Polymerized Membranes, a Review

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Abstract

Membranes are of great technological and biological as well as theoretical interest. Two main classes of membranes can be distinguished: Fluid membranes and polymerized, tethered membranes. Here, we review progress in the theoretical understanding of polymerized membranes, i.e. membranes with a fixed internal connectivity. We start by collecting basic physical properties, clarifying the role of bending rigidity and disorder, theoretically and experimentally as well as numerically. We then give a thorough introduction into the theory of self-avoiding membranes, or more generally non-local field theories with δ-like interactions. Based on a proof of perturbative renormalizability for non-local field theories, renormalization group calculations can be performed up to 2-loop order, which in 3 dimensions predict a crumpled phase with fractal dimension of about 2.4; this phase is however seemingly unstable towards the inclusion of bending rigidity. The tricritical behavior of membranes is discussed and shown to be quite different from that of polymers. Dynamical properties are studied in the same frame-work. Exact scaling relations, suggested but not demonstrated long time ago by De Gennes for polymers, are established. Along the same lines, disorder can be included leading to interesting applications. We also construct a generalization of the \( O(N) \)-model, which in the limit \( N \to 0 \) reduces to self-avoiding membranes in analogy with the \( O(N) \)-model, which in the limit \( N \to 0 \) reduces to self-avoiding polymers. Since perturbation theory is at the basis of the above approach, one has to ensure that the perturbation expansion is not divergent or at least Borel-summable. Using a suitable reformulation of the problem, we obtain the instanton governing the large-order behavior. This suggest that the perturbation expansion is indeed Borel-summable and the presented approach meaningful. Some technical details are relegated to the appendices. A final collection of various topics may also serve as exercises.

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1 Introduction and outline

One of the most challenging ideas in modern physics is the concept of universality: Certain properties of physical systems do not depend on microscopic details and furthermore are equivalent for seemingly unrelated problems. This is epitomized by systems undergoing symmetry breaking continuous phase transitions. The most powerful tools to reveal these relations are delivered by quantum field theory, which has celebrated an overwhelming success in nearly all areas of physics. The study of the $O(N)$-model, which is a field theory for the statistics of $N$-component spins with short-range interactions, has shown that their critical behavior is described by a set of exponents which are completely characterized by the dimension and the underlying symmetry (the number of components of the order parameter). Universality is ensured since the microscopic details are averaged out, and do not affect the large scale fluctuations. A variety of techniques have been developed to examine the critical behavior of this model; possibly the most successful one is the renormalization group procedure\[2\] which analytically justifies the concept of universality. The technically most convenient implementations are field theoretical methods, e.g. the $\varepsilon$-expansion about the upper critical dimension of 4, an expansion about the lower critical dimension of 2, and exact re-summations in the large $N$ limit. (For an review of these techniques, see Ref.[3].) The best studied method is the $\varepsilon$-expansion about the upper critical dimension of 4, where calculations have been performed up to fifth order. Together with resummation techniques which take care of the large-order behavior known from instanton calculus, this is a very powerful tool for extracting critical exponents.

On the other hand, field theories have strong connections to geometrical problems involving fluctuating lines. For example, the motion of particles in space-time describes a world-line. Summing over all world-lines, weighted by an appropriate action, is the Feynman path integral approach to calculating transition probabilities, which can alternatively be obtained from a quantum field theory. The latter can be extended to string theory, generalizing the sum over particle trajectories to the sum over trajectories of lines. Another example is the high-temperature expansion of the Ising model. The energy-energy correlation function can be expressed as a sum over all self-avoiding closed loops which pass through two given points. Self-avoidance is necessary in order not to overcount configurations. We face an important new theoretical concept, which is the subject of this review: Parameterizing the loop by its length, different parts of the loop interact with each other irrespective of their distance. Treating such phenomena in the framework of field-theory demands an enlargement of the concept of local field-theories to multi-local ones. The first direct such approach was developed in the context of self-avoiding polymers, which are formally equivalent to the loops appearing in the high-temperature expansion of the Ising model, by Edwards and Des Cloizeaux[4, 5, 6]. In this approach, hard self-avoidance is replaced by a soft short range repulsive interaction upon contact of the monomers. This interaction is then studied perturbatively by expanding about ideal random walks. Here too, the perturbative expansion can be reorganized into an expansion about the upper critical dimension of 4, which was shown[7] to be equivalent to the perturbation expansion of $\phi^4$-

![Figure 1: Budding of fluid membranes, from [1].](image-url)
theory in the limit $N \to 0$. This equivalence provides two apparently different approaches for calculating the same exponents.

