Finite size quantum physical systems

Atoms
Nuclei
Molecules

Quantum Dots

Cold gas in a trap?
Quantum Dot

1. Disorder (× – impurities)
2. Complex geometry
3. e–e interactions

Realizations:
- Metallic clusters
- Gate determined confinement in 2D gases (e.g. GaAs/AlGaAs)
- Carbon nanotubes
Quantum dots

Leo Kouwenhoven and Charles Marcus
Finite number $N$ of electrons:

$$\hat{H}\Psi_\alpha = E_\alpha \Psi_\alpha$$

No interactions between electrons $\rightarrow$ Shrodinger eqn in $d$ dimensions

In the presence of the interactions between electrons $\rightarrow$
Shrodinger equation in $dN$ dimensions
1. Disorder ($\times$ – impurities)
2. Complex geometry
3. $e-e$ interactions for a while

**Realizations:**
- Metallic clusters
- Gate determined confinement in 2D gases (e.g. $GaAs/AlGaAs$)
- Carbon nanotubes
- ...
I. Without interactions

Random Matrices, Anderson Localization, Quantum Chaos
1. Disorder (× – impurities)
2. Complex geometry

How to deal with disorder?

- Solve the Schrödinger equation exactly
- Start with plane waves, introduce the mean free path, and . . .

How to take quantum interference into account?
Integrable Systems

The variables can be separated and the problem reduces to $d$ one-dimensional problems

$d$ integrals of motion

Examples

1. A ball inside rectangular billiard; $d=2$
   - Vertical motion can be separated from the horizontal one
   - Vertical and horizontal components of the momentum, are both integrals of motion

2. Circular billiard; $d=2$
   - Radial motion can be separated from the angular one
   - Angular momentum and energy are the integrals of motion
**Classical Dynamical Systems with** \( d \) **degrees of freedom**

### Integrable Systems
- The variables can be separated \( \Rightarrow d \) one-dimensional problems \( \Rightarrow d \) integrals of motion
- Rectangular and circular billiard, Kepler problem, . . . , 1d Hubbard model and other exactly solvable models, . .

### Chaotic Systems
- The variables **can not** be separated \( \Rightarrow \) there is only one integral of motion - energy

#### Examples
- Sinai billiard
- Stadium
- Kepler problem in magnetic field
The variables *cannot* be separated → there is only one integral of motion - energy.

**Examples**

- Sinai billiard
- Stadium
- Kepler problem in magnetic field

Yakov Sinai  
Leonid Bunimovich  
Johnnes Kepler
Integrable $d$-dimensional systems

$d$ integrals of motion, $d$ quantum numbers

$\mathcal{I}_k \quad k = 1, 2, \ldots, d$

Chaotic $d$-dimensional systems

The only conserved quantity is the energy
Each eigenstate is characterized only by the eigenvalue of the Hamiltonian

Connection with the inverse problem:

Q: Why original conditions can not be used as the integrals of motion?

A: Not stable
Classical Chaos \( \hbar = 0 \)

- Nonlinearities
- Lyapunov exponents
- Exponential dependence on the original conditions
- Ergodicity

**Q:** What does it mean Quantum Chaos?

Quantum description of any System with a finite number of the degrees of freedom is a linear problem – Shrodinger equation
**RANDOM MATRICES**

$N \times N$ **ensemble of Hermitian matrices with random matrix element** $N \to \infty$

$E_\alpha$

$\nu(\epsilon) \equiv \left\langle \sum_\alpha \delta(\epsilon - E_\alpha) \right\rangle$

- spectrum (set of eigenvalues)
- density of states
- ensemble averaging

Gaussian ensembles (matrix elements are normally distributed)

Wigner Semicircle

$N \to \infty$
**RANDOM MATRICES**

\[ \mathbf{N} \times \mathbf{N} \]

**ensemble of Hermitian matrices**
**with random matrix element**

\[ N \rightarrow \infty \]

