Bose-Einstein condensate in double-well potential

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Outlook

• Double-well trap. Experimental realization.
• Josephson dynamics. Two regimes.
• Quantum fluctuations. Independent condensates.
• Attractive interaction
Classical waves approximation

\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + g\Psi|\Psi|^2 \]

E.P. Gross, 1961; L.P. Pitaevskii, 1961

\[ g = \frac{4\pi\hbar^2}{m} a, \quad a \text{ is s-wave scattering length.} \]

Plays role of the Maxwell equations in this problem, but contains \( \hbar \) explicitly, because dispersion law of the waves \( \omega = \hbar k^2 / 2m \) contains \( \hbar \).
Corrections due to quantum fluctuations of the wave function

For a harmonic trap corrections are of the order of

\[ \delta \sim \frac{(TF)^{6/5}}{N} \]

where \( TF \) is the "Thomas - Fermi parameter"

\[ TF = N \frac{a}{a_{ho}} \gg 1 \]

Values of \( \delta \) are typically \( \sim 1\% \)
Bose-Einstein condensates in a double-well potential.
Experiments:

(a) Setup for double-well potential, Y. Shin et al., 2004[1]. (b), (c): d=6, 13mcm
Variables of inter-wells dynamics

Numbers of atoms:

\[ N_a, \quad N_b \]

Phase difference:

\[ \Phi = \Phi_a - \Phi_b \]

Can they be measured?
Measurement of the phase difference. I

Interference of expanding condensates.

Density profile after expansion:

\[ n(x,t) = \left[ n_a + n_b + 2\sqrt{n_a n_b} \cos\left(\frac{md}{\hbar t} x + \Phi\right) \right] \]

The measurement is destructive.
Interference of condensates released from the double-well trap [1].
Measurement of the phase difference. II

"Interference in momentum space"
Momentum distribution of two identical condensates in a double-well trap:

\[
n(p_x) = 2 \left[ 1 + \cos\left( \frac{p_z d}{\hbar} + \Phi \right) \right] n_a(p_x)
\]

Measurement can be almost non-destructive!

Stimulated light scattering

Two laser beams with frequencies $\omega_1, \omega_2$ and wave vectors $\mathbf{k}_1, \mathbf{k}_2$ transfer energy $\hbar \omega = \hbar (\omega_1 - \omega_2)$ and momentum $\mathbf{q} = \hbar (\mathbf{k}_1 - \mathbf{k}_2)$ to condensate.

The number of scattered photons

$$\sim n(p_x) \text{ with } p_x = \frac{m}{q} \left( \hbar \omega - \frac{q^2}{2m} \right)$$
Interference in momentum space.

$S(q, E)$

$\nu - \nu_r \ [kHz]$
Setup for continuous phase measurement [3].
Josephson dynamics

In a good approximation:

\[ \Psi(x, t) = \Psi_a(x, N_a) e^{i\Phi_a} + \Psi_b(x, N_b) e^{i\Phi_b} \]

\[ N_a + N_b = N, \Phi = \Phi_a - \Phi_b \]

\[ N_{a,b}, \Phi_{a,b} \text{ are functions of } t. \]

The Josephson equation:

\[ \frac{\partial \Phi}{\partial t} = - \frac{1}{\hbar} (\mu_a - \mu_b) \]

\[ N_a, N_b = \text{const} : \Phi = (\mu_a - \mu_b)t / \hbar \]
Time dependence of the relative phase in presence of chemical potentials difference [1].
Frequency shift as a function of magnetic field gradient.

Continuous read-out of the relative phase of two condensates[3].
Josephson current

\[ \Psi(x,t) = \Psi_a(x,N_a)e^{i \Phi_a} + \Psi_b(x,N_b)e^{i \Phi_b} \]

\[ -\frac{\partial N_a}{\partial t} = I = \frac{i \hbar}{2} \left( \Psi \frac{d \Psi^*}{dx} - \Psi^* \frac{d \Psi}{dx} \right) = -\frac{E_J}{\hbar} \sin \Phi \]

\[ \Phi = \Phi_a - \Phi_b, \quad k = \frac{(N_a - N_b)}{2} \]

\[ k << N : \mu_a - \mu_b \approx E_c k \]

\[ \frac{\partial \Phi}{\partial t} = -\frac{E_C}{\hbar} k; \quad \frac{\partial k}{\partial t} = \frac{E_J}{\hbar} \sin \Phi \]