There is much work in the field theory community on generalizing results for fluctuating lines to entities of other internal dimensions $D$. The most prominent example is string theory, which describes $D = 2$ world sheets [8, 9, 10, 11]. An earlier example is provided by the correspondence between gauge theories and random surfaces[12, 13]. The low temperature expansion of the Ising model in $d$ dimensions also results in a sum over surfaces that are $d - 1$ dimensional. For $d = 3$, the surfaces are made out of plaquettes, the basic objects of lattice gauge theories. All these objects share the common property that not only fluctuations of shape but also topology changes occur and have to be summed over in the partition function. The biologically relevant representatives of this class of membranes are fluid membranes, which in general are formed by a lipid bi-layer. In contrast to fluid membranes are “tethered”, polymerized surfaces[14, 15], which have a fixed internal connectivity, and are thus simpler than their fluid counterparts. Experimental realizations are e.g. the network formed by spectrin in red blood cells or graphite mono-layers. These systems may be found in three quite different phases: a collapsed compact phase, a flat phase and an intermediate crumpled swollen phase with fractal dimension of about 2.4. Experimentally, the situation is still under debate (cf. section 2.5). In numerical simulations (cf. section 2.5), generically flat membranes are found (see figure 1). The reason why eventually no crumpled swollen phase may be observable is that the rigidity of tethered membranes is – in sharp contrast to fluid membranes – strongly enhanced by the effect of shear waves. Technically, integrating out these degrees of freedom renormalizes the rigidity, and if the initial rigidity is beyond a certain threshold, the membrane will become flat (see section 2.4). Intuitively this is analogous to a crumpled sheet of paper, which is much more rigid than an uncrumpled one.

Numerically, it has been observed that tethered membranes seemingly are always flat, even when starting with self-avoidance only. This can be traced back to the effective (entropic) bending rigidity which is always present in these models. However, since the largest membranes simulated so far consist of only $75 \times 75$ atoms in the simplest spring and bead model, which has the inconvenience of being rather rigid, and of about $25 \times 25$ atoms in the more sophisticated plaquette-models, simulations are far from being conclusive. These general physical, including numerical and experimental considerations are presented in

Figure 2: Polymerized tethered membrane in the flat phase, from [16].
more detail in section 2.

For theoretical analysis, it is convenient to further generalize to membranes of arbitrary (inner) dimension $D$, interpolating between polymers for $D = 1$ and membranes for $D = 2$. Simple power counting indicates that self-avoidance is relevant only for dimensions $d < d_c = 4D/(2 - D)$, making possible an $\varepsilon = 2D - d(2-D)/2 \sim (d_c(D) - d)$-expansion, which was first carried to 1-loop order about an arbitrary point on the line $\varepsilon = 0$ in Refs. [17, 18, 19, 20, 21]. To obtain results for polymers or membranes, one then has the freedom to expand about any internal dimension $D$, and the corresponding upper critical dimension of the embedding space[22]. This freedom can be used to optimize the calculation of critical exponents.

A major breakthrough in the understanding of these non-local field theories is the proof by David, Duplantier and Guitter, that the field-theory of a $D$-dimensional self-avoiding tethered membrane is renormalizable to all orders in perturbation theory. The main technical tool is the multilocal operator product expansion (MOPE), generalizing the concept of (local) operator product expansion (OPE), introduced into field theory long time ago by Wilson [23] and Kadanoff [24], to the multilocal situation. We shall present this technique in section 3. A collection of useful tools is given in section 4, and a condensed version of the above-mentioned proof in section 5.

These general arguments have been checked by explicitly going to 2-loop order [25, 26]. The calculation is technically difficult but it is valuable to understand the underlying principles. We therefore review these calculations in section 6, suggesting to the reader more concerned with applications to skip this section as well as section 5 with the discussion of the proof of perturbative renormalizability. The most important physical prediction of this calculation is that there exists a crumpled swollen phase with fractal dimension of about 2.4.

