Previous lecture. **BEC and long range order.**
Long range order and order parameter.
BEC fragmentation and role of interactions.

**This lecture:**
- Landau criterion (galilean vs rotational) and metastability
- hydrodynamic theory of superfluids
- propagation of sound and collective oscillations.
- expansion of a trapped gas.
- limits of applicability of hydrodynamic theory
LANDAU’S CRITERION FOR SUPERFLUIDITY
Fluid in the presence of moving trap (T=0)
Trap is assumed to be smooth at “macroscopic” level.

\[ H = H_0 - v \hat{p} \]

Assumption: adiabatic switch on of motion; fluid can absorb energy and momentum only through creation of elementary excitations.
Creation of excitation costs energy \( \epsilon(p) - vp \)

At T=0 no excitation is created (\( \iff \) fluid remains at rest) if \( \epsilon(p) - vp \geq 0 \)

i.e. if \( v \leq v_c \) with

\[ v_c = \min_p \frac{\epsilon(p)}{p} \neq 0 \]

Fluid at rest is not ground state of H. Ground state has \( J = Nmv \)

CONFIGURATION IS METASTABLE
Current will not decay (permanent current) if \( v \leq v_c \) with

\[
v_c = \min_p \frac{\varepsilon(p)}{p}
\]

Moving fluid is not ground state configuration
Ground state has \( J = 0 \)

CONFIGURATION IS METASTABLE
DO WE NEED METASTABILITY TO DEFINE SUPERFLUIDITY?

LANDAU CRITERION IN ROTATING TRAP

Dynamics in rotating frame governed by

\[ H = H_0 - \Omega \hat{l}_z \]

Fluid at rest if \( \Omega < \Omega_C \) where \( \Omega_C = \min_l \frac{\varepsilon(l)}{l} \)

\[ \varepsilon(l) \equiv \text{energy of elementary excitation} \]

\[ l \equiv \text{angular momentum of elementary excitation} \]

\( \Omega_C \neq 0 \) \( \leftrightarrow \) Landau criterion for superfluidity

Deep difference with respect to Galilean transformation: Fluid at rest in the presence of rotating trap can be ground state
SUPERFLUIDS ARE NOT ABLE TO ROTATE AT LOW ANGULAR VELOCITY.
AT HIGH ANGULAR VELOCITIES: QUANTIZED VORTICES

Quantized vortex is energetically favoured if
In general \( \Omega_v < \Omega_c \)

- \( 0 < \Omega < \Omega_v \) Fluid at rest is ground state
- Vanishing moment of inertia: \( L_z = \Theta \Omega = 0 \)
- \( \Omega_v < \Omega < \Omega_c \) Fluid at rest is metastable
- \( \Omega_c < \Omega \) Fluid at rest is unstable (Landau criterium)

\[ \Omega > \Omega_v = \frac{E_{\text{vortex}} - E_0}{L_z} \]
SOME QUESTIONS:

- Relationship between superfluidity and BEC
- Dispersion law of elementary excitations
- Actual value of critical angular velocity
- Structure of vortices
- Role of roughness of the trap in mesoscopic systems
- Actual consequences of energetic instability

Some general answers provided by macroscopic theories (this lecture)

In dilute atomic gases systematic answers provided by microscopic theory of order parameter (next lectures)
From the equation for the field operator to the hydrodynamic equations of superfluids

Equation for the field operator in uniform systems (bosonic field)

\[
i\hbar \frac{\partial}{\partial t} \hat{\Psi}(r, t) = [\hat{\Psi}(r, t), H] = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \int dr' \hat{\Psi}^+(r', t)V(r - r') \hat{\Psi}(r', t) \right] \hat{\Psi}(r, t)
\]

(exact equation also for strongly interacting systems)

If \( \hat{\Psi}(r, t) \) is solution, \( \hat{\Psi}(r - vt, t) \exp\left[ \frac{i}{\hbar} \left( mr v - \frac{1}{2} mv^2 t \right) \right] \)

is also solution (Galilean transformation with velocity \( v \))

In the presence of BEC expectation value of \( \hat{\Psi} \) (order parameter)

transforms from \( \sqrt{n_0} e^{-i\mu t/\hbar} \) to \( \sqrt{n_0} e^{iS(r, t)} \) with

\[
S(r, t) = \left[ mr v - \left( \frac{1}{2} mv^2 + \mu \right) t \right] / \hbar
\]
IRROTATIONALITY OF FLOW IS FUNDAMENTAL FEATURE OF SUPERFLUIDS

- quenching of moment of inertia (lecture 5)
- quantization of circulation and quantized vortices (lecture 6)

Relationship for superfluid velocity and equation for the phase are expected to hold also if density is slowly varying in space and time.
DERIVATION OF HD EQUATIONS AT T=0

At T=0 the current in equation of continuity \( \frac{\partial n}{\partial t} + \nabla j = 0 \) coincides with superfluid current \( j = n v_S \) \( (n = n_S) \).

