Title: What is superfluidity?
WHAT IS SUPERFLUIDITY?

In $^4$He – II

1. Frictionless flow through narrow pores
2. Hess-Fairbank effect
3. Persistent currents
4. Quantized circulation (vortices)
5. Zero (or much reduced) friction on object moving with velocity $< v_c$
6. Second sound
7. Josephson effect
8. ....

In BEC gases:

✓

(?)

✓

(?)

(✓)

Question: What is the relation of the above phenomena to (a) the existence of BEC (b) the nature of the excitation spectrum?

A common view: A necessary and sufficient condition for "superfluidity" is the Landau criterion

$v < c$ where $c \equiv \min. \varepsilon(p)/p$

but some prima facie problems:
not sufficient: glasses, He-I
not necessary: $^3$He-A, "gapless"
(and cuprate) superconductors

$^4$He - II

$\varepsilon(p) \uparrow$

$p \rightarrow$
SUPERFLUIDITY IN LIQUID $^4$HE

$^4$He liquefied: 1908

T < $T_\lambda$ (2.17 K): 1920

Frictionless flow below $T_\lambda$: 1938

Modern point of view:

Define

$$\omega_c \equiv \frac{\hbar}{mR^2} \equiv \text{quantum unit of rotation} \quad (\sim 10^{-4} \text{ Hz for } R \sim 1\text{ cm})$$

**EXPT. A**

(“Hess-Fairbank” effect)

walls rotate with
ang. velocity $\lesssim \omega_c$,
liquid stationary

**EQUILIBRIUM EFFECT**

**EXPT B**

(Persistent currents)

walls at rest,
liquid rotating with
ang. velocity $\gg \omega_c$.

**METASTABLE EFFECT**
GENERAL THEOREMS ON ROTATING SYSTEMS

(classical/QM’l)

Suppose the interparticle potential, isotropic

\[ H(t) = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} \sum_{ij} U(|r_i - r_j|) + \sum_i V(r_i - r_o(t)) \]

\[ \dot{r}_o(t) = \omega \times r_o(t) \]

potential of rotating walls

As calculated by observer rotating with walls:

\[ (r'_i \equiv r_i - r_o(t)) \]

\[ H_{rot} = \frac{1}{2m} \sum_i p_i' \cdot p_i' + \sum_i (p_i' \cdot (\omega \times r_i')) + \frac{1}{2} \sum_{ij} U(r_i' - r_j') + \sum_i V(r_i') \]

\[ \equiv p_i \quad = - \omega \cdot \mathbf{L} \quad \equiv |r_i - r_i'| \]

i.e., time-independent

\[ H_{rot} = H_{lab} - \omega \cdot \mathbf{L} \]

Thermodynamic equilibrium state obtained by minimizing

\[ F_{rot} \equiv \langle H_{rot} \rangle - TS = F_{lab} - \omega \cdot \mathbf{L} \]

1) All physical properties time-independent as viewed from rotating frame, but does not imply system stationary in that frame!

2) If \( U \) not isotropic, no frame in which \( H \) time-independent \( \Rightarrow \) no stationary state ("Couette flow")
1. Classical systems

Rewrite $H_{rot}$ in form

$$H_{rot} = \frac{1}{2m} \sum_i \left( p_i' - m\omega \times r_i' \right)^2 + \frac{1}{2} \sum_{ij} U(r_i' - r_j') + \sum_i V_{eff}(r_i')$$

$$V_{eff}(r_i) \equiv V(r_i) - \frac{1}{2} m (\omega \times r_i)^2$$

centrifugal potential

In classical mechanics, $\tilde{p}$ and $\tilde{r}$ independent variables

$\Rightarrow$ can change variables,

$$\tilde{r}_i \equiv r_i', \quad \tilde{p}_i \equiv p_i - m\omega \times r_i$$

$\Rightarrow$

$$\hat{H} = \frac{1}{2m} \sum_i \tilde{p}_i^2 + \frac{1}{2} \sum_{ij} U(|\tilde{r}_i - \tilde{r}_j|) + \sum_i V_{eff}(\tilde{r}_i)$$

$\equiv H_{lab} \ (V \rightarrow V_{eff})$

i.e. equilibrium in rotating frame is exactly as it would be if walls at rest but $V \rightarrow V_{eff}$ (effect of centrifugal term), and thus is at rest:

in classical mechanics, system in thermal equilibrium always rotates completely with walls

(“pseudo-Bohr-van Leeuwen theorem”)

$\Rightarrow$ to get HF effect, $\tilde{r}$ and $\tilde{p}$ must be non commutating.

