Macdonald processes and KPZ equation in a half-space

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Consider the (multiplicative noise) stochastic heat equation,

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + Z \xi,$$

where $Z(t,x)$, $x \in \mathbb{R}, t > 0$,

where $\xi$ is a Gaussian space time white noise. In one spatial dimension, one can make sense of it in an integrated sense.

$h := \log(Z)$ is the solution to the KPZ equation

$$\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + \xi.$$

**Question:** What is the probability distribution of the solution?

**The answer depends on**

1. The initial condition. In this talk we restrict to delta initial data.
2. The space $\mathbb{X}$. It may be the whole line $\mathbb{R}$, the torus $\mathbb{R}/\mathbb{Z}$ or an interval with Neumann/Dirichlet/other boundary conditions.
A beautiful answer when $\mathbb{X} = \mathbb{R}$

Consider the solution to the multiplicative SHE on the whole line $\mathbb{R}$,

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + Z \xi,$$

where $x \in \mathbb{R}, t > 0$,

with delta initial condition $Z(0, \cdot) = \delta_0$.

Theorem (Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn 2010)

For $u \in \mathbb{C}$ with $\text{Re}(u) > 0$,

$$\mathbb{E} \left[ e^{-u Z(t,0)} e^{t/24} \right] = \mathbb{E} \left[ \prod_{i=1}^{+\infty} \frac{1}{1 + u \exp ((t/2)^{1/3} \alpha_i^{\text{GUE}})} \right].$$

where $\{\alpha_i^{\text{GUE}}\}_{i \geq 1}$ are the limiting eigenvalues of the GUE scaled at the edge (Airy point process).

- The RHS is explicit and can be computed numerically. One can deduce large time limits, large deviations estimates.
Analogue when \( X = \mathbb{R}_+ \) ?

Consider now the solution \( Z(\tau, x) \) to the multiplicative SHE in a half-space,
\[
\partial_t Z = \frac{1}{2} \partial_{xx} Z + Z \xi, \quad \text{where } x \in \mathbb{R}_{\geq 0}, t > 0,
\]
with delta initial condition \( Z(0, \cdot) = \delta_0 \) for some boundary condition at \( x = 0 \) (Neumann, Dirichlet, mixed...).
What is the law of the solution? Can one find a function \( f \) and a matrix ensemble such that
\[
\mathbb{E} \left[ e^{-uZ(t, x)} \right] = \mathbb{E} \left[ \prod_{i=1}^{+\infty} f \left( u e^{(t/2)^{1/3}a_i} \right) \right] ?
\]

Based on results on symmetrized last passage percolation [Baik-Rains 2001], we expect a transition depending on boundary parameters between Tracy-Widom and Gaussian type statistics.
Moments

Consider mixed moments of $\mathbb{E}[Z(t,x_1)\ldots Z(t,x_k)]$. These are solutions of the delta Bose gas, which can be solved by Bethe ansatz. However, the moments grow too fast to characterize the distribution.

There are two rigorous approaches.

**KPZ equation through ASEP**

The height function of the asymmetric simple exclusion process (ASEP) converges to the KPZ equation [Bertini-Giacomin 1997] under a certain weak asymmetry scaling.

**KPZ equation through discrete directed polymers**

The free energy of directed polymer in $\mathbb{Z}^2$ at high temperature converges to the KPZ equation [Alberts-Khanin-Quastel 2010].

Both discretizations are marginals or limits of more general measures on partitions called (half-space) Macdonald processes.
Plan of the talk

1. Last passage percolation in a half-quadrant: the Baik-Rains transition
2. Definition of models and results
   - Limit theorems for Half-line ASEP
   - Law of the KPZ equation
   - Formulas for the inverse gamma polymer partition function in a half-quadrant
3. Half-space Macdonald measures
   - Definition
   - Hall-Littlewood measures and the stochastic six vertex model
   - Markov Dynamics and half-space Macdonald processes
   - Computing observables
(half-space) Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

- Hall-Littlewood process
- Determinantal/Pfaffian Schur process
- Stochastic six-vertex model
- Geometric last passage percolation (LPP)
- Exponential Last Passage Percolation TASEP
- KPZ equation on $\mathbb{R}$ or $\mathbb{R}_{>0}$

- $q = 0$
- $q = t$
- $t = 0$
- $q \to 1$

- special case
- continuous time
- weak asymmetry
- inverse-gamma polymer
- zero-temperature
- intermediate disorder
Last Passage Percolation in a half quadrant

Let $w_{ij}$ a family of i.i.d. exponential random variables with rate 1 when $i > j$ and $\alpha$ when $i = j$.