By adding a smoothly varying external field \( \mu \rightarrow \mu + V_{ext}(r) \) gradient of eq. for the phase yields

\[
\begin{aligned}
\frac{\partial}{\partial t} n + \nabla(n v_S) &= 0 \\
 m \frac{\partial}{\partial t} v_S + \nabla \left( \frac{1}{2} m v_S^2 + \mu(n) + V_{ext} \right) &= 0
\end{aligned}
\]

- Equations are **classical** (do not depend on Planck constant)
- Velocity field is **irrotational** (role of the phase)
- Condensate density does not enter HD eqs \( (n \) is total density)\)
- Applicable to **macroscopic** phenomena
- HD equations hold for both **Bose** and **Fermi** superfluids
- HD equations depend on **equation of state** \( \mu(n) \)
  (sensitive to quantum correlations, statistics, dimensionality, ...)

Hydrodynamic equations of superfluids (T=0) Closed equations for \( n \) and \( v_S \)
Simplest application of HD equations

density profile at equilibrium: \( \partial n / \partial t = 0, v_s = 0 \)

\[ \mu(n) + V_{\text{ext}} = \mu_0 \]

- density profile in Thomas-Fermi approximation
- \( \mu_0 \) is chemical potential fixed by normalization
- \( \mu_0 \) determines Thomas-Fermi radius \( R \) (where density vanishes) according to \( \mu_0 = V_{\text{ext}}(R) \)

Example:

\[ \mu = gn \]
\[ n(r) = \frac{1}{g} \left( \mu_0 - V_{\text{ext}}(r) \right) \]

Systematically employed in trapped dilute Bose gases at \( T=0 \)

\[ r < R \]

Less trivial applications of HD equations concern:

- **collective oscillations**
- **expansion** of the gas after switching off the trap
Collective oscillations in hydrodynamic theory

In linear regime \( n = n_{eq} + \delta n \) hydrodynamic equations take the form

\[
\frac{\partial^2}{\partial t^2} \delta n = \nabla (n_0 \nabla \left( \frac{\partial \mu}{\partial n} \delta n \right))
\]

In the most relevant case of dilute Bose gas \( (\mu = gn) \) one finds

\[
\frac{\partial^2}{\partial t^2} \delta n = \nabla (c^2(r) \nabla \delta n)
\]

with \( mc^2(r) = n \frac{\partial \mu}{\partial n} = \mu_0 - V_{ext}(r) \) (local sound velocity, vanishes at the border of the condensate)
Propagation of sound in BEC gases

In uniform medium HD theory gives trivial sound wave solution

\[ \delta n \propto e^{i(qz - \omega t)} \quad \text{with} \quad \omega = cq \quad \text{and} \quad c = \sqrt{gn/m} \quad \text{(Bogoliubov sound)} \]

In non uniform medium sound waves can propagate if wave length is smaller than size of the condensate. Condition is easily satisfied in elongated condensates. If wave length is larger than radial size one has to take average of density profile along radial direction. Simple integration yields (Zaremba, 1998)

\[ c = \sqrt{gn(0) / 2m} \]
Sound wave packets propagating in a BEC (Mit 97)

velocity of sound as a function of central density
Consider isotropic harmonic oscillator potential \( \omega_x = \omega_y = \omega_z \equiv \omega_{ho} \). One finds solutions in the form \( (n_r = 0,1,2..., l = 0,1,2...) \)

\[
\delta n = r^l Y_{lm}(\theta, \varphi) P_l^{2n_r}(r/R)
\]

number of radial nodes

\( l \)

angular momentum

\( m \)

third component ang. mom.

(R is the Thomas-Fermi radius fixed by condition \( n_{eq}(R) = 0 \))

DISPERSION LAW OF COLLECTIVE OSCILLATIONS \( (\mu(n) = gn) \)

\[
\omega(n_r, l) = \omega_{ho} (2n_r^2 + 2n_r l + 3n_r + l)^{1/2}
\]

(Stringari, 1996)

compare with prediction of non interacting harmonic oscillator model

\[
\omega(n_r, l) = \omega_{ho} (2n_r + l)
\]
Collective frequencies do not depend on value of coupling constant

On one side \( mc^2(0) = gn(0) = \mu_0 \), on the other side \( \mu_0 = \frac{1}{2} m \omega_{ho}^2 R^2 \)
so that \( c^2 \approx \omega_{ho}^2 R^2 \) and \( \omega = cq \approx \frac{c}{R} \approx \omega_{ho} \)

