3. Bogoliubov method for bosons

3.1 Considered regime: almost pure condensate

→ $T \ll T_c$
→ weak interactions : $n a^3 \ll 1$

for simplicity, canonical ensemble and contact interaction

$$H = \int d^3r \left[\hat{\psi}^\dagger h_1 \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

$$h_1 = -\frac{\hbar^2}{2m} \Delta_{\vec{r}} + U(\vec{r})$$

3.2 Idea of Bogoliubov method

* $\sim N$ particles in condensate mode $\phi(\vec{r})$ so split field operator as

$$\hat{\psi}(\vec{r}) = \phi(\vec{r})\, \hat{a}_\phi + \hat{\psi}_\perp(\vec{r})$$

Useful properties :

→ $\int \phi^* \hat{\psi}_\perp = 0$

→ $\hat{N} = \hat{a}_\phi^\dagger \hat{a}_\phi + \underbrace{\hat{\delta N}}_{= \int \hat{\psi}_\perp^\dagger \hat{\psi}_\perp}$

→ $\left[\hat{\psi}_\perp(\vec{r}), \hat{\psi}_\perp^\dagger(\vec{r}\,') \right] = \langle \vec{r} | Q | \vec{r}\,' \rangle$

where $Q = \mathbb{1} - |\phi\rangle\langle\phi|$ one-body projector on non-condensed modes

* treat $\hat{\psi}_\perp$ as a perturbation:

" $\hat{\psi}_\perp \ll \phi\,\hat{a}_\phi$ " in the physical state

e.g. $\int \langle \hat{\psi}_\perp^\dagger \hat{\psi}_\perp \rangle \ll \int \langle a_\phi^\dagger a_\phi \rangle |\phi|^2 = N_0$.

3.3 Zeroth order : pure condensate

$$\hat{\psi} \longrightarrow \phi\,\hat{a}_\phi$$

$$\hat{a}_\phi^\dagger \hat{a}_\phi \longrightarrow N$$

$$H_0 = E[N,\phi] = N \int \phi^* h_1 \phi + \frac{g}{2}(N-\cancel{1}) |\phi|^4$$

Minimize energy over ϕ with $\int |\phi|^2 = 1$:

$$\frac{\delta}{\delta\phi^*} \left\{ E[N,\phi] - N\mu \int \phi^*\phi \right\} = 0$$

μ: Lagrange multiplier

$$\mu\,\phi(\vec{r}) = \Big[h_1 + \underbrace{g N |\phi(\vec{r})|^2}_{\text{mean field term}} \Big]\, \phi(\vec{r}) \quad : \text{NLSE}$$

or Gross - Pitaevskii equation

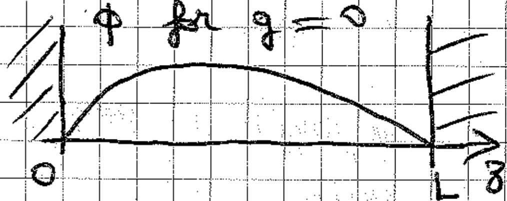
⚠ for $g \neq 0$, $\mu \neq \frac{E}{N}$ (because $1 \neq \frac{1}{2}$)

but $\mu = \partial_N E[N,\phi] = \frac{d}{dN} E[N,\phi]$

so μ = chemical potential.

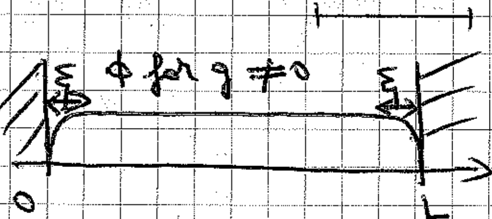
The NLSE was accurately tested experimentally / studied theoretically in a trap (a lecture in itself!)

