $\Rightarrow$ projection on ground state $|\phi_0\rangle$ of $\hat{\mathcal{H}}$ of energy ε_0

$$g_1(z) = \langle \psi^*(z)\psi(0)\rangle$$

$$\underset{t.l.}{=} \langle \phi_0 | (\hat{x} - i\hat{y})\, e^{-z(\hat{\mathcal{H}} - \varepsilon_0)/\hbar}\, (\hat{x} + i\hat{y}) | \phi_0 \rangle \qquad [z \geq 0]$$

* main results in therm. limit:

→ a single parameter T/T_{fluc}

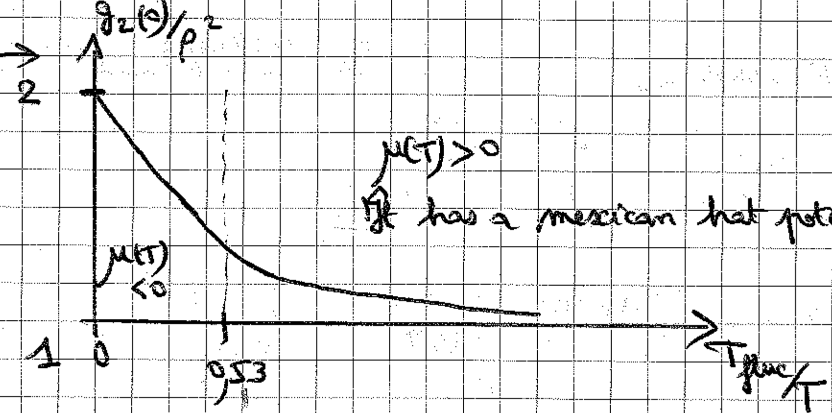
→ $g_1(z) \underset{z\to\infty}{\propto} e^{-|z|/l_{coh}}$

$g_2(z) - \rho^2 \underset{z\to\infty}{\propto} e^{-2|z|/l_{corr}}$

	$T \ll T_{fluc}$	$T \gg T_{fluc}$
l_{coh}	$\frac{\rho\lambda^2}{\pi}$	$\frac{\rho\lambda^2}{2\pi}$
l_{corr}	ξ	$\frac{\rho\lambda^2}{2\pi}$
	like improved Bog.	like ideal gas

NB. $\mu < k_B T$ and $k_B T < k_B T_{degen}$

with $\mu = T=0$ chemical potential $\simeq \rho g$



5.4 A semi-classical field model in 2D

NB. ρ depends on cut-off in 2D classical field model

* Glauber-P distribution:

$$\hat{\sigma} = \int \mathcal{D}\psi \; P[\psi] \; |coh:\psi\rangle\langle coh:\psi|$$

normalized Glauber coherent state such that

$$\hat{\psi}(\vec{r})\,|coh:\psi\rangle = \psi(\vec{r})\,|coh:\psi\rangle$$

* Fokker-Planck equation for P:

$\hat{\sigma} = e^{-\beta\hat{H}}$ as the result of imaginary time evolution:

$$\frac{d}{d\tau}\hat{\sigma}(\tau) = -\frac{1}{2}\{\hat{H}, \hat{\sigma}\}$$

$$\hat{\sigma}(0) = \mathbb{1}$$

Corresponding equation for P:

$$\partial_\tau P[\psi] = - E[\psi]\, P[\psi] \quad \leftarrow \text{decay term}$$

$$- \sum_{\vec{r}} \partial_{\psi(\vec{r})} (F P) + c.c. \quad \leftarrow \text{drift term}$$

$$- \frac{g_0}{4\ell^d} \partial^2_{\psi(\vec{r})} \left[\psi^2(\vec{r})\, P[\psi] \right] + c.c.$$

↘ (non positive) "diffusion" term

with $E[\psi] = \sum_{\vec{r}} \ell^d \left[\psi^* \left(-\frac{\hbar^2}{2m}\Delta - \mu\right)\psi + \frac{g_0}{2}|\psi|^4 \right]$

$$F[\psi] = -\frac{1}{2\ell^d} \partial_{\psi^*} E[\psi]$$

* various degrees of approximation:

→ keeping only the decay term

$$P[\psi] = e^{-\beta E[\psi]} \quad \text{classical field model}$$

→ keeping also the drift term: semi-classical

⟺ deterministic evolution for $\psi(\vec{r},\tau)$ and weight $w(\tau)$:

$$\partial_\tau \psi = F = -\frac{1}{2}\left[-\frac{\hbar^2}{2m}\Delta - \mu + g_0|\psi|^2 \right]\psi$$

imaginary time GPE

$$\frac{d}{d\tau} w(\tau) = - E[\psi(\tau)]\, w(\tau)$$

$$\hat{\sigma}_{SC} = \int \mathcal{D}\psi(0)\; W(\beta)\; |coh:\psi(\beta)\rangle\langle coh:\psi(\beta)|$$