Equivalently, wave function must satisfy SVBC.
GENERAL THEOREMS, cont.

In QM, many-body wave function must satisfy TDSE, (anti) symmetry and Single-Valuedness Boundary Condition:

\[ \Psi \left( r_1, r_2, \ldots, r_i, z_i, \theta_i, \ldots, r_N \right) \]

\[ = \Psi \left( r_1, r_2, \ldots, r_i, z_i, \theta_i + 2\pi, \ldots, r_N \right) \]

\[ (\forall \ r_1, r_2, \ldots, r_N) \]

\[ \Rightarrow \text{ in general, ps-Bohr-vL theorem can be violated} \]

\[ (\text{even for } \omega = 0) \]

Question: For \( \omega = 0 \), what is the max. equilibrium ang. momentum of system?

Suppose GSWF \( \Psi_0 \{ r_i \} \) satisfies SVBC, and TISE with eigenvalue \( E_0 \), and suppose ang. momentum is \( L_0 \).

Now consider

\[ \Psi_{\text{trial}}^{(r_i)} \equiv (\exp i \sum \theta_i) \Psi_0 \{ r_i \} \quad (l_i \rightarrow l_i - h) \]

still satisfies SVBC and (anti) symmetry

Since \( |\Psi'| = |\Psi| \), potential energy unchanged. KE is changed:

\[ \Delta \langle T \rangle = -\frac{\hbar}{m \langle r^2 \rangle} L + \frac{Nh^2}{2m \langle r^2 \rangle} \]

\[ \Rightarrow \text{ max. angular momentum in equilibrium is } \hbar/2 \text{ per particle.} \]
EXPLANATION OF HESS-FAIRBANK EFFECT IN TERMS OF BEC:

Walls rotating with ang. velocity
\[ \omega \lesssim \omega_c \iff \equiv \hbar/m R^2 \]
What does liquid do?

General principle: Average ang. velocity of atoms (\( \bar{\omega} \))
as close as possible to \( \omega \)

\[ \uparrow : \text{ Single-atom states must obey} \]
quantization condition: \( \omega = n\omega_c \) \( (\ell = n\hbar) \)

A. "Normal" (non-BEC) system:
many different single-particle
states occupied (typical value of
\( n \sim (kT/\hbar\omega_c)^{1/2} \sim 10^7 \))

\[ \Rightarrow \text{to get } \bar{\omega} = \omega, \text{just shift atoms} \]
slightly between states.

B. BEC system \( (T \ll T_c) \)
(almost) all atoms in
condensate \( \rightarrow \) must have same
value of \( n \) \( (n_0) \) \( \Rightarrow \bar{\omega} \equiv n_0 \omega_c \)

INTERACTIONS
"OPTIONAL"
\( ^4\text{He}: \) PERSISTENT CURRENTS

Initially, after walls stopped,
\[
\langle L \rangle = N_0 \ell_0 \hbar, \quad \ell \gg 1 \quad (\tilde{\omega} \gg \omega_c)
\]

But groundstate has \( \langle L \rangle = 0. \quad (\omega = 0) \)

Why no relaxation?

\[ \chi_0(r) = \chi_0(r) \exp i \phi(r) \]

condensate w.f.

Df: “winding no.” \( n = \oint \frac{\nabla \phi \cdot dl}{2\pi} \)

Initially, \( n = \ell_0 \): eq\( ^m \) state has \( n = 0. \)

To change \( n \), must depress \( |\chi_0(r)| \) to zero somewhere!