Surface modes \((n_r = 0)\) have dispersion \( \omega = \sqrt{l \omega_{ho}} \) [ \( l \omega_{ho} \) in ideal gas]

In presence of external force dispersion law of surface modes is given by \( \omega^2 = qF / m \) with F evaluated at the surface. One has \( \omega^2 = q \omega_{ho}^2 R \approx l \omega_{ho}^2 \)

Compare with surface modes in superfluid liquid helium where dispersion is fixed by surface tension

Kohn theorem

Center of mass \((n_r = 0, l = 1)\) frequency is equal to \( \omega_{CM} = \omega_{ho} \) like in non interacting case (consequence of transl. invariance of two-body force). Result is independent of temperature and statistics
Collective modes are measured with high precision in Bose-Einstein condensates. They confirm validity of HD theory. They can be used to test Casimir-Polder force, beyond mean field effects ....

Efforts are now produced to measure collective frequencies in superfluid Fermi gases (different equation of state $\Rightarrow$ different results for collective frequencies, see Lecture 8)
**m=0 axial compression mode at T=0 (Mit 97)**

exp: \( \omega = 1.57 \omega_z \)

theory: \( \omega = \sqrt{\frac{5}{2}} \omega_z = 1.58 \omega_z \)

**m=0 radial compression mode at T=0 (Ens 2001)**

exp: \( \omega = 2.07 \omega_z \)

theory: \( \omega = 2 \omega_z \)
Surface scissors $m=1$ mode measured at Oxford (1999)

\[ \omega_{\text{exp}} = 265.6 \pm 0.8 \text{Hz} \]

\[ \omega_{HD} = \sqrt{\omega_\perp^2 + \omega_z^2} = 265 \pm 2 \text{Hz} \]

(Guery-Odelin and Stringari, 1999)

Surface modes measured at MIT (2000)

<table>
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<tr>
<th>$\ell$</th>
<th>$\nu \ell$ (Hz)</th>
<th>$\nu_\ell/\nu_1$ (Expt.)</th>
<th>$\nu_\ell/\nu_1$ (Theor.)</th>
</tr>
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<td>1</td>
<td>90.1 ± 0.5</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>130.5 ± 2.5</td>
<td>1.45 ± 0.04</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>4</td>
<td>177 ± 5</td>
<td>1.96 ± 0.06</td>
<td>2</td>
</tr>
</tbody>
</table>
Expansion of condensate after switching off the trap

Compare predictions of ideal gas with hydrodynamic theory

**Ideal gas:** asymptotic expansion is fixed by momentum distribution. Result follows from time evolution of s.p. wave function

$$\varphi(r, t) = e^{i\pi/4} (m / 2\pi \hbar t)^{3/2} \int dr' \varphi(r', 0) \exp[i(m / 2\hbar t)(r - r')^2]$$

For large times one finds: $n(r, t) = (m / t)^3 n(p = mr / t)$ independent of statistics

Momentum distribution of ideal Bose-Einstein condensed gas in axial ($\omega_x = \omega_y \equiv \omega_\perp$) trap:

$$n(p) = N_0 (\hbar \alpha n \omega_\hbar)^{-3/2} \exp[-\frac{p_\perp^2}{\hbar m \omega_\perp} - \frac{p_z^2}{\hbar m \omega_z}]$$

$p_\perp \equiv p_x + p_y$

to be compared with isotropic distribution of non condensed gas

$$n(p) \propto \exp[-\frac{p^2}{2mk_B T}]$$ (in classical limit)
Anisotropy in $n(p) \rightarrow$ anisotropy in asymptotic density distribution $n(r)$

Furthermore expansion inverts deformation of density distribution (from cigar to disc; from disc to cigar).

Asymptotic anisotropy of expanding BEC (JILA 1995)

However, ideal gas is, in general, bad approximation. Role of interactions should be included.
Role of interactions on the expansion (predictions of HD theory)

In the presence of harmonic trapping hydrodynamics equations have simple scaling solution holding also in non linear regime (Castin and Dum; Kagan et al. 1996)

\[
n(r, t) = n_0 \left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2}\right)
\]

\[
\nu_S(r, t) = \frac{1}{2} \nabla \left(\alpha_x x^2 + \alpha_y y^2 + \alpha_z z^2\right)
\]

where \( R_i(t) = b_i(t) R_i(0) \) and \( b_i \) are scaling factors. \( \alpha_i, b_i \) are related each other by equation of continuity. Hydrodynamic eqs. finally yield eqs.