5.5 Test of SC field model

* exact for ideal Bose gas

$$P[\psi] \propto e^{-\sum_k |\alpha_k|^2/n_k} \qquad n_k = \frac{1}{e^{\beta(E_k-\mu)}-1}$$

→ no blackbody catastrophe

* test on Bogoliubov type model :

$$\hat{H} = (E_k+\mu)\, a^\dagger a + \frac{\mu}{2}(a^2 + a^{\dagger 2})$$

→ $E_k \to 0$:

$$\langle \hat{H} \rangle_{exact} = k_B T - \frac{\mu}{2}$$

$$\langle \hat{H} \rangle_{SC} = \frac{1}{2} k_B T + \frac{\mu}{e^{2\beta\mu}-1}$$

$$= k_B T - \frac{\mu}{2} + \frac{1}{6}\beta\mu^2 + \ldots$$

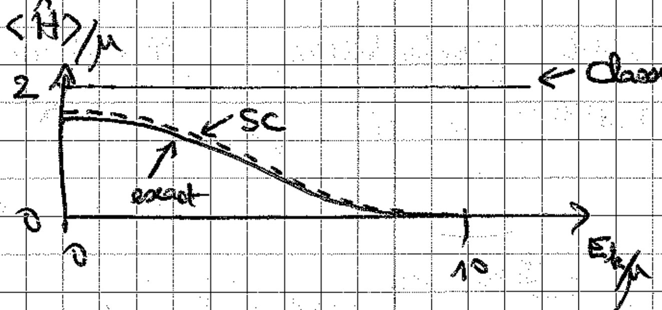
$\beta\mu \ll 1$ ↑ classical field

→ $E_k \to \infty$: $\qquad (\mathcal{E}_k = [E_k(E_k+2\mu)]^{1/2})$

$$\langle H \rangle_{exact} \simeq \mathcal{E}_k\, e^{-\beta\mathcal{E}_k} - \frac{\mu^2}{4E_k}$$

$$\langle H \rangle_{SC} \simeq \cosh(\beta\mu)\, \mathcal{E}_k\, e^{-\beta\mathcal{E}_k}$$

no possible UV divergence in SC (even better than quantum theory ...)

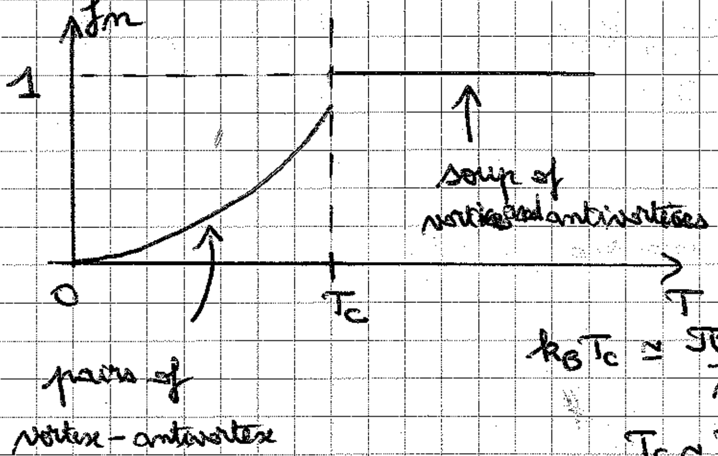


$k_B T = 2\mu$

5.6 Results in 2D

→ expected in therm. limit ($g_0 > 0$) : BKT transition

[BKT = Berezinskii – Kosterlitz – Thouless]



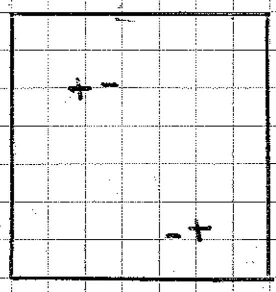
$$f_n = \frac{\langle \hat{P}_x^2 \rangle}{N m k_B T}$$

$$k_B T_c \simeq \frac{\pi \hbar^2 \rho}{m} \frac{1}{\ln(C \rho \xi_1^2)}$$

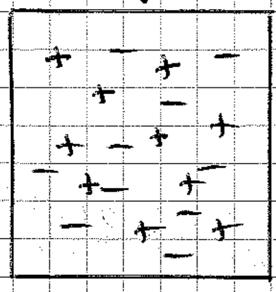
$T_c \sim T_{fluc}$!

→ numerics (finite size system) :

low T



high T



at low T, vortex density drops $\sim$ as activation law:

$$\rho_{v,+} \propto \rho \, e^{-\Delta/k_B T}$$

→ analytics:

$$\Delta(T)_{\text{therm. limit}} = \pi \frac{\hbar^2 \rho}{m} f(\xi/\lambda)$$

$$f(\xi/\lambda) \sim \frac{1}{\ln(\xi/\lambda)} \qquad \xi \ggg \lambda$$

then $\Delta(T_c) \simeq k_B T_c$ (weakly interacting limit $\xi \ggg \lambda$)