Columbia University

MATHUN2010
Linear Algebra
Spring 2017

Midterm I

Instructor: Guillaume Barraquand

Time: February 15, 2017. 10:10am – 11:25am

Your name: ______________________________

UNI: ____________________

Exercise:	1	2	3	4	5	Total
Points:	10	6	18	10	6	50
Score:						

Instructions:

- **Please write your UNI on every page.**
- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are **not** allowed.

Exercise 1 . . . . 10 points

Determine whether the following statements are true (**T**) or false (**F**). **You do not need to justify your answer for this question.**

(a) ___ For two invertible matrices A and B of size $n \times n$, we have

$$(ABA^{-1})^3 = AB^3A^{-1}.$$

(b) ___ For two matrices A and B of size $n \times n$, we have

$$(A+B)^2 = A^2 + 2AB + B^2.$$

(c) ___ If $A^2 = I_2$, then A must be either I_2 or $-I_2$.

(d) ___

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} = 3$$

(e) ___ There exists a 3×4 matrix with rank 4.

Exercise 2 . . . . 6 points

Find the inverse of

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Exercise 3 . . . .

Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this exercise, we will show that for a nonnegative integer n,

$$A^n = \begin{pmatrix} 1 & n & u_n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix},$$

where u_n is a certain sequence to be determined. Let B be the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) (2 points) Find $\ker(B)$.

(b) (1 point) Is B invertible? ◯ Yes ◯ No

(c) (2 points) Compute B^2.

(d) (2 points) Compute B^3.

(e) (2 points) Show that for all $n \geqslant 4$, $B^n = 0$.

(f) (2 points) Using the fact that $A = I_3 + B$, Compute A^n for a positive integer n.

Hint: You may use without justification that for two $n \times n$ square matrices M and N such that $MN = NM$,

$$(M+N)^n = \sum_{i=0}^{n} \binom{n}{i} M^i N^{n-i},$$

with the understanding that $M^0 = N^0 = I_n$.

(g) (2 points) To help you, we suggest possible choices for the value of u_n (tick the correct one): ○ $\dfrac{n(n^2+3n+5)}{6}$ ○ n ○ $\dfrac{n(n-1)}{2}$ ○ $\dfrac{n(n+1)}{2}$

○ $\dfrac{6-8n+12n^2-n^3}{6}$ ○ $\dfrac{n^3+14n-6}{6}$ ○ n^2-n+1

(h) (3 points) Find the inverse of A.

(i) (2 points) Is the formula for A^n also valid when n is negative? ◯ Yes ◯ No

(j) (Bonus) Find a general formula for

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 4 . . . .

For $\theta \in \mathbb{R}$, let

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) (4 points) For $\alpha, \beta \in \mathbb{R}$, compute and simplify the product $R_\alpha R_\beta$.

Hint: You can do the computation using the usual product rule and simplify. Otherwise you can use block matrices and guess the product using an appropriate result from the course.

(b) (2 points) Let P be the subspace

$$P = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Interpret geometrically the linear transformation of P defined by

$$\overrightarrow{x} \mapsto R_\theta \overrightarrow{x}$$

for $\overrightarrow{x} \in P$.

(c) (1 point) For any $z \in \mathbb{R}$, compute

$$R_\theta \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$

(d) (3 points) Interpret geometrically the linear transformation of $\mathbb{R}^3$ defined by

$$\overrightarrow{x} \mapsto R_\theta \overrightarrow{x}.$$

Hint: any $\overrightarrow{x} \in \mathbb{R}^3$ *can be decomposed as*

$$\overrightarrow{x} = \overrightarrow{p} + \overrightarrow{p}^{\perp},$$

where $\overrightarrow{p} \in P$ *and* $\overrightarrow{p}^{\perp}$ *is of the form* $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ *for a certain* z.

Exercise 5 . . . . 6 points

Can one find a matrix of size 10×10 with 92 ones among its entries that is invertible? If your answer is yes, give an example. If your answer is no, explain why and give an example of such a matrix with rank 9.

Extra space.