- **Spectral Rigidity**
- **Level repulsion**

\[ E_\alpha \]
\[ \delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle = \frac{1}{\nu} \]
\[ \langle \ldots \ldots \rangle \]
\[ S_\alpha \equiv \frac{E_{\alpha+1} - E_\alpha}{\delta_1} \]
\[ P(s) \]

- spectrum (set of eigenvalues)
- mean level spacing
- distribution function of spacings between the nearest neighbors

**Noncrossing rule**

\[ P(s = 0) = 0 \]
\[ P(s \ll 1) \propto s^\beta \quad \beta = 1, 2, 4 \]
Noncrossing rule (theorem)

Suggested by Hund (Hund F. 1927 Phys. v.40, p.742)


Usually textbooks present a simplified version of the justification due to Teller (Teller E., 1937 J. Phys. Chem 41 109).

Arnold V. I., 1972 Funct. Anal. Appl.v. 6, p.94

### RANDOM MATRICES

\[ N \times N \quad \text{ensemble of Hermitian matrices with random matrix element} \quad N \rightarrow \infty \]

**Dyson Ensembles**

<table>
<thead>
<tr>
<th>Matrix elements</th>
<th>Ensemble</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>real</td>
<td>orthogonal</td>
<td>1</td>
</tr>
<tr>
<td>complex</td>
<td>unitary</td>
<td>2</td>
</tr>
<tr>
<td>2 \times 2 matrices</td>
<td>simplectic</td>
<td>4</td>
</tr>
</tbody>
</table>
1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If $H_{12}$ is real (orthogonal ensemble), then for $s$ to be small two statistically independent variables ($H_{22} - H_{11}$ and $H_{12}$) should be small and thus $P(s) \propto s \beta = 1$
1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If $H_{12}$ is real (orthogonal ensemble), then for $s$ to be small two statistically independent variables ($H_{22} - H_{11}$ and $H_{12}$) should be small and thus $P(s) \propto s$ \quad \beta = 1$

3. Complex $H_{12}$ (unitary ensemble) \quad both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent \quad three independent random variables should be small \quad $P(s) \propto s^2$ \quad \beta = 2
Poisson – completely uncorrelated levels

Wigner-Dyson; GOE

Poisson

Gaussian Orthogonal Ensemble

Orthogonal $\beta=1$

Unitary $\beta=2$

Simplectic $\beta=4$

$P(s)$
No conservation laws $\Rightarrow$ no quantum numbers except the energy
$N \times N$ matrices with random matrix elements. $N \to \infty$

Spectral Rigidity
Level repulsion

$P(s << 1) \propto s^\beta \quad \beta = 1, 2, 4$

Dyson Ensembles

<table>
<thead>
<tr>
<th>Matrix elements</th>
<th>Ensemble</th>
<th>$\beta$</th>
<th>Realizations</th>
</tr>
</thead>
<tbody>
<tr>
<td>real</td>
<td>orthogonal</td>
<td>1</td>
<td>T-inv potential</td>
</tr>
<tr>
<td>complex</td>
<td>unitary</td>
<td>2</td>
<td>broken T-invariance (e.g., by magnetic field)</td>
</tr>
<tr>
<td>$2 \times 2$ matrices</td>
<td>simplectic</td>
<td>4</td>
<td>T-inv, but with spin-orbital coupling</td>
</tr>
</tbody>
</table>
Main goal is to classify the eigenstates in terms of the quantum numbers.

For the nuclear excitations this program does not work.

N. Bohr, Nature 137 (1936) 344.
Main goal is to classify the eigenstates in terms of the quantum numbers.

For the nuclear excitations this program does not work.

Study spectral statistics of a particular quantum system – a given nucleus.

