# Quantum gases

## 0. Introduction

gas : $n b^3 \ll 1$ $\quad$ $n$ density, $b$ interaction range

$\neq$ liquid (like superfluid helium)

## 0.1 History

theory : 1925 : Einstein : BEC
50's : Bogoliubov, Beliaev : weakly repulsive Bose gas
BCS : condensation of pairs in a weakly attractive Fermi gas

expt : 1995 : first BEC in atomic gases (Rb, Na)
2001 : first degenerate atomic Fermi gases
strongly interacting regime can be reached for fermionic atoms

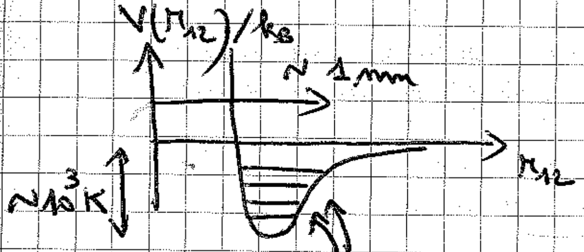
## 0.2 Why 70 years ?

at $T < T_c$ equilibrium state not a gas but a solid (or a liquid for He)

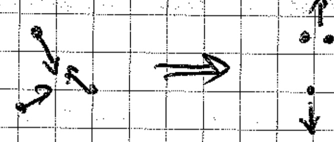
way out : limit of low density $n$

elastic collision rate $n \sigma v$ $\gg$ recombination rate to deeply bound dimers $\propto n^2$

metastability !




deeply bound dimers



price to pay : low $T$ since $n\lambda^3 \sim 1$ $\quad \lambda^2 = \dfrac{2\pi\hbar^2}{m k_B T}$

$n \sim 10^{14} - 10^{15}$ at$/cm^3$

$T_c \sim 1\,\mu K$

$N_0 \sim 10^6$ (alkali) to $10^9$ (H)

80's : laser cooling (Phillips, Chu, Cohen-Tannoudji)

90's : evaporative cooling in magnetic traps

## 0.3 Versatile systems

many parameters adjustable :

- external potential $V(\vec{r})$
  - harmonic ; 3D, 2D, 1D cases
  - periodic ; mimics condensed matter physics

[in this lecture : often a box for simplicity, with periodic boundary conditions]

- interaction strength can be controlled at will (Feshbach resonance in B field)

$f_k$ = scattering amplitude of 2 atoms of relative momentum $k$

$$\lim_{k\to 0} f_k = -a \qquad a = \text{scattering length}$$

$a$ can be varied almost from $-\infty$ to $+\infty$:

$a = 0$ : almost ideal gas

$|a| = \infty$ : maximally interacting gas
done with fermions
(bosons short lived for $|a| = \infty$)

# 1. The ideal Bose gas

## 1.1 Basic facts

* Hamiltonian $\quad H = \sum_{i=1}^{\hat{N}} h(i) \qquad h = \frac{p^2}{2m} + U(\vec{r})$

a single spin state

$$\hat{\psi}(\vec{r}) = \sum_\alpha \phi_\alpha(\vec{r})\, a_\alpha \qquad [a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$$

eigenmode of $h$
energy $\varepsilon_\alpha$

$$H = \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha$$

* thermal equilibrium : grand canonical ensemble

$$\hat{\rho} = \frac{1}{\Xi} e^{-\beta(H-\mu\hat{N})} = \frac{1}{\Xi} e^{-\beta \sum_\alpha (\varepsilon_\alpha - \mu) a_\alpha^\dagger a_\alpha}$$

$\beta = \frac{1}{k_B T} \qquad \mu$ = chemical potential $< \varepsilon_0$

* saturation of the excited modes population :

$$n_\alpha = \langle a_\alpha^\dagger a_\alpha \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} - 1} < \frac{1}{e^{\beta(\varepsilon_\alpha - \varepsilon_0)} - 1}$$

$$N' \equiv \sum_{\alpha \neq 0} n_\alpha < N'_{max}(T) = \sum_{\alpha \neq 0} \frac{1}{e^{\beta(\varepsilon_\alpha - \varepsilon_0)} - 1}$$

if $N > N'_{max}$, at least $N - N'_{max}(T)$ particles in the ground mode : a condensate of macroscopic number

Interesting regime : many eigenmodes with $\varepsilon_\alpha < k_B T$.

