4. Extension of Bogoliubov theory to 1D and 2D for weakly interacting bosons

4.1 Motivations:

* In 1D and 2D, no BEC in thermodynamic limit for $T>0$
(Bogoliubov; Mermin, Wagner; Hohenberg)

* easily seen from Bogoliubov theory:

$$f_{nc} = \left(\frac{\langle \delta \hat{N} \rangle}{N}\right)_{t.l.} = \frac{1}{\rho} \int \frac{d^d k}{(2\pi)^d} \left[\left(U_k^2 + V_k^2\right) n_k + V_k^2\right]$$

$k \to 0$: $U_k^2,\ V_k^2 \propto \frac{1}{k}$ $\qquad n_k = \frac{1}{e^{\beta \epsilon_k} - 1} \propto \frac{1}{k}$ ($T>0$)

	$T=0$	$T>0$
1D	$f_{nc} = \infty$	$f_{nc} = \infty$
2D	$f_{nc} \propto \frac{1}{\rho \xi^2}$	$f_{nc} = \infty$

* a possible formalism: Popov theory (functional integral) book 80's

* simpler approach (with Mora): Bogoliubov type theory in phase-modulus representation:

$$\hat{\psi}(\vec{r}) = e^{i\hat{\theta}(\vec{r})} \sqrt{\hat{\rho}(\vec{r})}$$

if $\hat{\rho}$ has weak fluctuations $\delta\hat{\rho}$ and $\hat{\theta}$ has slow spatial fluctuations, quadratic hamiltonian in $(\delta\hat{\rho}, \text{grad}\,\hat{\theta})$ is obtained.

4.2 How to describe interactions in low d?

answer: scattering length in dimension d

* defined from zero energy 2-body scattering state

$$\Delta \phi(\vec{r}) = 0 \quad \text{out of potential}$$

3D: $\phi(r) = A + B/r = 1 - a_{3D}/r$

2D: $\phi(r) = A + B \ln r = \ln(r/a_{2D})$

1D: $\phi(r) = A + Br = 1 + r/2a_{1D}$

this defines Bethe-Peierls models

$a_{1D} = f(a_{3D}, a_{ho})$: Olshanii

$a_{2D} = f(a_{3D}, a_{ho})$: Holzmann, Petrov, Shlyapnikov

NB 1D : $\Longleftrightarrow$ $V(x) = g_{1D}\,\delta(x)$ (Lieb – Liniger) $\qquad g_{1D} = \dfrac{\hbar^2}{m a_1}$

* lattice model :

2D : $\quad g_0 = \dfrac{2\pi\hbar^2}{m}\,\dfrac{1}{\ln(C_{2D}\, l/a_{2D})} \qquad C_{2D} = \dfrac{1}{\pi}\, e^{\overset{\text{Euler}}{-\gamma} + \overset{\text{Catalan}}{2G/\pi}}$

1D : $\quad g_0 = \dfrac{g_{1D}}{1 + l/\pi^2 a_{1D}}$

4.3 Constraints on the parameters

4.3.1 To have weak density fluctuations

$$\mathrm{Var}\,\hat\rho \ll \rho^2$$

$\hat\rho(\vec r) = \hat\psi^\dagger(\vec r)\,\hat\psi(\vec r)$

$\hat\rho^2$ not normally ordered $\qquad [\hat\psi(\vec r), \hat\psi^\dagger(\vec r')] = \dfrac{\delta_{\vec r,\vec r'}}{l^d}$

$$\Rightarrow \quad \mathrm{Var}\,\hat\rho = \frac{\rho}{l^d} + \underbrace{\left[\langle \hat\psi^{\dagger 2}\hat\psi^2\rangle - \rho^2\right]}_{g_2(\vec 0)}$$

impose $\quad \rho\, l^d \gg 1$

and

$|g_2(\vec 0)| \ll \rho^2 \quad \}$ to be checked a posteriori

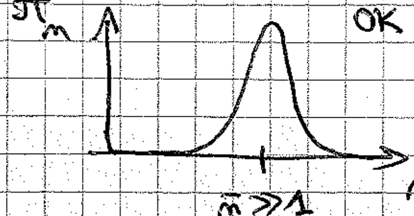
4.3.2 To have a well defined phase operator

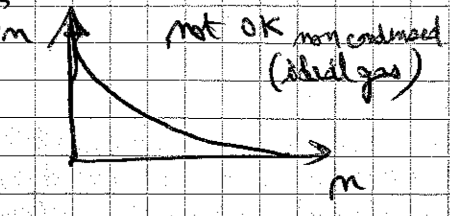
suppose $\quad \hat\psi(\vec r) = e^{i\hat\theta(\vec r)}\sqrt{\hat\rho(\vec r)}$

and $\quad [\hat\rho(\vec r), \hat\theta(\vec r')] = i\,\dfrac{\delta_{\vec r,\vec r'}}{l^d}$

then $\quad e^{i\hat\theta(\vec r)}\,|n : \vec r\rangle = |n-1 : \vec r\rangle \quad \ldots$ even if $n = 0$