(a) Electron in atom:

Schrödinger eqn. linear \( \Rightarrow \) nodes cost no extra energy, e.g.

\[
\psi(t) = a(t) \psi_p + b(t) \psi_s \quad \left\{ \begin{array}{l}
t \rightarrow -\infty: \quad a = 1, \quad b = 0 \\
t \rightarrow +\infty: \quad a = 0, \quad b = 1 \\
\end{array} \right.
\]

\[
\langle E \rangle(t) = |a(t)|^2 E_p + |b(t)|^2 E_s = \text{monotonically decreasing}
\]

(b) BEC (\(^4\text{He}\)):

Extra term in energy: \( \langle V \rangle = V_0 \int |\chi_0(r)|^4 \, dr \)

\( \Rightarrow \) energy NOT monotonically decreasing!

(REPULSIVE) INTERACTIONS ESSENTIAL!
DECAY FROM p-STATE TO s-STATE

\[ \psi_\circ (t = -\infty) = f_p(r) \exp i\theta \]
\[ \equiv \psi_p(\theta) \]

irrelevant

\[ \psi (t = +\infty) = f_s(r) \times \text{const} (\theta) \equiv \psi_s(\theta) \]

Try interpolation formula

\[ \psi(\theta:t) = a(t)\psi_p(\theta) + b(t)\psi_s(\theta) \]

\[ |a(t)|^2 + |b(t)|^2 = 1, \quad a(-\infty) = 1, \quad a(+\infty) = 0 \]

note: irrespective of phase of \(b\), must be node at some value of \(\theta\) at time s.t. \(|b(t)| = |a(t)|\).

Then:

(a) If system obeys linear (Schrödinger) equation:

\[ E(t) = |a(t)|^2 E_p + |b(t)|^2 E_s \]

\[ = \text{const.} + \Delta E_{ps} (|a(t)|^2 - |b(t)|^2) \]

\[ \frac{dE(t)}{dt} < 0, \quad \forall t. \]

(b) If extra GP-like term

\[ E_{cp} = g \int |\psi(\theta)|^4 \, d\theta \quad (g > 0): \]

then

\[ E_{cp}(t) = \text{const.} + g |a(t)|^2 \cdot |b(t)|^2 \]

nonmonotonic!

\[ \frac{dE(t)}{dt} \not< 0 \text{ (necessarily)} \]
A "toy" model to illustrate the basic phenomena of superfluidity in an annulus:

Single-particle states: \( \psi_0 = \text{const.} \) (s-state)  
\[ \psi_1 = \text{const.} \ e^{i\theta} \] (p-state)

If we consider \( 0 < \omega < \omega_c \) and interactions (etc.) weak enough, only those two states are relevant.

Characteristic energies:

(a) single-particle splitting \((2)(E_1 - E_0) = \frac{\hbar^2}{mR^2} = \hbar \omega_c\)

(b) asymmetry energy
\[ -\int \psi_1^* (r) \ V_{\text{ext}} (r) \ \psi_0 (r) \ dr \equiv V_0 (> 0) \]

(c) mean-field interaction energy per particle
\[ \frac{4\pi N\alpha h^2}{m} \int dr \ |\psi_0(r)|^4 \equiv g \] (can have either sign)

Limit of weak asymmetry and interaction:
\[ g, \ V_0 \ll \hbar \omega_c, \ g/V_o \text{ arbitrary} \]
WHAT IS SUPERFLUIDITY? (cont.)

Under stated conditions, the effective Hamiltonian is

\[ \hat{H}_{\text{eff}} = \hat{H} - \omega \cdot \hat{L} = -\hbar \delta \omega \left( a_1^+ a_1 - a_0^+ a_0 \right) - V_0 \left( a_0^+ a_1 + \text{H.c.} \right) + g(a_0^+ a_0 a_1^+ a_1) \]

- s.p. rotational splitting
- asymmetry en.
- const. "Fock" term

Note: asymmetry en. favors 0-1 mixing, Fock term opposes it.

GP ansatz for many-body w.f. at \( T = 0 \):

\[ \Psi_N = (a_0^+ \cos \frac{\chi}{2} \exp i\Delta \phi/2 + a_1^+ \sin \frac{\chi}{2} \exp -i\Delta \phi /2)^N |\text{vac}\rangle \]

\( \Delta \phi \) only affects term in \( V_0 \), which is minimized by choice \( \Delta \phi = 0 \) (i.e. \( \psi(\theta) \) min. where \( V(r) \) most repulsive).