In a large box with hard (∞) walls in $z = 0, L$:



(a sinus fonction)

ideal gas condensate very sensitive to the vessel shape



$N|\phi|^2 \simeq n$ except at distance $\lesssim \xi$ from the hard walls

ξ = healing length

$$\frac{\hbar^2}{m\xi^2} = \mu \simeq ng$$

3.4 First order

$$H_1 = \int \phi^* a_\phi^\dagger h_1 \hat\Psi_\perp + g|\phi|^2 \phi^* \hat a_\phi^\dagger \underbrace{\hat a_\phi^\dagger \hat a_\phi}_{\hat N} \hat\Psi_\perp + h.c.$$

$$H_1 = \int \phi^* \left[h_1 + Ng|\phi|^2\right] \hat a_\phi^\dagger \hat\Psi_\perp + h.c.$$

$$= \int \mu \phi^* \hat a_\phi^\dagger \hat\Psi_\perp + h.c. = 0$$

NLSE

3.5 Second order : Bogoliubov theory

Two types of contribution :

(1) revisiting the 0^{th} order calculation $\hat{\psi} \to \phi \hat{a}_\phi$

now keep $\hat{a}_\phi^\dagger \hat{a}_\phi = \hat{N} - \hat{\delta N}$ up to first order in $\hat{\delta N}$

$$H_2^{n_\phi \neq N} = \int \phi^* h_1 \phi \left(-\hat{\delta N}\right) + \frac{g}{2} \int |\phi|^4 \left(-2 \hat{N} \hat{\delta N}\right)$$

$$\text{NLSE} \Rightarrow \quad H_2^{n_\phi \neq N} = -\mu \, \hat{\delta N}$$

an effective grand canonical ensemble for the non-condensed particles

(2) from explicit $\hat{\psi}_\perp$ factors :

$$H_2^{\psi_\perp} = \int \hat{\psi}_\perp^\dagger h_1 \hat{\psi}_\perp + \frac{g}{2} \int \Big[\left(\hat{\psi}_\perp^{\dagger 2} \phi^2 \hat{a}_\phi^2 + h.c. \right) + 4 |\phi|^2 \hat{\psi}_\perp^\dagger \underbrace{\hat{a}_\phi^\dagger \hat{a}_\phi}_{\hat{N}} \hat{\psi}_\perp \Big]$$

How to eliminate $\hat{a}_\phi$?

Phase representation (Girardeau) :

$$\hat{a}_\phi = \hat{A}_\phi \sqrt{\hat{n}_\phi} \qquad \text{with} \quad \hat{A}_\phi = \frac{1}{\sqrt{\hat{n}_\phi + 1}} \hat{a}_\phi$$

$\to$ $\hat{A}_\phi$ annihilates one particle in the condensate mode without $\sqrt{n}$ amplitude

$$\hat{A}_\phi\,|n:\phi\rangle = |n-1:\phi\rangle \quad \text{if } n>0$$
$$= 0 \quad \text{otherwise}$$

$\rightarrow$ $\hat{A}_\phi$ is almost unitary:

$$\hat{A}_\phi \hat{A}_\phi^+ = \mathbb{1}$$
$$\hat{A}_\phi^+ \hat{A}_\phi = \mathbb{1} - |0:\phi\rangle\langle 0:\phi|$$

in considered regime, probability of having empty condensate mode is negligible: ANEC.

then " $\hat{A}_\phi = e^{i\hat{\theta}}$ " $\quad \hat{\theta}$ hermitian phase operator

<u>Elimination of $\hat{a}_\phi$</u>:

$$\hat{\Lambda}(\vec{r}) \equiv \hat{A}_\phi^+ \hat{\psi}_\perp(\vec{r}) = e^{-i\hat{\theta}}\hat{\psi}_\perp(\vec{r})$$

$\rightarrow$ $\hat{\Lambda}$ conserves the total number of particles

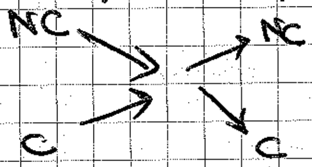
$\rightarrow$ $[\hat{\Lambda}(\vec{r}), \hat{\Lambda}^+(\vec{r}\,')] = \langle \vec{r}|Q|\vec{r}\,'\rangle$

Finally
$$H_{Bog} = E[N,\phi] + \int d^3\vec{r}\,\Big[\hat{\Lambda}^+(h_1-\mu)\hat{\Lambda} + 2gN|\phi|^2\hat{\Lambda}^+\hat{\Lambda} + \frac{gN}{2}\left(\phi^2\hat{\Lambda}^{+2} + \phi^{*2}\hat{\Lambda}^2\right)\Big]$$

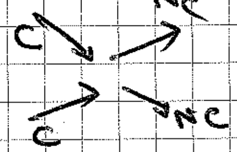
" Hartree - Fock" mean field term for non-condensed particles $(2 = 1+1)$

squeezing term

simple interpretation :



Hartree-Fock term



and vice-versa

squeezing term