Particular nucleus

$^{166}$Er

Spectra of several nuclei combined (after spacing) rescaling by the mean level

N. Bohr, Nature 137 (1936) 344.
Main goal is to classify the eigenstates in terms of the quantum numbers

For the nuclear excitations this program does not work

E.P. Wigner  

Study spectral statistics of a particular quantum system - a given nucleus

<table>
<thead>
<tr>
<th>Random Matrices</th>
<th>Atomic Nuclei</th>
</tr>
</thead>
<tbody>
<tr>
<td>• <em>Ensemble</em></td>
<td>• <em>Particular quantum system</em></td>
</tr>
<tr>
<td>• <em>Ensemble averaging</em></td>
<td>• <em>Spectral averaging (over $\alpha$)</em></td>
</tr>
</tbody>
</table>

Nevertheless

Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics
Why the random matrix theory (RMT) works so well for nuclear spectra
Why the random matrix theory (RMT) works so well for nuclear spectra

Original answer: These are systems with a large number of degrees of freedom, and therefore the “complexity” is high.

Later it became clear that there exist very “simple” systems with as many as 2 degrees of freedom (d=2), which demonstrate RMT-like spectral statistics.
Chaotic Systems

The variables cannot be separated $\Rightarrow$ there is only one integral of motion - energy

Examples

Sinai billiard

Stadium

Kepler problem in magnetic field

Yakov Sinai

Leonid Bunimovich

Johnnes Kepler
**Integrable d-dimensional systems**

$d$ integrals of motion, $d$ quantum numbers

$I_k \quad k = 1, 2, \ldots, d$

**Chaotic d-dimensional systems**

The only conserved quantity is the energy

Each eigenstate is characterized only by the eigenvalue of the Hamiltonian

**Connection with the inverse problem:**

**Q:** Why original conditions cannot be used as the integrals of motion?

**A:** Not stable
\[ \hbar \neq 0 \]

Bohigas – Giannoni – Schmit conjecture

Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai’s billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal-invariant systems whose classical analogs are \( K \) systems show the same fluctuation properties as predicted by GOE.
Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai’s billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal–invariant systems whose classical analogs are $K$ systems show the same fluctuation properties as predicted by GOE.
Bohigas – Giannoni – Schmit conjecture

Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai’s billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal-invariant systems whose classical analogs are $K$ systems show the same fluctuation properties as predicted by GOE.

\[ \hbar \neq 0 \]
**Q:** What does it mean Quantum Chaos?

*Two possible definitions*

- Chaotic classical analog
- Wigner - Dyson-like spectrum
Classical \hspace{20mm} Quantum

Integrable \hspace{1mm} ? \hspace{1mm} Poisson

Chaotic \hspace{1mm} ? \hspace{1mm} Wigner-Dyson

\[ P(s) \]

\[ 0 \leq s \leq 3 \]
Poisson to Wigner-Dyson crossover
Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential - disordered conductor

Scattering centers, e.g., impurities
Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential - disordered conductor

- Scattering centers, e.g., impurities
- As well as in the case of Random Matrices (RM) there is a luxury of ensemble averaging.
- The problem is much richer than RM theory
- There is still a lot of universality.

Anderson localization (1956)

At strong enough disorder all eigenstates are localized in space
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

\[ f = 3.04 \, \text{GHz} \quad f = 7.33 \, \text{GHz} \]

\begin{itemize}
  \item \textit{Anderson Insulator}
  \item \textit{Anderson Metal}
\end{itemize}
Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential - disordered conductor

🌟 Scattering centers, e.g., impurities

Models of disorder: Randomly located impurities

$$U(\vec{r}) = \sum_i u(\vec{r} - \vec{r}_i)$$
Poisson to Wigner-Dyson crossover

**Important example:** quantum particle subject to a random potential – disordered conductor

*Scattering centers, e.g., impurities*

**Models of disorder:**
- Randomly located impurities
- White noise potential
- **Anderson model** – tight-binding model with *on-site* disorder
- **Lifshits model** – tight-binding model with *off-diagonal* disorder

\[ U(\vec{r}) = \sum_i u(\vec{r} - \vec{r}_i) \]

\[ u(\vec{r}) \rightarrow \lambda \delta(\vec{r}) \quad \lambda \rightarrow 0 \quad c_{im} \rightarrow \infty \]
Anderson Model

