Superfluidity versus Bose-Einstein condensation

C. Trefzger, M2 – ICFP

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Consider a dilute gas of bosonic particles of mass m confined in a box of size $[0, L]^d$ where d is the dimension of the space. We will impose periodic boundary conditions on the wavefunction of the system. The Hamiltonian of the gas is then written as follows

$$H_0 = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{\text{int}}(\mathbf{r}_i - \mathbf{r}_j), \tag{1}$$

where $\mathbf{p}_i$ is the momentum operator of particle i and $\mathbf{r}_i$ its position operator. In dimension $d = 3$, the particles interact through a contact potential $V_{\text{int}}(\mathbf{r}_1 - \mathbf{r}_2) = g\,\delta(\mathbf{r}_1 - \mathbf{r}_2)$ where $g = 4\pi\hbar^2 a/m$. Furthermore, at $T = 0$ we suppose for the moment that all the atoms of the gas form a Bose-Einstein condensate (BEC).

1. **Landau's criterion for superfluidity**. Consider an impurity of mass M that moves through the atomic gas with a velocity $\mathbf{v}$. The atomic gas is initially at $T = 0$ and the interaction between the impurity and the atoms of the gas are supposed to be weak.

   (a) Consider an elementary process where the impurity, of mass M, and initial velocity equal to $\vec{v}$, creates an excitation in the gas of energy $\hbar\omega$ and momentum $\hbar\vec{k}$. Use energy conservation and momentum conservation to show that one cannot create such an excitation if the velocity of the impurity is lower than ω/k.

   (b) Derive the excitation spectrum in the Bogoliubov approximation.

   (c) Using the Bogoliubov spectrum determine the minimal velocity $|\vec{v}|$ (Landau's critical velocity) at which Bogoliubov excitations can be created by the impurity and therefore the impurity can be slowed down by the gas. This absence of "slowing down" for small velocities is a manifestation of the superfluidity of the gas (see e.g. A. P. Chikkatur, A. Görlitz, D. M. Stamper-Kurn, S. Inouye, S. Gupta, and W. Ketterle, Phys. Rev. Lett. **85**, 483 (2000)).

2. **Another point of view on the Landau's criterion : Thermodynamical instability**

Stiring potential

We apply now a perturbing potential that breaks the translationary invariance of the gas along the x-axis, a potential that moves at a constant velocity $\mathbf{v} = v\,\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector oriented in the x-axis :

$$W(t) = \sum_{i=1}^{N} \mathcal{W}(\mathbf{r}_i - \mathbf{v}t). \tag{2}$$

(Note that the velocity $\mathbf{v}$ in this context is not an operator but a vector with real components.).

a) Write the Schrödinger equation on the state vector of the system $|\psi(t)\rangle$ in the presence of the perturbation, in terms of the operators H_0 and $W(t)$.

b) We introduce the time-dependent unitary transformation

$$U(t) = e^{i\mathbf{P}\cdot\mathbf{v}t/\hbar}, \tag{3}$$

where we have introduced the total momentum operator of the gas :

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i. \tag{4}$$

After this unitary transformation, the state vector of the system is

$$|\tilde{\psi}(t)\rangle \equiv U(t)|\psi(t)\rangle. \tag{5}$$

Show that the position operator of the particle j under the unitary transformation transforms as follows :

$$U(t)\mathbf{r}_j U^\dagger(t) = \mathbf{r}_j + \mathbf{v}t. \tag{6}$$

c) Write the Schrödinger equation satisfied by $|\tilde{\psi}\rangle$ and deduce the new Hamiltonian $\tilde{H}$.

d) From an important property of $\tilde{H}$, what is the aim of having done such a unitary transformation ?

e) We now make the limit $\mathcal{W} \to 0$. Write the operator $\tilde{H}$ in this limit.

Bogoliubov and thermodynamical instability In this section we suppose that the gas interacts weakly enough and that it is at a low enough temperature such that we can use Bogoliubov theory to describe it. We therefore suppose that a condensate is present in the mode of wavevector $\mathbf{k}_0 = \mathbf{0}$ of the box.

a) In second quantization write the total momentum operator along x, P_x, in the space of wavevectors therefore in terms of the operators $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$. The operator $a_{\mathbf{k}}$ annihilates a particle in the plane wave of wavevector $\mathbf{k}$.

b) We recall the decomposition of the field Λ on the Bogoliubov modes :

$$\Lambda(\mathbf{r}) = e^{-i\theta}\psi_\perp(\mathbf{r}) = \sum_{\mathbf{k}\neq\mathbf{0}} u_{\mathbf{k}}(\mathbf{r})b_{\mathbf{k}} + v^*_{\mathbf{k}}(\mathbf{r})b^\dagger_{\mathbf{k}}, \tag{7}$$

where the modes $(u_{\mathbf{k}}, v_{\mathbf{k}})$ belong to the family $\mathcal{F}_+$ seen in the theory lectures. Express $e^{-i\theta}a_{\mathbf{k}}$, for $\mathbf{k} \neq \mathbf{0}$, as a function of the operators b and $b^\dagger$.

c) Show that the quantity $k_x b_{\mathbf{k}} b_{-\mathbf{k}}$ is an odd function of $\mathbf{k}$. Deduce that the sum over $\mathbf{k} \neq \mathbf{0}$ of this quantity vanishes.

d) Show that the total momentum operator of the gas along x is simply given by

$$P_x = \sum_{\mathbf{k}\neq\mathbf{0}} \hbar k_x b^\dagger_{\mathbf{k}} b_{\mathbf{k}}. \tag{8}$$

e) Show that there is thermodynamical instability for $v > c_s$ where c_s is the sound velocity, that is the speed at which sound propagates into the gas.

3. **Quantum depletion** Due to interactions the ground state of the system is not a pure BEC

   (a) Calculate the modes v_k et u_k of the Bogoliubov approach.

   (b) In the Bogoliubov approximation determine the non-condensed fraction for a gas at zero temperature $T = 0$. Deduce the condition of validity for the Bogoliubov approach to be valid.

   We give the following integral :

$$\int_0^\infty dq\ q^2 \left(\frac{q^2 + 1/2}{q\sqrt{q^2+1}} - 1 \right) = \frac{1}{6}.$$