Consider directed paths $\pi$ from the box $(1,1)$ to $(n,m)$ in the half quadrant. We define the last passage percolation time $H(n,m)$ by

$$H(n,m) = \max_{\pi} \sum_{(i,j) \in \pi} w_{ij}.$$
Baik-Rains transition

Last passage percolation can be studied within the framework of determinantal/Pfaffian point processes.

Theorem (Baik-Rains 2001, Baik-B.-Corwin-Suidan 2016)

- When $\alpha > 1/2$,
  \[ \frac{H(n,n) - 4n}{2^{4/3}n^{1/3}} \Rightarrow \mathcal{L}_{\text{GSE}}, \]

- When $\alpha = 1/2$,
  \[ \frac{H(n,n) - 4n}{2^{4/3}n^{1/3}} \Rightarrow \mathcal{L}_{\text{GOE}}, \]

- When $\alpha < 1/2$,
  \[ \frac{H(n,n) - cn}{c'n^{1/2}} \Rightarrow \mathcal{N}, \]

- Far from the diagonal, we obtain a transition between Tracy-Widom GUE and Gaussian statistics.

- Scaling $\alpha$ close to $1/2$ and $(n,m)$ close to the boundary, we obtain crossover ensemble related to RMT.

- Multipoint correlations (along space-like paths) can also be characterized.
Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

- Hall-Littlewood process
- Determinantal/Pfaffian Schur process
- Geometric last passage percolation (LPP)
- Exponential Last Passage Percolation TASEP
- ASEP
- KPZ equation on $\mathbb{R}$ or $\mathbb{R}_{>0}$

- $q = 0$
- $q = t$
- $q \rightarrow 1$
- $t = 0$
- $q \rightarrow 0$
- $t = 0$

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- $q = 0$
- $t = 0$
- $q \rightarrow 1$
Half-line ASEP

Let $R > L \geq 0$, and consider the asymmetric simple exclusion process (ASEP) on the positive integers with open boundary condition:

One can characterize the system by the function

$$N_x(\tau) = \# \{ \text{particles on the right of site } x \text{ at time } \tau \}.$$

Under weakly asymmetric scaling, [Corwin-Shen 2016, Parekh 2017](see Hao’s talk) showed that $N_x(t)$ converges to the KPZ equation on the positive reals with Neumann boundary condition,

$$\begin{cases}
\partial_\tau h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + \xi \\
\partial_x h(\tau, x) \bigg|_{x=0} = a \in \mathbb{R}.
\end{cases}$$

This is equivalent to saying that $h = \log Z$ where $Z$ solves the mSHE with mixed Robin boundary condition $Z$, $\partial_x Z(\tau, x) \bigg|_{x=0} = a \ Z(\tau, 0)$. 
Previous results on half-line ASEP

[prev img]

- [Liggett 1975] classified the stationary measures when

\[ \frac{\alpha}{R} + \frac{\gamma}{L} = 1. \]

Then \( \rho = \alpha/R \) is the average density enforced at the boundary. There is a transition at \( \rho = 1/2 \) between product-form Bernoulli measure and spatially correlated stationary measures (which can be expressed using Matrix Product Ansatz).

- [Tracy-Widom 2013] used coordinate Bethe ansatz to find formulas for the transition probabilities, but these do not seem amenable for asymptotic analysis.

- We analyze half-line ASEP through a half space version of the **stochastic six-vertex model**. This approach was first developed in the full-space [Borodin-Corwin-Gorin 2014, Aggarwal-Borodin 2016, Aggarwal 2016, Borodin-Olshanski 2016]
Recall results for TASEP

When $\gamma = L = 0$ (no left jumps), the model is equivalent to last passage percolation. Without loss of generality, we may set $R = 1$ and recall

$$N_x(t) = \# \{ \text{particles on the right of } x \text{ at time } t \}.$$

**Theorem (Baik-Rains 2001, Baik-B.-Corwin-Suidan 2016)**

- When $\alpha > 1/2$,
  $$\frac{N_0(t) - \frac{t}{4}}{2^{-4/3}t^{1/3}} \xrightarrow{t \to \infty} -\mathcal{L}_{\text{GSE}}.$$