- Lattice - tight binding model
- Onsite energies $\varepsilon_i$ - random
- Hopping matrix elements $I_{ij}$

$-W < \varepsilon_i < W$
uniformly distributed

$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$

Anderson Transition

$I < I_c$

Insulator
All eigenstates are localized
Localization length $\xi$

$I > I_c$

Metal
There appear states extended all over the whole system
Localization of single-electron wave-functions:

\[
\left[ -\frac{\nabla^2}{2m} + U(r) - \epsilon_F \right] \psi_\alpha(r) = \xi_\alpha \psi_\alpha(r)
\]

\(\psi_\alpha(x)\) extended
\(\zeta_{loc}\) localized
Localization of single-electron wave-functions:

\[
\begin{bmatrix}
\nabla^2 \\
\frac{-1}{2m} + U(r) - \epsilon_F
\end{bmatrix}
\psi_\alpha(r) = \xi_\alpha \psi_\alpha(r)
\]

- \(d=1\); All states are \textit{localized}
- \(d=2\); All states are \textit{localized}
- \(d>2\); \textit{Anderson transition}
Anderson Transition

$I < I_c$

**Insulator**

All eigenstates are localized

Localization length $\xi$

- The eigenstates, which are localized at different places will not repel each other

- Poisson spectral statistics

$I > I_c$

**Metal**

There appear states extended all over the whole system

- Any two extended eigenstates repel each other

- Wigner – Dyson spectral statistics
Exact diagonalization of the Anderson model

\[
\left[ -\frac{\nabla^2}{2m} + W U(\mathbf{r}) - \varepsilon_\alpha \right] \psi_\alpha(\mathbf{r}) = 0
\]
Q: What does it mean Quantum Chaos?

Two possible definitions

Chaotic classical analog

Wigner - Dyson-like spectrum

Are the two definitions equivalent?

Maybe not because of the localization!
Quantum particle in a random potential (Thouless, 1972)

Energy scales

1. Mean level spacing

\[ \delta_1 = \frac{1}{\nu} \times L^d \]

2. Thouless energy

\[ E_T = \frac{hD}{L^2} \]

\[ g = \frac{E_T}{\delta_1} \]

Thouless conductance

\[ g = \frac{Gh}{e^2} \]

\( E_T \) has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems).

\( D \) is the diffusion constant,

\( L \) is the system size,

\( d \) is the number of dimensions.
Thouless Conductance and One-particle Spectral Statistics

Localized states
Insulator
Poisson spectral statistics

Extended states
Metal
Wigner-Dyson spectral statistics

\( N \times N \) Random Matrices

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

\( N \to \infty \)

Quantum Dots with Thouless conductance \( g \)

\( g \to \infty \)
Scaling theory of Localization
(Abrahams, Anderson, Licciardello and Ramakrishnan 1979)

\[ g = \frac{E_T}{\delta_1} \]

Dimensionless Thouless conductance

\[ g = \frac{Gh}{e^2} \]

\[ L = 2L = 4L = 8L \ldots \]

without quantum corrections

\[ E_T \propto L^{-2} \quad \delta_1 \propto L^{-d} \]

\[ \frac{d(\log g)}{d(\log L)} = \beta(g) \]
\[ \log \beta(g) = \log \log L \]

### β - function

\[ \frac{d \log g}{d \log L} = \beta(g) \]

#### Unstable fixed point

Metal – insulator transition in 3D

All states are localized for \( d=1,2 \)
Conductance $g$
Anderson transition in terms of pure level statistics

![Graph showing the probability distribution function $P(s)$ with different curves for metal, critical, and insulator phases, along with a graph showing the scaling of level spacing variance with disorder $W$.](image)
Integrable

All chaotic systems resemble each other.