$\rightarrow$ assume never empty lattice cell





4.3.3 To recover continuous space physics

cut-off momentum $\propto \frac{1}{\ell}$ should be large enough

$$\ell \ll \lambda$$

$$\ell \ll \xi \qquad \left(\frac{\hbar^2}{m\xi^2} = \mu\right)$$

compatible with $\rho\lambda^d \gg 1$ if $\rho\lambda^d \gg 1$ degenerate regime

$\rho\xi^d \gg 1$ weakly interacting regime

4.4 Perturbative Expansion of lattice hamiltonian

* Model Hamiltonian : grand canonical

$$\hat{H} = \sum_{\vec{r}} \ell^d \left[-\frac{\hbar^2}{2m} \hat{\psi}^\dagger \Delta \hat{\psi} + (V(\vec{r}) - \mu)\hat{\rho} + \frac{g_0}{2} \hat{\rho}\left(\hat{\rho} - \frac{1}{\ell^d}\right)\right]$$

discrete Laplacian $\Delta f = \sum_{j \in d \text{ direction of space}} \frac{f(\vec{r}+\ell\vec{e}_j) + f(\vec{r}-\ell\vec{e}_j) - 2f(\vec{r})}{\ell^2}$

$$\Rightarrow e^{i[\theta(\vec{r}) - \theta(\vec{r}')]} \quad \text{between neighboring points}$$

* two small parameters:

$$\hat{\rho} = \rho_0 + \hat{\delta\rho} \qquad \varepsilon_1 = \frac{|\delta\rho|}{\rho_0}$$

$$\varepsilon_2 = |\ell \,\vec{\text{grad}}\,\theta|$$

one can a posteriori chose ℓ so that $\varepsilon_1 \sim \varepsilon_2 \sim \frac{1}{\sqrt{\rho_0 \ell^d}}$

* expansion of $\hat{H}$:

$$\hat{\rho}^{1/2} = \rho_0^{1/2} + \frac{\hat{\delta\rho}}{2\rho_0^{1/2}} - \frac{\hat{\delta\rho}^2}{8\rho_0^{3/2}} + \dots$$

$$e^{i[\theta(\vec{r})-\theta(\vec{r}')]} = 1 + i[\theta(\vec{r}) - \theta(\vec{r}')] - \frac{1}{2}[\theta(\vec{r}) - \theta(\vec{r}')]^2 + \dots$$

* zeroth order:

$$H_0 = \sum_{\vec{r}} \ell^d \rho_0^{1/2} \left[-\frac{\hbar^2}{2m}\Delta + V(\vec{r}) + \frac{1}{2} g_0\rho_0 - \mu\right] \rho_0^{1/2}$$

so $\rho_0^{1/2}(\vec{r})$ solves Gross-Pitaevskii like equation !

solution contains a number of particles $N_0 = \sum_{\vec{r}} \ell^d \rho_0(\vec{r})$ function of μ.

inversion : $N_0(\mu) \Longrightarrow \mu(N_0)$

$\rho_0(\mu, \vec{r}) \Longrightarrow \rho_0(N_0, \vec{r})$

* first order : vanishes
* second order : setting $\hat{B}(\vec{r}) = \dfrac{\delta\hat{\rho}(\vec{r})}{2\rho_0^{1/2}(\vec{r})} + i\,\rho_0^{1/2}(\vec{r})\,\hat{\theta}(\vec{r})$,

one has $\left[\hat{B}(\vec{r}), \hat{B}^\dagger(\vec{r}\,')\right] = \dfrac{\delta_{\vec{r},\vec{r}\,'}}{l^d}$ and

$$H_2 = l^d \sum_{\vec{r}} \hat{B}^\dagger\left(-\frac{\hbar^2}{2m}\Delta + U + 2g_0\rho_0 - \mu\right)\hat{B} + \frac{g_0\rho_0}{2}\left(\hat{B}^2 + \hat{B}^{\dagger 2}\right)$$