- When $\alpha = 1/2$,
  $$\frac{N_0(t) - \frac{t}{4}}{2^{-4/3}t^{1/3}} \xrightarrow{t \to \infty} -\mathcal{L}_{\text{GOE}}.$$

- When $\alpha < 1/2$,
  $$\frac{N_0(t) - t\alpha(1-\alpha)}{\sigma t^{1/2}} \xrightarrow{t \to \infty} \mathcal{N}(0,1).$$

$\mathcal{L}_{\text{GSE}}/\mathcal{L}_{\text{GOE}}$ is the Tracy Widom GSE/GOE distribution (Gaussian symplectic/orthogonal ensemble).
Laplace transform of ASEP current

We assume

1. Ligget’s condition $\frac{\alpha}{R} + \frac{\gamma}{L} = 1$.
2. The boundary enforces a density of particles $\alpha/R = 1/2$ at the origin.

Theorem (B.-Borodin-Corwin-Wheeler 2017)

Let $t = L/R \in (0, 1)$. For any time $\tau > 0$ and $u < 0$,

$$
\mathbb{E} \left[ \frac{1}{(ut^{N_0(\tau)}, t^2)_{\infty}} \right] = \text{Pf}[J + f_u \cdot K]_{\ell^2(Z_{\geq 0})} := 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{x_1, \ldots, x_k \in Z_{\geq 0}} f_u(x_1) \ldots f_u(x_k) \text{Pf}[K(x_i, x_j)]_{i, j=1}^k
$$

where $K$ is a certain kernel (a variant of the Pfaffian Schur process kernel) and

$$f_u(x) = \frac{(ut^{x+1}; t^2)_{\infty}}{(ut^x; t^2)_{\infty}} - 1.$$
Asymptotic analysis of the Fredholm Pfaffian yields:

**Theorem (B.-Borodin-Corwin-Wheeler 2017)**

We assume

1. **Ligget’s condition** $\frac{\alpha}{R} + \frac{\gamma}{L} = 1$.
2. The boundary enforces a density of particles $\frac{\alpha}{R} = \frac{1}{2}$ at the origin.

Then

$$
\frac{N_0 \left( \frac{\tau}{R-L} \right) - \frac{\tau}{4}}{2^{-4/3} \tau^{1/3}} \xrightarrow{\tau \to \infty} \mathcal{L}_{\text{GOE}}.
$$

For other values of $\alpha$, or the current away from the boundary, we expect the same results as for TASEP (but cannot prove it yet).
Law of KPZ equation in half-space

Consider the KPZ equation on $\mathbb{R}_+$ with Neumann boundary condition,

$$\begin{cases}
\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + \xi \\
\partial_x h(t,x) \big|_{x=0} = a \in \mathbb{R}.
\end{cases}$$

Using the convergence of half-line ASEP height function to the KPZ equation [Corwin-Shen 2016, Parekh 2017] we obtain:

**Theorem (B.-Borodin-Corwin-Wheeler 2017)**

Assume the boundary parameter $a = -1/2$. For $u \in \mathbb{C}$ with $\Re(u) > 0$,

$$\mathbb{E} \left[ e^{-uZ(t,0)} e^{t/24} \right] = \mathbb{E} \left[ \prod_{i=1}^{+\infty} \sqrt{\frac{1}{1 + 4u \exp \left( (t/2)^{1/3} a_i^{\text{GOE}} \right)}} \right],$$

where $\{a_i^{\text{GOE}}\}_{i=1}^{\infty}$ forms the GOE point process (i.e. the sequence of rescaled eigenvalues of a large Gaussian real symmetric matrix).

See also recent results by [Krajenbrink-Le Doussal 2018, Gueudre-Le Doussal 2012] in the case $a = +\infty$. 
Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

- Hall-Littlewood process
- Stochastic six-vertex model
- Determinantal/Pfaffian Schur process
- Geometric last passage percolation (LPP)
- Exponential Last Passage Percolation TASEP
- Whittaker process
- inverse-gamma polymer

- q = 0
- q = t
- t = 0
- t = 0
- q = 0
- q → 1

- special case
- continuous time
- weak asymmetry
- intermediate disorder

- KPZ equation on \( \mathbb{R} \) or \( \mathbb{R}_{>0} \)
Inverse-gamma directed polymer