All integrable systems are integrable in their own way

Chaotic
Disordered Systems: 

\[ E_T > \delta_1; \quad g > 1 \]  
**Anderson metal; Wigner-Dyson spectral statistics**

\[ E_T < \delta_1; \quad g < 1 \]  
**Anderson insulator; Poisson spectral statistics**

**Q:** Is it a generic scenario for the Wigner-Dyson to Poisson crossover?

**Speculations**

Consider an **integrable** system. Each state is characterized by a set of **quantum numbers**.

It can be viewed as a point in the **space of quantum numbers**. The whole set of the states forms a **lattice** in this space.

A **perturbation** that violates the integrability provides matrix elements of the **hopping** between different sites (**Anderson model**)!
Consider an integrable system. Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?).

Weak enough hopping - Localization - Poisson
Strong hopping - transition to Wigner-Dyson
The very definition of the localization is not invariant – one should specify in which space the eigenstates are localized.

Level statistics is invariant:

- Poissonian statistics
  \[ \exists \text{ basis where the eigenfunctions are localized} \]

- Wigner-Dyson statistics
  \[ \forall \text{ basis the eigenfunctions are extended} \]
**Example 1**

**Doped semiconductor**

- Low concentration of donors
  - Electrons are localized on donors \( \Rightarrow \text{Poisson} \)

- Higher donor concentration
  - Electronic states are extended \( \Rightarrow \text{Wigner-Dyson} \)

---

**Example 2**

**Rectangular billiard**

- Two integrals of motion
  - \( p_x = \frac{\pi n}{L_x} \), \( p_y = \frac{\pi n}{L_x} \)

- Lattice in the momentum space
- Line (surface) of constant energy

- Ideal billiard
  - Localization in the momentum space \( \Rightarrow \text{Poisson} \)

- Deformation or smooth random potential
  - Delocalization in the momentum space \( \Rightarrow \text{Wigner-Dyson} \)
Diffusion and Localization in Chaotic Billiards

Fausto Borgonovi,1,3,4 Giulio Casati,2,3,5 and Baowen Li6,7

1Dipartimento di Matematica, Università Cattolica, via Trieste 17, 25121 Brescia, Italy
2Università di Milano, sede di Como, Via Luctia 3, Como, Italy
3Istituto Nazionale di Fisica della Materia, Unità di Milano, via Celoria 16, 20133 Milano, Italy
4Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Pavia, Italy
5Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Milano, Italy
6Department of Physics and Centre for Nonlinear and Complex Systems, Hong Kong Baptist University, Hong Kong
7Center for Applied Mathematics and Theoretical Physics, University of Maribor, Koper, 2000 Maribor, Slovenia

(Received 29 July 1996)

\[ \varepsilon \equiv \frac{a}{R} \]

\[ \varepsilon > 0 \] Chaotic stadium

\[ \varepsilon \rightarrow 0 \] Integrable circular billiard

Angular momentum is the integral of motion

\[ \hbar = 0; \quad \varepsilon << 1 \]

Diffusion in the angular momentum space

\[ D \propto \varepsilon^{5/2} \]

Localization and diffusion in the angular momentum space

Poisson

\[ \varepsilon = 0.01 \]

\[ g = 0.012 \]

Wigner-Dyson

\[ \varepsilon = 0.1 \]

\[ g = 4 \]
1D Hubbard Model on a periodic chain

\[ H = t \sum_{i,\sigma} \left( c_{i,\sigma}^+ c_{i+1,\sigma}^+ c_{i+1,\sigma} c_{i,\sigma} \right) + U \sum_{i,\sigma} n_{i,\sigma} n_{i,-\sigma} + V \sum_{i,\sigma,\sigma'} n_{i,\sigma} n_{i+1,\sigma'} \]

\( V = 0 \) \quad Hubbard model \quad \text{integrable}

\( V \neq 0 \) \quad extended Hubbard model \quad \text{nonintegrable}

12 sites
3 particles
Zero total spin
Total momentum \( \pi/6 \)