\[
\dot{b}_i + \omega_i^2 b_i - \omega_i^2 \frac{1}{b_i b_x b_y b_z} = 0
\]

\[
\nu_S(r, t) = \frac{1}{2} \nabla \left(\alpha_x x^2 + \alpha_y y^2 + \alpha_z z^2\right)
\]

\[
\alpha_i = \frac{\dot{b}_i}{b_i}
\]

small oscillations:
\[
b_i = 1 + \delta b_i
\]
(one recovers HD results)

expansion:
\[
\ddot{b}_i - \omega_i^2 \frac{1}{b_i b_x b_y b_z} = 0
\]
\[
b_i(0) = 1
\]
\[
\dot{b}_i(0) = 0
\]
For strongly anisotropic trap \( \omega_x = \omega_y \equiv \omega_\perp \gg \omega_z \) (cigar configuration), equation for the expansion admits analytic solution (Castin, Dum 1996)

\[
\tau = \omega_\perp t
\]

\[
b_\perp = \sqrt{1 + \tau^2}
\]

\[
b_z = 1 + \lambda \left( \tau \arctan \tau - \ln \sqrt{1 + \tau^2} \right)
\]

\[
\lambda = \frac{\omega_z}{\omega_\perp}
\]

- Expansion is much faster in radial than in axial direction (HD forces proportional to density gradients).
- Initial cigar geometry transformed into disc geometry (and vice versa).
- From quantitative point of view hydrodynamic expansion significantly differs from prediction of non-interacting model.
- Expansion in interacting gases provides info on HD forces, not on momentum distribution (unless one switches off interaction).
- Clear experimental evidence of hydrodynamic expansion.
Experiments probe HD nature of the expansion with high accuracy. Aspect ratio $\equiv R_\perp / Z$.
Some questions about hydrodynamic theory:

Can hydrodynamic theory provide value of critical angular velocity? (Landau’s criterion for superfluidity)

Can the hydrodynamic behaviour be used as a proof of superfluidity?

Can hydrodynamic theory describe interference fringes of two expanding and overlapping condensates?
Can hydrodynamic theory provide value of critical angular velocity? (Landau’s criterion for superfluidity)

\[ \Omega_C = \min_l \frac{\varepsilon(l)}{l} \]

In uniform systems HD theory predicts \( v_C = c \) for critical velocity. In trapped gases lowest energy states, for given angular momentum, are surface states with \( \varepsilon(l) = \sqrt{l} \omega_{ho} \), so that

\[ \Omega_C = \min_l \frac{\omega_{ho}}{\sqrt{l}} = 0 \]

Hydrodynamics actually fails for large angular momentum (large \( l \rightarrow \) rapid variations of density profile). More microscopic theory (Lecture 3) needed to evaluate \( \Omega_C \)
Can the hydrodynamic behaviour be used as a proof of superfluidity?

Superfluid hydrodynamics vs collisional hydrodynamics

Collisional hydrodynamics predicts equation for the velocity field:
\[ m \frac{\partial}{\partial t} v + \nabla \left( \frac{1}{2} mv^2 + V_{ext} \right) + \frac{\nabla P}{n} - v \times (\nabla \times v) = 0 \]
(holds in collisional regime $\omega \tau \ll 1$)

Compare with prediction of superfluid hydrodynamics ($\nabla P = n \nabla \mu$ at $T=0$)
\[ m \frac{\partial}{\partial t} v_s + \nabla \left( \frac{1}{2} mv_{s}^2 + V_{ext} \right) + \frac{\nabla P}{n} = 0 \]

For irrotational velocity flow the two equations coincide. Collective oscillations of irrotational nature are identical (provided equation of state is the same)

Rotational effects provide more stringent test of superfluidity (see Lectures 6,7)
Hydrodynamic theory describes correctly expansion of single condensate and provides corresponding density and velocity profiles.

Description of fringes produced by two expanding condensates requires explicit equation for wave function of the condensate.

Hydrodynamic theory can be derived (see Lecture 3) from equation for the condensate wave function (Gross-Pitaevskii equation) neglecting quantum pressure term (Thomas-Fermi approximation).

Resulting HD approximation is appropriate for describing sound propagation and collective oscillations in large N samples as well as expansion of single condensate. It instead rules out possibility of describing interference phenomena. (Fringes of matter waves depends explicitly on Planck constant).
This lecture

Lecture 2. **Superfluidity and hydrodynamics.**
Landau criterion (galilean vs rotational). Hydrodynamic theory of superfluids. Collective oscillations and expansion

Next lecture

Lecture 3. **Equation for the order parameter.**
Gross-Pitaevskii theory. Healing length.
Time dependent theory. Bogoliubov equations.