Let $\alpha_1, \alpha_2, \ldots$ and $\alpha_\circ$ be positive parameters.

$$w_{i,i} \sim \text{Gamma}^{-1}(\alpha_\circ + \alpha_i)$$

$$w_{i,j} \sim \text{Gamma}^{-1}(\alpha_i + \alpha_j)$$

The partition function of the half-quadrant inverse-gamma polymer is

$$Z(n,m) = \sum_{\pi:(1,1)\to(n,m)} \prod_{(i,j) \in \pi} w_{i,j}$$

- At zero temperature, the free energy (i.e. log($Z$)) converges to passage times of half-space LPP.
- At high temperature, the free energy converges to the KPZ equation [Wu 2018] with Neumann boundary condition (with boundary parameter controlled by $\alpha_\circ$).
Laplace transform formula of the inverse-gamma polymer

- Using Macdonald processes, these formulas can be proved and generalized.

**Theorem ([B-Borodin-Corwin 2018])**

For $m \geq n$ and any $u > 0$, we have

\[
\mathbb{E} \left[ e^{-uZ(m,n)} \right] =
\frac{1}{n!} \int_{r-i\infty}^{r+i\infty} \frac{dz_1}{2i\pi} \cdots \int_{r-i\infty}^{r+i\infty} \frac{dz_n}{2i\pi} \frac{1}{\prod_{1 \leq i < j \leq n} \Gamma(z_i - z_j)} \prod_{i=1}^{n} \frac{\Gamma(z_i + \alpha_j)}{\Gamma(\alpha_i + \alpha_j)} \prod_{i,j=1}^{n} \Gamma(z_i - \alpha_j)
\]

\[
\prod_{i=1}^{n} \left( u^{\alpha_i - \alpha} \frac{\Gamma(\alpha_i + z_i)}{\Gamma(\alpha_i + \alpha_i)} \prod_{j=n+1}^{m} \frac{\Gamma(\alpha_j + z_i)}{\Gamma(\alpha_j + \alpha_i)} \right)
\]

where $r$ is such that $r + \alpha_\circ > 0$ and $r > \alpha_i$ for all $1 \leq i \leq n$. 

Almost a Fredholm Pfaffian

If the parameters $\alpha_i > 0$ are sufficiently close to each other, for any $m \geq n \geq 1$, $u > 0$,

$$
\mathbb{E} \left[ e^{-uZ(m,n,\tau)} \right] = \sum_{k=0}^{n} \frac{1}{k!} \int_{r-i\infty}^{r+i\infty} \frac{dz_1}{2i\pi} \cdots \int_{r-i\infty}^{r+i\infty} \frac{dz_k}{2i\pi} \oint \frac{dv_1}{2i\pi} \cdots \oint \frac{dv_k}{2i\pi} \\
\times \prod_{1 \leq i < j \leq k} \frac{(z_i - z_j)(v_i - v_j)\Gamma(v_i + v_j)\Gamma(-z_i - z_j)}{(z_j + v_i)(z_i + v_j)\Gamma(v_j - z_i)\Gamma(v_i - z_j)} \\
\times \prod_{i=1}^{k} \left[ \frac{\pi}{\sin(\pi(v_i + z_i))} G_{n,m}(v_i) \frac{\Gamma(2v_i)}{\Gamma(v_i - z_i) z_i + v_i} u^{z_i + v_i} \right],
$$

for well chosen $r$ and a certain explicit function $G_{n,m}(z)$.

- Using the approximation $\Gamma(z) \approx 1/z$ close to $z = 0$, the series would become a Fredholm Pfaffian. However, without that approximation, the series is not summable as $n$ goes to infinity.
- A formal saddle point asymptotic analysis leads to the Baik-Rains transition as for LPP.
Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

Hall-Littlewood process

Determinantal/Pfaffian Schur process

Stochastic six-vertex model

Geometric last passage percolation (LPP)

Exponential Last Passage Percolation TASEP

ASEP

KPZ equation on $\mathbb{R}$ or $\mathbb{R}_{>0}$

$q = 0$

$t = 0$

$q = t$

$q \rightarrow 1$

special case

continuous time

weak asymmetry

zero-temperature

inverse-gamma polymer

intermediate disorder
Macdonald measures

▶ An integer partition $\lambda$ is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Symmetric Macdonald polynomials $P_\lambda, Q_\lambda$ are multivariate symmetric polynomials whose coefficients are rational functions in two parameters $q, t \in (0, 1)$. They degenerate to Schur functions $s_\lambda$ when $q = t$.