\( U=4 \quad V=0 \)

\( U=4 \quad V=4 \)
Finite number $N$ of electrons:

$$\hat{H}\Psi_\alpha = E_\alpha \Psi_\alpha$$

No interactions between electrons $\rightarrow$ Shrodinger eqn in $d$ dimensions

**Integrable** system – each energy is conserved
**Poissonian** many-body spectrum

In the presence of the interactions between electrons $\rightarrow$
Shrodinger eqn in $dN$ dimensions
Finite number $N$ of electrons:

$$\hat{H}\Psi_\alpha = E_\alpha \Psi_\alpha$$

No interactions between electrons $\Rightarrow$ Shroedinger eqn in $d$ dimensions

**Integrable** system – each energy is conserved

**Poissonian** many-body spectrum

In the presence of the interactions between electrons $\Rightarrow$ Shroedinger eqn in $dN$ dimensions

**Q:** Can interaction between the particles drive this system into chaos and make it **ergodic**?

**Random Matrics** statistics of nuclear spectra
II. With interactions

Fermi Liquid and Disorder
Zero Dimensional Fermi Liquid
Fermi Liquid

- Fermi statistics
- Low temperatures
- Not too strong interactions
- Translation invariance

What does it mean?
2. Substantial renormalizations. For example, in a Fermi gas:

\[ \partial n / \partial \mu, \quad \gamma = c / T, \quad \chi / g \mu_B \]

are all equal to the one-particle density of states. These quantities are different in a Fermi liquid.
1. **Resistivity is proportional to** $T^2$:


...The increase of the resistance caused by the interaction between the electrons is proportional to $T^2$ and at low temperatures exceeds the usual resistance, which is proportional to $T^5$.

... the sum of the moments of the interaction electrons can change by an integer number of the periods of the reciprocal lattice. Therefore the momentum increase caused by the electric field can be destroyed by the interaction between the electrons, not only by the thermal oscillations of the lattice.
1. Resistivity is proportional to $T^2$:


Umklapp electron – electron scattering dominates the charge transport (?!)

2. Jump in the momentum distribution function at $T=0$.

2a. Pole in the one-particle Green function

$$G(\varepsilon, \vec{p}) = \frac{Z}{i\varepsilon_n - \xi(\vec{p})}$$

Fermi liquid $= 0<Z<1$ (?!)

Signatures of the Fermi Liquid state ?!
Can Fermi - liquid survive without the momenta

Does it make sense to speak about the Fermi - liquid state in the presence of a quenched disorder
1. Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, $l$. The step in the momentum distribution function is broadened by this uncertainty.
Q: Does it make sense to speak about the Fermi–liquid state in the presence of a quenched disorder?

1. Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, $l$. The step in the momentum distribution function is broadened by this uncertainty.

2. Neither resistivity nor its temperature dependence is determined by the umklapp processes and thus does not behave as $T^2$.

3. Sometimes (e.g., for random quenched magnetic field) the disorder averaged one-particle Green function even without interactions does not have a pole as a function of the energy, $\varepsilon$. The residue, $Z$, makes no sense.

Nevertheless even in the presence of the disorder:

I. Excitations are similar to the excitations in a disordered Fermi-gas.

II. Small decay rate

III. Substantial renormalizations
Quantum Dot

1. Disorder (×impurities)
2. Complex geometry
3. e–e interactions

Realizations:
- Metallic clusters
- Gate determined confinement in 2D gases (e.g. GaAs/AlGaAs)
- Carbon nanotubes
At the same time, we want the typical energies, $\varepsilon$, to exceed the mean level spacing, $\delta_1$:

$$\delta_1 \ll \varepsilon \ll E_T$$

$$g \equiv \frac{E_T}{\delta_1} \gg 1$$
Set of one particle states. $\sigma$ and $\alpha$ label correspondingly spin and orbit.

