BASICS OF BOSE-EINSTEIN CONDENSATION THEORY

Y. Castin, LKB - ENS, Paris

OUTLINE

• atoms: waves and particles
• the Bose law
• when the Bose gas becomes degenerate
• how to reach Bose-Einstein condensation
• atomic interactions and Gross-Pitaevskii equation
**ATOMS: WAVES AND PARTICLES**

**Analogy with optics:**

<table>
<thead>
<tr>
<th>Object</th>
<th>optics</th>
<th>atomic physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>field</td>
<td>$E(r, t), B(r, t)$</td>
<td>$\phi(r, t)$</td>
</tr>
<tr>
<td>equation of motion</td>
<td>$(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) B = 0$</td>
<td>$i\hbar \partial_t \phi = -\frac{\hbar^2}{2m} \Delta \phi$</td>
</tr>
<tr>
<td>particle</td>
<td>photon</td>
<td>atom</td>
</tr>
<tr>
<td>energy</td>
<td>$\hbar \omega$</td>
<td>$\frac{1}{2}mv^2$</td>
</tr>
<tr>
<td>momentum</td>
<td>$\hbar k$</td>
<td>$p = mv$</td>
</tr>
<tr>
<td>wavelength</td>
<td>$\lambda = \frac{2\pi}{k} = \frac{\hbar}{\hbar k}$</td>
<td>$\lambda = \frac{\hbar}{p}$</td>
</tr>
<tr>
<td>dispersion relation</td>
<td>$\omega = ck$</td>
<td>$\omega = \frac{\hbar k^2}{2m}$</td>
</tr>
</tbody>
</table>
Values in an ordinary gas:

- **equipartition of energy:**
  \[
  \frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} k_B T
  \]

- **sodium atoms at 300 K:**
  \[
  \Delta v_x = 300 \text{ m/s}
  \]
  \[
  \lambda = 5 \times 10^{-11} \text{ m}
  \]

With Sisyphus cooling:

\[
\lambda \sim 1 \mu\text{m}.
\]
ATOMIC MODES IN A BOX

Energy levels of an atom in a box:

- **periodic boundary conditions:**
  \[ \phi(x + L, y, z) = \phi(x, y + L, z) = \phi(x, y, z + L) = \phi(x, y, z). \]

- **quantisation of wavevectors:**
  \[ \phi(x, y, z) \propto e^{i(k_xx + k_yy + k_zz)} \]
  \[ k_\alpha = \frac{2\pi}{L}q_\alpha \]

- **quantisation of energy:**
  \[ \epsilon_k = \frac{\hbar^2}{2mL^2} \left( q_x^2 + q_y^2 + q_z^2 \right) \]
THE BOSE LAW

Indistinguishable particles in quantum theory are:

- **bosons:**
  \[ P_\sigma |\psi\rangle_B = |\psi\rangle_B \]

- **or fermions:**
  \[ P_\sigma |\psi\rangle_F = \epsilon(\sigma) |\psi\rangle_F. \]

Configuration defined by a set of occupation numbers \( \{n_\alpha\} \)

**Example:** two spin 1/2 particles of opposite spin:

- **bosons:**
  \[ |\psi\rangle_B \propto |+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle \]
- **fermions:**
  \[ |\psi\rangle_F \propto |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \]

\[ |+\rangle \otimes |-\rangle \quad \text{meaningless} \]
Thermodynamics of the ideal Bose gas:

\[
\text{Proba}(\{n_\alpha\}) = \frac{1}{\Xi} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu)n_\alpha}
\]

where \( \beta = 1/(k_B T) \) and \( \mu \) is the chemical potential.

Bose law for the occupation number:

\[
\langle n_\alpha \rangle = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}
\]

so that

\(-\infty < \mu < \epsilon_0.\)
Lower limit for $\mu$ is non-degenerate regime:

$$\langle n_{\mathcal{V}} \rangle \simeq \rho \lambda^3 e^{-\beta \hbar^2 k^2 / 2m}$$

in a large box, where

$$\lambda = \sqrt{\frac{2\pi \hbar^2}{mk_B T}}$$

is the thermal de Broglie wavelength.