▶ Macdonald functions satisfy a Cauchy type summation identity. For two sets of parameters $a = \{a_i\}, b = \{b_i\}$,

$$\sum_\lambda P_\lambda(a)Q_\lambda(b) = \Pi(a; b)$$

where $\Pi(a; b) = \prod_{i,j} \frac{(ta_ib_j;q)_\infty}{(a_ib_j;q)_\infty}$.

▶ This leads to the definition of Macdonald measures [Borodin-Corwin 2011] which generalizes the Schur measure [Okounkov 2001]. These are measures on partitions such that

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Pi(a; b)} P_\lambda(a)Q_\lambda(b).$$
Half-space Macdonald measures

- Macdonald functions also satisfy a Littlewood type summation identity
  \[ \sum_{\lambda' \text{ even}} b^\text{el}_\lambda P_\lambda(a) = \Phi(a) \]
  where \( \Phi(a) = \prod_{i<j} \frac{(ta_ia_j;q)_\infty}{(a_ia_j;q)_\infty} \) and \( \lambda' \) even means \( \lambda_1 = \lambda_2, \lambda_3 = \lambda_4 \ldots \)

- More generally, we define
  \[ \mathcal{E}_\lambda(b) = \sum_{\mu' \text{ even}} b^\text{el}_\mu Q_{\lambda/\mu}(b), \]
  so that
  \[ \sum_\lambda \mathcal{E}_\lambda(b) P_\lambda(a) = \prod(a;b)\Phi(a). \]

- We define the half-space Macdonald measure by
  \[ \mathbb{P}^{q,t}(\lambda) = \frac{1}{\prod(a,b)\Phi(a)} P_\lambda(a)\mathcal{E}_\lambda(b). \]

Half-space Macdonald measures degenerate when \( q = t \) to Pfaffian Schur measures [Borodin-Rains 2005].
Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

- Determinantal/Pfaffian Schur process
  - q = t
  - t = 0
  - q = 0
  - q → 1

- Geometric last passage percolation (LPP)
  - q = t
  - t = 0

- Exponential Last Passage Percolation TASEP
  - q = t
  - t = 0

- Whittaker process
  - q → 1

- Stochastic six-vertex model
  - special case
  - continuous time

- Hall-Littlewood process
  - q = 0

- q-Whittaker process
  - q = 0

- KPZ equation on R or R_0
  - weak asymmetry
  - intermediate disorder

- inverse-gamma polymer
  - zero-temperature
Six vertex model in a half-quadrant

Let \(a_1, a_2, \cdots \in (0, 1)\) and \(t \in (0, 1)\) be some parameters.

\[
\begin{align*}
P(\rightarrow) &= 1, \\
P(\cdots) &= \delta = \frac{1 - a_x a_y}{1 - t a_x a_y}, \\
P(\cdots) &= 1 - \delta, \\
P(\cdots) &= t \delta, \\
P(\cdots) &= 1 - t \delta, \\
P(\cdots) &= 1.
\end{align*}
\]

We use the boundary weights [Kuperberg 2000, Wheeler-Zinn-Justin 2016]

\[
\begin{align*}
P(\cdots) &= P(\rightarrow) = 1, \\
P(\cdots) &= P(\cdots) = 0.
\end{align*}
\]

For small \(\delta\), paths are discretizations of the trajectories of particles in half-line ASEP.
Let $h(x,y)$ be the number of outgoing vertical arrows from the vertices on the left of $(x,y)$.

Let $\ell(\lambda) = \lambda'_1$ be the number of nonzero components in a partition $\lambda$ following the half-space Hall-Littlewood measure (i.e. Macdonald measure when $q = 0$).