* in a box with $L \gg \lambda$ :

- 3D : $\displaystyle\sum_{\vec{k} \neq 0} \longrightarrow \int d^3\vec{k}$

$$N'_{max}(T) \simeq \zeta(3/2) \left(\frac{L}{\lambda}\right)^3$$

a phase transition in thermodynamic limit

$$(n\lambda^3)_{crit} = \zeta(3/2) = 2{,}612\ldots$$

- 1D : $\Sigma \to \int$ leads to divergence : $N'_{max}$ dominated by discrete low energy modes

use approximation $\displaystyle n_k^{max} \simeq \frac{k_B T}{\hbar^2 k^2 / 2m}$ $\qquad k = \frac{2\pi}{L} q, \quad q \in \mathbb{Z}^*$

$$\sum_{q \in \mathbb{Z}^*} \frac{1}{q^2} < +\infty \implies N'_{max} \simeq \frac{\pi}{3} \left(\frac{L}{\lambda}\right)^2$$

BEC disappears in thermodynamic limit (here $T \neq 0$)

- 2D : both high and low $k$ contribute

$$N'_{max} \simeq 2\left(\frac{L}{\lambda}\right)^2 \ln \frac{L}{\lambda}$$

no BEC in thermodynamic limit (here $T \neq 0$)

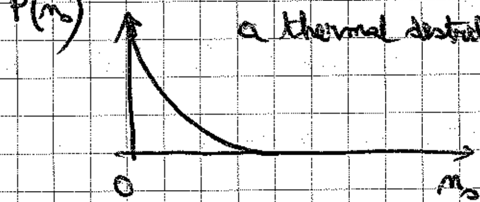
## 1.2 Beyond BEC theory when a condensate is present

* if $N_0 \sim N$, GC ensemble has unphysically large fluctuations in the condensate mode

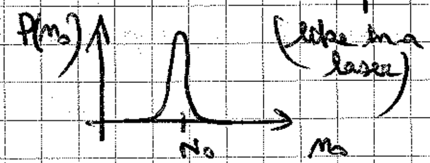
$$\mathrm{Var}\, \hat{n}_0 = \langle a_0^\dagger a_0 a_0^\dagger a_0 \rangle - \langle a_0^\dagger a_0 \rangle^2 \underset{\text{Wick}}{=} N_0(N_0+1) \simeq N_0^2$$

probability distribution of $n_0$:

$$\hat{\sigma}_0 \propto e^{-\beta(\varepsilon_0 - \mu)\, a_0^\dagger a_0}$$




one rather expects (like in a laser)




* way out : use canonical distribution

$$\hat{\sigma}_C = \frac{1}{Z} e^{-\beta \sum_\alpha \varepsilon_\alpha \hat{n}_\alpha} P_N$$

$P_N$ projects on $N$-particle subspace

eliminate condensate mode :

$$\hat{n}_0 = \hat{N} - \sum_{\alpha \neq 0} \hat{n}_\alpha = \hat{N} - \delta\hat{N}$$

$$\hat{\sigma}_C = \frac{1}{Z'} e^{-\beta \sum_{\alpha \neq 0} (\varepsilon_\alpha - \varepsilon_0) \hat{n}_\alpha}\, Y\Big(N - \sum_{\alpha \neq 0} \hat{n}_\alpha\Big)$$

$Y$ : Heaviside function

approximation of never empty condensate (ANEC):

$$Y(\ ) \simeq 1 \qquad \text{OK in large } N \text{ limit}$$

$\Rightarrow$ a GC ensemble for the excited modes
reservoir = condensate mode
chemical potential = $\varepsilon_0$

$$n_\alpha^c \simeq \frac{1}{e^{\beta(\varepsilon_\alpha-\varepsilon_0)}-1}$$

$$\mathrm{Var}_c\,\hat{n}_0 = \mathrm{Var}_c\,\hat{\delta N} \simeq \sum_{\alpha\neq 0} n_\alpha^c\,(1+n_\alpha^c)$$

In a box $(L \gg \lambda)$ : $\quad n_{\vec{k}}^c \simeq \dfrac{k_BT}{\varepsilon_{\vec{k}}}$

$$\mathrm{Var}_c\,\hat{n}_0 \propto \left(\frac{L}{\lambda}\right)^4$$

* test of ANEC : 1D ideal Bose gas in harmonic trap, canonical ensemble exactly solvable

for $N \to +\infty$, $k_BT$ and $\hbar\omega$ fixed:


$$n_1 - \frac{1}{e^{\beta\hbar\omega}-1} \begin{cases} \propto \dfrac{1}{N} & GC^{(*)} \\ = 0 & \text{ANEC} \\ = O\left(N e^{-\beta N\hbar\omega}\right) & C \end{cases}$$

(*) because
$$|\mu_{GC} - \varepsilon_0| \sim \frac{k_BT}{N_0} \sim \frac{k_BT}{N} \text{ in this limit}$$