\[ \hat{H}_0 = \sum_{\alpha} \varepsilon_{\alpha} a^{+}_{\alpha,\sigma} a_{\alpha,\sigma} \]
\[ \hat{H}_{\text{int}} = \sum_{\alpha,\beta,\gamma,\delta,\sigma,\sigma'} M_{\alpha\beta\gamma\delta} a^{+}_{\alpha,\sigma} a^{+}_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'} \]

$\varepsilon_{\alpha}$ - one-particle orbital energies

$M_{\alpha\beta\gamma\delta}$ - interaction matrix elements

**Nuclear Physics**
- $\varepsilon_{\alpha}$ are taken from the shell model
- $M_{\alpha\beta\gamma\delta}$ are assumed to be random

**Quantum Dots**
- $\varepsilon_{\alpha}$
- $M_{\alpha\beta\gamma\delta}$ RANDOM; Wigner-Dyson statistics
The same statistics of the random spectra and one-particle wave functions (eigenvectors) 

Thouless Conductance and One-particle Quantum Mechanics 

Localized states 
Insulator 
Poisson spectral statistics 

Extended states 
Metal 
Wigner-Dyson spectral statistics 

$N \times N$ Random Matrices 

$N \to \infty$ 

Quantum Dots with dimensionless conductance $g$ 

$g \to \infty$
Matrix Elements

\[ \hat{H}_{\text{int}} = \sum_{\alpha,\beta,\gamma,\delta} M_{\alpha\beta\gamma\delta} a_\alpha^+ a_\beta^+ a_\gamma a_\delta, \]

- **Diagonal** - \(\alpha,\beta,\gamma,\delta\) are equal pairwise
  - \(\alpha = \gamma\) and \(\beta = \delta\) or \(\alpha = \delta\) and \(\beta = \gamma\) or \(\alpha = \beta\) and \(\gamma = \delta\)

- **Offdiagonal** - otherwise

It turns out that in the limit \(g \to \infty\)

- **Diagonal matrix elements are much bigger than the offdiagonal ones**
  \[ M_{\text{diagonal}} >> M_{\text{offdiagonal}} \]

- **Diagonal matrix elements in a particular sample do not fluctuate** - selfaveraging
$\Psi_\alpha (x)$ is a random function that rapidly oscillates as long as $T$-invariance is preserved.

$|\psi_\alpha (x)|^2 \geq 0$

$\psi_\alpha (x)^2 \geq 0$

$M_{\alpha\beta\gamma\delta} = \frac{\lambda}{\nu} \int d\vec{r} \psi_\alpha (\vec{r}) \psi_\beta^*(\vec{r}) \psi_\gamma (\vec{r}) \psi_\delta^*(\vec{r})$

$U(\vec{r}) = \frac{\lambda}{\nu} \delta(\vec{r})$

$\lambda$ is dimensionless coupling constant

$\nu$ is the electron density of states

Toy model: Short range $e-e$ interactions

$\psi_\alpha (\vec{r})$ is one-particle eigenfunctions

$\Psi_\alpha (x)$ is a random function that rapidly oscillates
In the limit \( g \to \infty \)

- **Diagonal matrix elements are much bigger than the offdiagonal ones**
  
  \[ M_{\text{diagonal}} \gg M_{\text{offdiagonal}} \]

- **Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging**

\[
M_{\alpha\beta\alpha\beta} = \frac{\lambda}{V} \int d\vec{r} |\psi_\alpha(\vec{r})|^2 |\psi_\beta(\vec{r})|^2
\]

\[
|\psi_\alpha(\vec{r})|^2 \Rightarrow \frac{1}{\text{volume}}
\]

**More general:** finite range interaction potential \( U(\vec{r}) \)

\[
M_{\alpha\beta\alpha\beta} = \frac{\lambda}{V} \int |\psi_\alpha(\vec{r}_1)|^2 |\psi_\beta(\vec{r}_2)|^2 U(\vec{r}_1 - \vec{r}_2) d\vec{r}_1 d\vec{r}_2
\]