Equations of "physical pendulum"
Physical pendulum

$\Phi$ is tilt angle, $k$ is angular momentum
Two regimes

\[ t = 0 : \quad \Phi = 0, \ k = k_0 \]

I \( k_0 < \sqrt{2E_J / E_C} \) - oscillations

\[ k_0 \ll \sqrt{2E_J / E_C} : \omega_J = \sqrt{E_J E_C} / \hbar \]

II \( k_0 > \sqrt{2E_J / E_C} \) - "rotation"

\[ k_0 \gg \sqrt{2E_J / E_C} : k \approx k_0, \ \Phi \approx -tE_C k_0 / \hbar \]

S. Raghavan et al., 1999.
Tunneling dynamics of a condensate in a double-well potential [2].
Population and phase evolution. $z=2k/N$
Quantization

Hamiltonian form of equations:

\[ H = E_C k^2 / 2 - E_J \cos \Phi \]

\[ \frac{\partial \Phi}{\partial t} = -\frac{\partial H}{\partial (\hbar k)} \cdot \frac{\partial (\hbar k)}{\partial t} = \frac{\partial H}{\partial \Phi} \]

\[ [\hat{k}, \hat{\Phi}] = -i, \quad \hat{k} = -i \frac{\partial}{\partial \Phi} \]
Quantum fluctuations

Uncertainty relation \( \langle k^2 \rangle \langle \sin^2 \Phi \rangle \geq \langle \cos \Phi \rangle^2 / 4 \):

- Strong tunneling, \( E_J \gg E_C \):
  \[ 1 - \langle \cos \Phi \rangle \approx 2 \langle \Phi^2 \rangle = \sqrt{E_C / E_J} \ll 1 \]
- Weak tunneling, \( E_J \ll E_C \):
  \[ \langle \cos \Phi \rangle \approx 2E_C / E_J \ll 1 \]
Observation of phase fluctuations

"Interference in momentum space"

Momentum distribution, averaged with respect of different runs of the experiment (for $\bar{k} = \bar{\Phi} = 0$):

$$\langle n(p_x) \rangle = 2\left[1 + \langle \cos \Phi \rangle \cos\left(\frac{p_z d}{\hbar}\right)\right] n_a(p_x)$$

Quantity $\alpha = \langle \cos \Phi \rangle$ is visibility factor.

It defines amplitude of fringes.

In the absence of fluctuations $\alpha = 1$
Visibility factor
Independent condensates [3]
Independent condensates. Phase shift.
Corrected Hamiltonian

Weak interaction: $E_C \rightarrow 0$

$E_C \rightarrow \text{const}, E_J \rightarrow \delta \sqrt{N^2 / 4 - k^2}$

Corrected Hamiltonian

$$H = E_C k^2 / 2 - \delta \sqrt{N^2 / 4 - k^2} \cos \Phi$$

$$H \approx (E_C + 2\delta / N)k^2 / 2 + N\delta \Phi^2 / 4$$

No interaction: $E_C = 0$

$$H \approx (2\delta / N)k^2 / 2 + N\delta \Phi^2 / 4$$

$$\langle \Phi^2 \rangle = \left\langle -\frac{\partial^2}{\partial k^2} \right\rangle = 1 / \left(4 \left\langle k^2 \right\rangle \right) = 1 / N \neq 0$$

Visibility: $\alpha = 1 - \frac{\langle \Phi^2 \rangle}{2} - \frac{2 \left\langle k^2 \right\rangle}{N^2} + \frac{1}{N} = 1$
Direct calculation of the momentum distribution

\[ \Psi(p) = \psi_a(p) a + \psi_b(p) b \]

\[ \alpha = \frac{2}{N} \langle [b^+ a] \rangle \]

\[ b^+ a \mid k \rangle = \sqrt{\frac{N}{2+k}(N/2-k+1)} \mid k-1 \rangle \]

\[ |gs\rangle = \sum_k C(k) \mid k \rangle \]

\[ \alpha = \frac{2}{N} \sum_k \sqrt{\frac{N}{2-k}(N/2+k+1)} C(k) C(k+1) \]
Parameters of experiments

[1]: $N \sim 10^5$, $\omega_J = 2\pi \times 1$ Hz,
$\delta / h = 5 \times 10^{-4}$ Hz, $E_J / E_C \sim 2500$

[2] $N \sim 2300$, $\omega_J = 2\pi \times 25$ Hz,
$\delta / h = 2$ Hz, $E_J / E_C \sim 1.7 \times 10^5$

[3] $N \sim 10^6$, $\delta / h < 1$ mHz