2. Which model for the interaction potential?

2.1. Basic postulate

In gaseous limit $n b^3 \ll 1$, at low temperature (say $n\lambda^3 > 1$), short range details of $V(\vec{r})$ are not important.

Only low k two-body scattering amplitude f_k matters.

$\Rightarrow$ take a convenient model such that

$$f_k^{\text{model}} \simeq f_k^{\text{true}} \quad \text{at low } k \quad (\text{in particular } k b \ll 1)$$

2.2 Reminder of scattering theory in 3D

$*$ the pb: Schrödinger equation at $E = \frac{\hbar^2 k^2}{m} \geq 0$

$$E\,\phi(\vec{r}) = \left[-\frac{\hbar^2}{m}\Delta_{\vec{r}} + V(\vec{r})\right]\phi(\vec{r})$$

with boundary conditions at $r \to \infty$:

$$\phi(\vec{r}) \simeq \underbrace{e^{i\vec{k}.\vec{r}}}_{\substack{= \langle \vec{r}|\vec{k}\rangle \\ \text{incoming wave}}} + \underbrace{f_{\vec{k}}(\vec{n})\,\frac{e^{ikr}}{r}}_{\substack{\text{scattered wave in direction} \\ \vec{n} = \vec{r}/r}}$$

* formal solution for $f_{\vec{k}}(\vec{n})$:

$$f_{\vec{k}}(\vec{n}) = -\frac{m}{4\pi\hbar^2} \langle k\vec{n} | T(E+i0^+) | \vec{k} \rangle$$

T matrix : $T(z) = V + V\,G(z)\,V \qquad z \in \mathbb{C}$

resolvent $\quad G(z) = \dfrac{1}{z - \left[\frac{p^2}{m} + V(\vec{r})\right]}$

* Born expansion (in powers of V) :

$$T(z) = V + V\,G_0(z)\,V + V\,G_0(z)\,V\,G_0(z)\,V + \dots$$

Born approximation $\qquad G_0(z) = \dfrac{1}{z - \frac{p^2}{m}}$

* pole of $f_{\vec{k}}$ for $k = iq$, $q > 0$, gives bound state energy $\quad E_{dim} = -\dfrac{\hbar^2 q^2}{m}$

* low k limit :

→ $f_{\vec{k}}(\vec{n})$ becomes rotationally invariant : s-wave scattering

→ $$f_k = \frac{-1}{\frac{1}{a} + ik - \frac{1}{2}k^2 r_e + \dots}$$

imposed by unitarity

a = scattering length (any sign)

r_e = effective range (any sign)

in all my lectures, zero range limit

$$k^2 |r_e| \ll \left|\frac{1}{a} + ik\right| \qquad \text{for relevant } k$$

in practice, a is the only parameter [⚠ only in weakly interacting regime for $N > 2$ bosons]

2.3 Convenient models

2.3.1 Contact potential (Fermi)

$$V(\vec{r}) = g\,\delta(\vec{r}) = g\,|\vec{r} = \vec{0}\rangle\langle\vec{r} = \vec{0}|$$

has a meaning only in the Born approximation
[OK only in 1D]

$$f_0 = -a = -\frac{m}{4\pi\hbar^2}\langle\vec{k} = \vec{0}|V|\vec{k} = \vec{0}\rangle \Rightarrow g = \frac{4\pi\hbar^2}{m}a$$

[next order in Born expansion gives ∞]

2.3.2 Bethe - Peierls model

→ potential replaced by contact conditions:

$$\exists A \,/\, \quad \phi(\vec{r}) \underset{r\to 0}{=} A\left[\frac{1}{r} - \frac{1}{a}\right] + O(r)$$

→ large r form of $\phi(\vec{r})$ valid $\forall r > 0$:

$$1 + f_k\left[\frac{1}{r} + ik\right] = A\left[\frac{1}{r} - \frac{1}{a}\right]$$

$$f_k = \frac{-1}{\frac{1}{a} + ik}$$

$\rightarrow$ bound state : $k = iq$ $\qquad \frac{1}{a} = q$

$\Rightarrow$ one bound state for $a>0$, $\quad E_{dim} = -\frac{\hbar^2}{ma^2}$

$\rightarrow$ which Schrödinger equation ?

$$\Delta \frac{1}{r} = -4\pi\, \delta(\vec{r})$$

$$\frac{\hbar^2}{m} \Delta\phi \text{ contains } -\frac{\hbar^2}{m} 4\pi A \delta(\vec{r}) = g\, \delta(\vec{r})\, \partial_r (r\phi)$$

$$E\, \phi(\vec{r}) = -\frac{\hbar^2}{m} \Delta\phi(\vec{r}) + \underbrace{g\, \delta(\vec{r})\, \partial_r (r\phi)}_{\text{a regularized } \delta \,:\, \text{Huang}}$$

⚠ for bosons, regularization works up to 2-body correlations, not for 3-body correlations.

2.3.3 Lattice model

$$\vec{r} \in l\, \mathbb{Z}^3$$

$$\vec{k} \in D = \left[-\frac{\pi}{l}, \frac{\pi}{l}\right[^3$$

$$V = g_0\, |\vec{r} = \vec{0}\rangle\langle \vec{r} = \vec{0}| \qquad \text{(in CoM frame)}$$

* adjust g_0 to have correct scattering length :

$$\frac{1}{g_0} + \int_D \frac{d^3k}{(2\pi)^3} \frac{m}{\hbar^2 k^2} = \frac{1}{g}$$

<u>Hint</u>: in calculation of $\langle \vec{k} | T(E + i0^+) | \vec{k}' \rangle$, $\langle \vec{r} = \vec{0} | G | \vec{r} = \vec{0} \rangle$ appears, obtained from $G = G_0 + G_0 V G$

* discussion: renormalisation of coupling constant

bare g_0 depends on momentum cut-off $\propto \frac{1}{l}$

$$g_0 = \frac{g}{1 - C\,a/l} \qquad C = 2,44\ldots$$

$\rightarrow$ Born regime: $|a|/l \ll 1$, $g_0 \simeq g$

$g_0 > 0$ for $a > 0$

$\rightarrow$ limit $l \rightarrow 0$: $\quad g_0 \sim -\frac{4\pi}{C} \frac{\hbar^2 l}{m} < 0$

- bad for $N > 2$ bosons: ground state not a gas but collapsed state

Hint: $\langle N : \vec{r} = \vec{0} | H | N : \vec{r} = \vec{0} \rangle \underset{l \to 0}{\longrightarrow} -\infty$

- no problem for fermions (believed, not proved):
  - ground state = gas
  - QMC with no sign problem
  - equivalence with Bethe Peierls for $l \rightarrow 0$