**Theorem (B.-Borodin-Corwin-Wheeler 2017)**

$$h(n,m)^{(d)} = \lambda'_1,$$

where

$$\mathbb{P}(\lambda) \propto P_\lambda(a_1,\ldots,a_n)\mathcal{E}_\lambda(a_{n+1},\ldots,a_m).$$

Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

- \( q = 0 \)
  - Hall-Littlewood process
    - special case
      - Stochastic six-vertex model
        - continuous time
          - ASEP
            - weak asymmetry
            - KPZ equation on \( \mathbb{R} \) or \( \mathbb{R}_{\geq 0} \)
  - \( t = 0 \)
    - \( q = t \)
      - Determinantal/Pfaffian Schur process
        - Exponential Last Passage Percolation TASEP
          - zero-temperature
            - inverse-gamma polymer
              - intermediate disorder
  - \( q = 0 \)
    - \( q \rightarrow 1 \)
      - \( q \)-Whittaker process
        - Geometric last passage percolation (LPP)
          - Whittaker process
            - inverse-gamma polymer
              - intermediate disorder
Consider a path $\gamma$ as on the left

- vertex $v \mapsto \lambda^v$ a random partition,
- edge $e \mapsto \rho_e$ a (set of) variable(s).

(More generally a specialization of the symmetric functions).

The Macdonald process (generalizing the Schur process [Okounkov-Reshetikhin 2003]) is a probability measure on the sequence of partitions $\Lambda := (\lambda^v)_{v \in \gamma}$ such that

$$\mathbb{P}(\Lambda) \propto \prod_{e \in \gamma} \text{weight}(e),$$

where

$$\text{weight}\left( \frac{\mu}{\rho} \downarrow \lambda \right) = Q_{\lambda/\mu}(\rho) \quad \text{and} \quad \text{weight}\left( \rho \uparrow \lambda \downarrow \mu \right) = P_{\lambda/\mu}(\rho).$$
The **half-space Macdonald process** is a probability measure on the sequence of partitions \( \Lambda := (\lambda^v)_{v \in \gamma} \) such that

\[
\mathbb{P}(\Lambda) \propto \mathcal{E}_{\lambda_\circ}(\rho_\circ) \prod_{e \in \gamma} \text{weight}(e),
\]

where the weight of off-diagonal edges are chosen as in the Macdonald process.

For any \( v \in \gamma \), the law of \( \lambda^v \) is a half-space Macdonald measure.
Make the path grow by unit boxes in the bulk and half-boxes along the diagonal.

We define Markov dynamics on sequences of partitions $\Lambda$ mapping a half-space Macdonald process to another half-space Macdonald process along a new path.

For well chose-dynamics, $\lambda_1^{(n,m)} \approx \log Z(n,m)$, where $Z(n,m)$ is the inverse-gamma partition function and $\lambda$ is distributed according to the $q$-Whittaker process with $q \to 1$.

\[ \mathbb{P}(\lambda) \propto P_\lambda(q^{\alpha_1}, \ldots, q^{\alpha_n}) \mathcal{E}_\lambda(q^{\alpha_n}, q^{\alpha_{n+1}}, \ldots, q^{\alpha_m}). \]
Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]

q = 0
Hall-Littlewood process

q = t
Determinantal/Pfaffian Schur process

q → 1
q-Whittaker process

q = 0

Stochastic six-vertex model

Geometric last passage percolation (LPP)

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ASEP

KPZ equation on \( \mathbb{R} \) or \( \mathbb{R}_{>0} \)

continuous time

special case

zero-temperature

inverse-gamma polymer

weak asymmetry

intermediate disorder
Computing observables of Macdonald measures?

Assume that we have an operator $D_n$ acting on functions of $n$ variables $a_1, \ldots, a_n$, diagonalized by Macdonald polynomials $P_\lambda(a)$, with eigenvalue $d_\lambda$. By acting on both sides of the Cauchy-Littlewood identity

$$\sum_{\lambda} P_\lambda(a) E_\lambda(b) = \Pi(a;b) \Phi(a),$$

and dividing by $\Pi(a;b) \Phi(a)$, we obtain

$$\mathbb{E}^{q,t}[d_\lambda] = \frac{D_n \Pi(a;b) \Phi(a)}{\Pi(a;b) \Phi(a)}.$$

Families of such operators exist (Macdonald difference operators, Noumi’s $q$-integral operator) and the resulting observables characterize the law of $\lambda$ [Borodin-Corwin 2011, Borodin-Corwin-Gorin-Shakirov 2012, B.-Borodin-Corwin 2018].
A variant of Noumi's \( q \)-integral operator