The same conclusion
Universal (Random Matrix) limit - Random Matrix symmetry of the correlation functions:

All correlation functions are invariant under arbitrary orthogonal transformation:

\[
\tilde{\psi}_\mu (\vec{r}) = \sum_\nu \int d\vec{r}_1 O^\nu_\mu (\vec{r}, \vec{r}_1) \psi_\nu (\vec{r}_1)
\]

\[
\int d\vec{r}_1 O^\nu_\mu (\vec{r}, \vec{r}_1) O^\eta_\nu (\vec{r}_1, \vec{r}') = \delta_{\mu \eta} \delta (\vec{r} - \vec{r}')
\]
There are only three operators, which are quadratic in the fermion operators $a^+, a$, and invariant under RM transformations:

\[
\hat{n} = \sum_{\alpha, \sigma} a^+_{\alpha, \sigma} a_{\alpha, \sigma}
\]

\[
\hat{S} = \sum_{\alpha, \sigma_1, \sigma_2} a^+_{\alpha, \sigma_1} \tilde{\sigma}_{\sigma_1, \sigma_2} a_{\alpha, \sigma_2}
\]

\[
\hat{K}^+ = \sum_{\alpha} a^+_{\alpha, \uparrow} a^+_{\alpha, \downarrow}
\]

**total number of particles**

**total spin**

????
Charge conservation (gauge invariance)
- no $\hat{K}$ or $\hat{K}^+$ only $\hat{K} \hat{K}^+$

Invariance under rotations in spin space
- no $\hat{S}$ only $\hat{S}^2$

Therefore, in a very general case

$$\hat{H}_{int} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{K}^+\hat{K}.$$  

Only three coupling constants describe all of the effects of e-e interactions
In a very general case only three coupling constants describe all effects of electron-electron interactions:

\[
\hat{H} = \sum_\alpha \varepsilon_\alpha n_\alpha + \hat{H}_{\text{int}}
\]

\[
\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + JS^2 + \lambda_{\text{BCS}}\hat{K}^+\hat{K}.
\]

I.L. Kurland, I.L. Aleiner & B.A., 2000

See also

H. Baranger & L.I. Glazman, 1999
In a very general case only three coupling constants describe all effects of electron-electron interactions:

\[
\hat{H} = \sum_{\alpha} \varepsilon_\alpha n_\alpha + \hat{H}_{int}
\]

\[
\hat{H}_{int} = eV\hat{n} + E_c \hat{n}^2 + JS^2 + \lambda_{BCS} \hat{K}^+ \hat{K}.
\]

For a short range interaction with a coupling constant \(\lambda\)

\[
E_c = \frac{\lambda \delta_1}{2} \quad J = -2\lambda \delta_1 \quad \lambda_{BCS} = \lambda \delta_1 (2 - \beta)
\]

where \(\delta_1\) is the one-particle mean level spacing.
\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]

\[ \hat{H}_0 = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} \]

\[ \hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + JS^2 + \lambda_{\text{BCS}} \hat{K}^+ \hat{K}. \]

Only one-particle part of the Hamiltonian, \( \hat{H}_0 \), contains randomness!
\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]

\[ \hat{H}_0 = \sum_\alpha \varepsilon_\alpha n_\alpha \]

\[ \hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{\text{BCS}} \hat{K}^+ \hat{K}. \]

- \( E_c \): determines the charging energy (Coulomb blockade)
- \( J \): describes the spin exchange interaction
- \( \lambda_{\text{BCS}} \): determines effect of superconducting-like pairing
\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]
\[ \hat{H}_0 = \sum_\alpha \varepsilon_\alpha n_\alpha \]
\[ \hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{\text{BCS}}\hat{K}^+\hat{K}. \]

I. Excitations are similar to the excitations in a disordered Fermi-gas.

II. Small decay rate

III. Substantial renormalizations

Isn’t it a Fermi liquid?

Fermi liquid behavior follows from the fact that different wave functions are almost uncorrelated.