Bogoliubov like Hamiltonian ! modal decomposition

$$\hat{B}(\vec{r}) = -i\hat{Q}\,\rho_0^{1/2}(\vec{r}) + \hat{P}\,\partial_{N_0}\rho_0^{1/2}(\vec{r}) + \sum_\alpha \hat{b}_\alpha u_\alpha(\vec{r}) + \hat{b}_\alpha^\dagger v_\alpha^*(\vec{r})$$

collective phase of the field

zero energy mode

$= \hat{N} - N_0$

anomalous mode

regular Bogoliubov mode same spectrum as Bog. theory

$$[\hat{P}, \hat{Q}] = -i$$

$$H_2 = -\frac{1}{2} N_0 \frac{d\mu}{dN_0} - \sum_\alpha \varepsilon_\alpha \langle v_\alpha | v_\alpha \rangle + \frac{1}{2}\frac{d\mu}{dN_0}\hat{P}^2 + \sum_\alpha \varepsilon_\alpha \hat{b}_\alpha^\dagger \hat{b}_\alpha$$

* why cubic/third order required ?

$\langle \hat{\delta\rho} \rangle_{H_2} = 0$; no correction to pure quasicondensate at second order !

done by Mora and Y. C. ; this was long .

4.5 Applications of the formalism

* equation of state : $D = [-\pi/l, \pi/l[^d$

$$\frac{\mu}{g_0} = \rho + \int_D \frac{d^dk}{(2\pi)^d}\left[(U_k+V_k)^2 n_k + V_k(U_k+V_k)\right]$$

$\rightarrow$ 3D : $l \to 0$! $\dfrac{1}{g_0}$ diverges, $\int_D V_k(U_k+V_k)$ diverges

replacing g_0 by its expression

$$\frac{\mu}{g} = \rho + \int_D \frac{d^3k}{(2\pi)^3}\left[(U_k+V_k)^2 n_k + \underbrace{V_k(U_k+V_k) + \frac{m\mu}{\hbar^2k^2}}\right]$$: one can set $l \to 0$

$$\underset{k\to\infty}{\sim} -\frac{m\mu}{\hbar^2k^2}$$

* → 2D: same 'miracle'

$$\rho = \frac{m\mu}{4\pi\hbar^2}\ln\left(\frac{4\hbar^2}{a_{2D}^2 m\mu e^{2\gamma+1}}\right) - \int \frac{d^2k}{(2\pi)^2}(U_k+V_k)^2 n_k$$

→ 1D: no pb

* density fluctuations: $g_2(\vec{r}) = \langle \hat\psi^\dagger(\vec{r})\hat\psi^\dagger(\vec{0})\hat\psi(\vec{0})\hat\psi(\vec{r})\rangle$

$$g_2(\vec{r}) = \rho^2 + 2\rho\int_D \frac{d^dk}{(2\pi)^d}\left[\underbrace{(U_k+V_k)^2 n_k}_{\text{thermal term}} + \underbrace{V_k(U_k+V_k)}_{\text{quantum term}}\right]\cos\vec{k}.\vec{r}$$

estimates for $\frac{g_2(\vec{0})-\rho^2}{\rho^2}$:

	QT	thermal, $k_BT<\mu$	thermal, $k_BT>\mu$
$d=1$	$-\frac{1}{\rho\xi}$	$\frac{(k_BT/\mu)^2}{\rho\xi}$	$\frac{k_BT/\mu}{\rho\xi}$
$d=2$	$-\frac{\ln(\xi/l)}{\rho\xi^2}$	$\frac{(k_BT/\mu)^3}{\rho\xi^2}$	$\frac{\frac{k_BT}{\mu}\ln\frac{k_BT}{\mu}}{\rho\xi^2}$
$d=3$	$-\frac{1}{\rho\xi^2 l}$	$\frac{(k_BT/\mu)^4}{\rho\xi^3}$	$\frac{1}{\rho\lambda^3}$

* coherence properties:

⚠ keep density fluctuations to have ~~cut off~~ finite limit for $l \to 0$

$$g_1(\vec{r}) = \langle\hat\psi^\dagger(\vec{r})\hat\psi(\vec{0})\rangle$$

$$\ln\left[g_1(\vec{r})/\rho\right] = -\frac{1}{\rho}\int\frac{d^dk}{(2\pi)^d}\left[(U_k^2+V_k^2)n_k + V_k^2\right](1-\cos\vec{k}.\vec{r})$$

large r behavior

1D: $T=0$ $\quad g_1(r) \propto \rho\left(\frac{\xi}{r}\right)^{1/2\pi\rho\xi}$

$T>0$: $\quad g_1(r) \propto e^{-r/(\rho\lambda^2/\pi)}$

2D: $T=0$: BEC $\quad g_1(r) \to \rho_{cond}$

$T>0$: $\quad g_1(r) \propto \frac{1}{r^{1/\rho\lambda^2}}$

4.6 Moving quasi-condensates

not treated