Then:

\[ E(\chi)N = \delta \omega \cos \chi - V_0 \sin \chi + \frac{g}{2} \sin^2 \chi \]

\[ L(\chi)/N = \frac{1}{2} (1 - \cos \chi) \]

(a) For \( \delta \omega \) close to \( -\omega_c/2 \) (i.e. \( \omega \to 0 \)):

since \( |\delta \omega| \gg V_0, g, \cos \chi \approx 1 \)

and so:

\[ L = 0 \quad \left( + o \left( V_0, g \right)^2 \right) \]

so (almost) complete Hess-Fairbank effect

\[ \mathcal{H}_z = -\delta \omega \quad \text{"easy" axis} \]

\[ \mathcal{H}_x \equiv V_0 \]
(b) Behavior near $\delta \omega = 0$ ($\omega \approx \omega_c/2$):

A. $g < V_o$

$\delta \omega > 0$

Behavior nonhysteretic.

B. $g > V_o$

Behavior hysteretic (supercurrent metastable)

Hysteresis terminated for $|\delta \omega| > \delta \omega_{\text{crit}}$.

$$(\delta \omega_{\text{crit}})^{2/3} = g^{2/3} - V_o^{2/3}$$
WHAT IS SUPERFLUIDITY? (cont.)

Conclusions we can draw within the “toy” model
\( g, V_o \ll \hbar \omega_c \) but \( g/V_o \) arbitrary:

**Hess-Fairbank effect** occurs for any \( g/V_c \), but

**metastability of supercurrents** requires both

(a) \( g > V_o \) (interaction repulsive & sufficiently strong) and

(b) \( \delta \omega \) not too large.

Thus, **metastability of superflow \( \neq \) HF effect**!

More realistic model for \(^4\text{He-II} (+ \) probably BEC gases):

\( g \gg \hbar \omega_c, V_o \).

Then many flow states (“winding no.” \( n \gg 1 \)) metastable, and

(a) \( L/N \) always close to \( nh \)

(b) **Criterion for metastability is**

\[ g > n^2 \hbar^2/mR^2 \]

Since \( v_s = n\hbar/mR \) and \( c_s = \sqrt{g/m} \), this is equivalent to

\( v_s < c_s \)

which for this simple model is just the Landau criterion. However, **in general criterion for sup\( \gamma \). \( \neq \) Landau one.**
EFFECT OF A HYPERFINE DEGREE OF FREEDOM ON
MESTABILITY OF SUPERFLOW

(T-L. Ho, 1982)

can p-state \((\Psi(\chi) = \exp i\chi)\)

decay to s-state \((\Psi(\chi) = \text{const})?\)

(a) Scalar OP (recap):

\[ p \rightarrow s \text{ requires system to go through} \]

states with \(|\Psi(\chi)|^2 \neq \text{const.} \text{ actually, must have node!}\)

for interaction repulsive, free energy barrier.

(b) Spin \(\frac{1}{2}\): \((E \text{ ind. of spin direction})\)

\[ \Psi_{\text{in}} = \exp i\chi |\uparrow\rangle \rightarrow \Psi_{\text{f}} = \text{const.} \langle \chi|?\]

Try: family of states parametrized by \(\theta, 0 \leq \theta \leq \pi:\)

\[ \Psi(\theta) = \cos \frac{\theta}{2} |\Psi_{\text{in}}\rangle + \sin \frac{\theta}{2} |\Psi_{\text{f}}\rangle \leftarrow \text{const.} \langle \chi|\downarrow\rangle \]

i.e. (apart from overall phase factor)

\[ \Psi(\theta; \chi) = \cos \frac{\theta}{2} \exp i\chi/|2|\uparrow\rangle + \sin \frac{\theta}{2} \exp -i\chi/|2|\downarrow\rangle \]

original state rotated through < \(\theta\) around axis in xy-plane making

\(\angle \chi\) with x-axis!

\[ \Rightarrow |\Psi(\theta)|^2 = \text{const} \Rightarrow E = E_o \text{ (no barrier!)} \]

(c) spin \(F\): can only change winding no. in this way by \(2m_F\)

[Formally: \(\Pi_1(SU(2)) = Z_{2m}\)]