We define an operator \( M^z_n \) on symmetric functions of \( n \) variables by

\[
M^z_n f(x_1, \ldots, x_n) = \int_{-\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds_1}{2\pi i} \cdots \int_{-\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds_n}{2\pi i} (-z)^{s_1+\ldots+s_n} \prod_{i<j} \frac{q^{s_j}x_i - q^{s_i}x_j}{x_i - x_j}
\]

\[
\times \prod_{i,j} \frac{(tx_i/x_j)_\infty (q^{s_j+1}x_i/x_j)_\infty}{(qx_i/x_j)_\infty (tq^{s_j}x_i/x_j)_\infty} \prod_{i=1}^{n} \Gamma(-s_i)\Gamma(1+s_i) f(q^{-s_1}x_1, \ldots, q^{-s_n}x_n),
\]

for \( \epsilon > 0 \) is small enough. We have the eigenrelation

\[
M^z_n P_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{(q^{-\lambda_i} t^iz)_\infty}{(q^{-\lambda_i} t^{-1}z)_\infty} P_\lambda(x).
\]

This comes from

- The Pieri rule for Macdonald polynomials \( P_\lambda Q_{(r)} \) which yields an operator written as a linear combination of \( q \)-shifts.
- Rewriting sums as Mellin-Barnes type integrals (to avoid dealing with issues of divergence of moments).

Applying \( M^z_n \) to \( \Pi(x; b)\Phi(x) \) yields the Laplace transform formulas for the inverse gamma polymer partition function.
Refined Cauchy identity

Recall that for usual Macdonald measures,

\[ p_{q,t}(\lambda) = \frac{1}{\Pi(a;b)} P_\lambda(a_1,\ldots,a_n) Q_\lambda(b_1,\ldots,b_m), \]

**Proposition ([Warnaar 2008])**

For \( u \in \mathbb{C}, \)

\[
\frac{1}{\Pi(a,b)} \sum_\lambda \prod_i \left( 1 - u q^{\lambda_i} t^{n-i} \right) P_\lambda(a) Q_\lambda(b) = \frac{\det \left[ \begin{array}{cc} 1 & -u \\ 1-a_i b_j & 1-t a_i b_j \end{array} \right]}{\det \left[ \begin{array}{c} 1-a_i b_j \end{array} \right]}.
\]

It implies that

\[
\mathbb{E}_{q,t} \left[ \prod_{i=1}^n \left( 1 - u q^{\lambda_i} t^{n-i} \right) \right]
\]

does not depend on \( q \)!
Comparing the \( q = 0 \) and \( q = t \) cases yields identities relating functionals of Schur (\( q = t \)) and Hall-Littlewood (\( q = 0 \)) random partitions.
Refined Littlewood identity

For half-space Macdonald measures,

\[ \mathbb{P}^{q,t}(\lambda) = \frac{1}{\Phi(a)} P_{\lambda}(a_1, \ldots, a_n) b^e_{\lambda} \mathbb{1}_{\lambda \text{'even}}, \]

**Proposition ([Rains 2015], [Betea-Wheeler-Zinn-Justin 2015])**

For \( u \in \mathbb{C} \),

\[
\frac{1}{\Phi(a)} \sum_{\lambda \text{ even}} \prod_{i \text{ even}} \left(1-uq^{\lambda_i}t^{n-i}\right) b^e_{\lambda} P_{\lambda}(a_1, \ldots, a_n) = \frac{\text{Pf} \left[ \frac{a_i - a_j}{1-ta_i a_j} - u \frac{a_i - a_j}{1-ta_i a_j} \right]}{\text{Pf} \left[ \frac{a_i - a_j}{1-ta_i a_j} \right]}.
\]

It implies that

\[
\mathbb{E}^{q,t} \left[ \prod_{i \text{ even}} \left(1-uq^{\lambda_i}t^{n-i}\right) \right]
\]

does not depend on \( q \)! Comparing the \( q = 0 \) and \( q = t \) cases yields identities relating functionals of (half-space) Schur and Hall-Littlewood random partitions.
Conclusion

- Large-scale statistics of the KPZ equation in a half-space and several models in its universality class (ASEP, directed polymers, stochastic six vertex models) can be studied via half-space Macdonald processes almost as well as their full-space counterparts.

- Through half-line ASEP, one obtains a beautiful formula characterizing the law of the KPZ equation on $\mathbb{R}_+$, but the result is restricted to the height at 0 and a specific boundary corresponding to the critical case in the Baik-Rains transition.

- Through the inverse gamma polymer in a half-quadrant, one obtains exact formulas, without restrictions on parameter ranges. However, analyzing these asymptotically remains a challenge.
Thank you