Energy minimization in the BCS state and excitation spectrum (II)

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1 BCS Hamiltonian and excitation spectrum

To describe fermionic superfluidity one often uses the so-called BCS Hamiltonian, which is a quadratic Hamiltonian in the field operator. In order to find it we decompose

\[
\hat{\psi}_\uparrow \hat{\psi}_\downarrow = \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle + (\hat{\psi}_\uparrow \hat{\psi}_\downarrow - \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle), \tag{1}
\]

and we neglect the terms which in the interaction energy are quadratic in the “fluctuations” \((\hat{\psi}_\uparrow \hat{\psi}_\downarrow - \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle)\).

a) Determine the BCS Hamiltonian starting from the model Hamiltonian of question 1 a) of the previous lecture, where we will replace \(\mu \text{ par } \overline{\mu}\).

b) In the Heisenberg point of view the equations for the time-evolution of the field operators are liner:

\[
\begin{pmatrix}
\hat{\psi}_\uparrow(x) \\
\hat{\psi}_\downarrow(x) \\
\hat{\psi}_\uparrow(x) \\
\hat{\psi}_\downarrow(x)
\end{pmatrix}
\begin{pmatrix}
\dot{\hat{\psi}}_\uparrow(x) \\
\dot{\hat{\psi}}_\downarrow(x) \\
\dot{\hat{\psi}}_\uparrow(x) \\
\dot{\hat{\psi}}_\downarrow(x)
\end{pmatrix}
= \mathcal{L}
\begin{pmatrix}
\hat{\psi}_\uparrow(x) \\
\hat{\psi}_\downarrow(x) \\
\hat{\psi}_\uparrow(x) \\
\hat{\psi}_\downarrow(x)
\end{pmatrix}; \tag{2}
\]

Calculate the operator \(\mathcal{L}\). Is it Hermitian? We will use a \(2 \times 2\) block notation where we introduce the spin base \(| \downarrow \rangle, | \uparrow \rangle\).

c) Show that

\[
x \rightarrow \begin{pmatrix}
\tilde{u}_k(x) \\
0 \\
0 \\
\tilde{v}_k(x)
\end{pmatrix}
= \begin{pmatrix}
U_k \\
0 \\
0 \\
V_k
\end{pmatrix} e^{ikx} / \sqrt{\mathcal{L}}, \tag{3}
\]

with \(U_k\) and \(V_k\) reals, are eigenvectors of \(\mathcal{L}\) and calculate the corresponding eigenvalues \(\lambda_k\). Show that the constants \(U_k\) and \(V_k\) are related to the \(u_k\) et \(v_k\) of the BCS ansatz. In particular we get: \(U_k = u_k = 1 / \sqrt{1 + \Gamma_k^2}\) and \(V_k = -v_k = -\Gamma_k / \sqrt{1 + \Gamma_k^2}\).
Show that
\[
x \to \begin{pmatrix} 0 & \tilde{u}_k(x) \\ -\tilde{v}_k(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & U_k \\ -V_k & 0 \end{pmatrix} \frac{e^{ikx}}{\sqrt{L}} \tag{4}
\]
are also eigenvectors of $\mathcal{L}$.

d) Show that we can write
\[
\hat{\psi}_\uparrow(x) = \sum_k \tilde{u}_k(x) b_{k\uparrow} - \tilde{v}_k(x) b_{k\downarrow}^\dagger \tag{5}
\]
\[
\hat{\psi}_\downarrow(x) = \sum_k \tilde{u}_k(x) b_{k\downarrow} + \tilde{v}_k(x) b_{k\uparrow}^\dagger, \tag{6}
\]
and that
\[
Ua_{k\uparrow}U^\dagger = b_{k\uparrow}. \tag{7}
\]
e) Show that $|\psi_{BCS}\rangle$ is the vacuum of the operators $b_{k\sigma}$.

f) Put the BCS Hamiltonian in the canonical form as a function of the $b_{k\sigma}$ and the $b_{k\sigma}^\dagger$.

g) For $g = 0$ draw the spectrum $k \to \lambda_k$ and give a physical interpretation in terms of creation of holes and particles.

h) Now we consider weak interactions such that $\Delta \ll \epsilon_F$. Draw $\lambda_k$ in this case and show that there is a gap of size $\Delta$ that appears in the spectrum of excitation.