The coherence length of the gas is $\sim \lambda$. 
WHEN THE BOSE GAS BECOMES DEGENERATE
\[ \rho \lambda^3 \gg 1 \]

Saturation of excited state population:

\[
N' \equiv \sum_{\alpha \neq 0} \langle n_\alpha \rangle < \sum_{\alpha \neq 0} \frac{1}{e^{\beta (\varepsilon_\alpha - \varepsilon_0)} - 1} \equiv N'_{\text{max}}
\]

For a large cubic box

\[
L \gg \lambda, \quad \text{i.e.} \quad k_B T \gg \frac{\hbar^2}{2 m L^2},
\]

\[
N'_{\text{max}} = \sum_{\tilde{k} \neq 0} \frac{1}{e^{\beta \hbar^2 k^2 / 2m} - 1} \simeq 2.612 \frac{L^3}{\lambda^3}
\]
If \( N > N'_{\text{max}} \) …

… there are at least \( N - N'_{\text{max}} \) atoms in the ground mode of the box.

A condensate forms if:

\[
\rho \lambda^3 > 2.612 \ldots \quad \text{Einstein, 1925}
\]

Totally counter-intuitive for Boltzmann statistics.

In a harmonic potential:

\[
N'_{\text{max}} \approx 1.202 \left( \frac{k_B T}{\hbar \bar{\omega}} \right)^3
\]

where \( \bar{\omega} \) is the geometric mean of the trap frequencies.
Even in a trap one has for \( N = N'_{\text{max}} \):

\[
\rho(\bar{0}) \lambda^3 \approx 2.612
\]

Below \( T_c \): condensate fraction

\[
\frac{N_0}{N} \approx \frac{N - N'_{\text{max}}}{N} \approx 1 - \left( \frac{T}{T_c} \right)^{3/2} \quad \text{box}
\]

\[
\approx 1 - \left( \frac{T}{T_c} \right)^3 \quad \text{harmonic trap}
\]

Realistic examples:

\[ T/T_c = 1/2 \quad \text{everyday} \]

\[ T/T_c = 1/4 \quad \text{the good days} \]
BEC in position space:  \( k_B T = 20\hbar \omega \quad N = 500 \text{ to } 32000 \)
Results of JILA:
HOW TO REACH BOSE-EINSTEIN CONDENSATION

The problem of solidification:

- For air with pressure 1 atm:
  \[ T_c \approx 0.4K \]
  but then one expects a solid phase.

- He\(^4\) does not solidify. Experiences superfluid transition at \( \sim 2K \)
  but is a liquid, not a gas (condensate fraction \(< 0.1\)).

- Only polarized hydrogen is gaseous at 1 atm, 0 K.
Low density route: use of metastability

- 2-body elastic collisions ensure thermalisation:
  \[ \gamma_{\text{elas}} \propto \rho \]

- 3-body collisions form molecules:
  \[ \gamma_{\text{inel}} \propto \rho^2 \]
  but are much slower at low density!

- the obtained condensate is metastable.

- Price to pay: ultralow temperatures
  \[ T_c \sim 40nK \text{ to } 1\mu K. \]
How to cool?

- laser cooling alone not yet succeeded:

\[ \lambda \sim \lambda_{\text{opt}} = \frac{2\pi}{k_L} \]

and bad effects of light when \( \rho \lambda_{\text{opt}}^3 \sim 1 \).

- forced evaporative cooling: remove atoms in high energy tails, let gas rethermalize, and so on

- efficient if

\[ \frac{\gamma_{\text{elas}}}{\gamma_{\text{loss}}} > 100. \]
EFFECT OF ATOMIC INTERACTIONS

- Not ideal Bose gas!
- Coherence length!!
How to characterize the interaction potential?

by its $r$ dependence:

$$\phi(r) = C_0 + C_1/r$$

Typical values

$$a = 50 \text{ nm (}^{87}\text{Rb}) \quad a = -1.5 \text{ nm (}^{7}\text{Li})$$

but $a$ can be tuned.

by its scattering length:

$$-\frac{\hbar^2}{m} \Delta \phi(r) = 0$$

$$\phi(r) \propto 1 - \frac{a}{r}$$
THE GROSS-PITAEVSKII EQUATION

\[ i\hbar \partial_t \phi(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \Delta + U(\vec{r}) + g N_0 |\phi(\vec{r}, t)|^2 - \mu \right] \phi(\vec{r}, t) \]

Comes from mean field for model interaction potential

\[ V(\vec{r}) = g \delta(\vec{r}) \partial_r (r \cdot) \]

with coupling constant \( g = \frac{4\pi \hbar^2}{m} a. \)

Explains almost everything, including superfluidity.