Another important question is whether non-leading terms play a role for the critical behavior of tethered membranes. This is certainly the case at the tricritical point, which separates the crumpled swollen from the compact phase, and which is analyzed in section 9. In contrast to polymers, whose tricritical behavior is dominated by the 3-point self-repulsion (which formally punishes triple intersection of the polymer with itself), in the case of the membrane ($D = 2$), this role is played by a modified 2-point interaction, not proportional to a $\delta$-interaction, but to its second derivative [27]. Subdominant operators may also play a role at the self-avoiding fixed point, at finite $\varepsilon$, i.e. well below the upper critical dimension [28].

It is well known that different dynamical models can lead to the same static behavior [29]. In the case of polymers, people have paid most attention to purely diffusive dynamics (Rouse model, model A) eventually including the effect of hydrodynamics (Zimm model). For a long time, the question whether these dynamical models are renormalizable, stayed open. As discussed in section 11, the methods mentioned above finally allowed to settle this question [30, 31].

Somehow surprisingly, the same kind of model also applies to the dynamics of an extended elastic object, be it a polymer or a membrane, in quenched disordered. Technically, averaging over disorder generates non-local interactions on the polymer, with interactions proportional to the disorder correlations. The latter may be taken to be $\delta$-distributions. In this respect, it is worth recalling that self-avoidance can also be generated by averaging over all realizations of an (imaginary) random potential, in which the polymer or membrane is fluctuating. In section 12, we review the analysis of a $D$-dimensional membrane (with $D = 0$ for a particle, $D = 1$ for a polymer and $D = 2$ for a membrane), in a quenched random force field with both potential and non-potential parts. In contrast to the pure potential case, this situation is accessible perturbatively [32, 33].

As is well-known, string theory is defined as the sum over all closed manifolds with arbitrary topology. Excluding from this sum self-intersecting configurations is a formidable task beyond current technical capabilities. For polymerized membranes, i.e. with non-fluctuating metric, this sum can indeed be taken, generalizing the high temperature expansion of the $O(N)$-model mentioned above from a gas of self-
avoiding loops of fugacity $N$, to a similar gas of closed fluctuating manifolds of internal dimension $D$ [34, 35]. As will be discussed in section 13, this generalization is not unique, leaving space for adaptation of the model to the situation in question. Among others, the model contains a novel mechanism not present in standard field theory, which turns first order transitions into second order ones (“reverse Coleman-Weinberg mechanism”). The model further contains a 1-loop fixed point for the random bond Ising model and finally allows for an intriguing conjecture regarding the nature of droplets dominating Ising criticality.

So far, these models have only been treated via perturbative techniques. An important question is, whether the theory is meaningful beyond perturbation expansion. This is a difficult issue, which so far is only partially answered for the case of self-avoiding polymers. A little bit easier to answer is the question, whether the perturbative series is well defined. For the case of the $O(N)$-model, it has been shown by Lipatov [36, 37], that the series is divergent, but can be resummed using a Borel-transform. For tethered membranes, the situation is difficult, since the usual instanton methods do not apply. In section 14 we show, what the analog of the instanton for the $\phi^4$-theory is, and why this implies that the perturbation series is also Borel-summable [38].

Finally let us point out that even though the primary aim of this review is to present from a unified viewpoint the theoretical concepts of multilocal field-theories, an effort is made to motivate the physical models and experimental relevance. On the other hand, the real progress which goes beyond today’s interest, lies in the fundamental technical achievements, and the author feels that skipping technically important details, would render this review much less useful. In order to keep the text readable, the central ideas are given before embarking on technical calculations, and wherever this is possible, we try to sketch how the techniques developed will be useful later.

The general structure of this review is therefore organized so that relevant material, which is necessary to place the following more technical parts in the physical context, is collected in section 2. The next section is devoted to the necessary elementary technical tools. The following sections are more specialized and can mostly be read independently, only necessitating section 3, and eventually 4.
2 Basic properties of membranes

2.1 Fluid membranes

Let us start by characterizing the different possible types of membranes. One very popular class of membranes are fluid membranes. We all know of soap-bubbles from childhood days. Biologically more relevant are bilayers of lipid molecules that are composed of a hydrophilic head and two hydrophobic chains. As shown in figure 4, in water the hydrophobic chains group together and form a lipid bilayer. This is the basis of most of the biologically relevant membranes.

For an analytical description, one needs the coordinate \( \vec{r}(x) \) of the membrane as a function of an internal parameter \( x \), characterized by the mapping

\[
\vec{r} : x \in \mathbb{R}^2 \longrightarrow \vec{r}(x) \in \mathbb{R}^d
\]

and by the induced metric

\[
g_{\alpha\beta} = \partial_\alpha \vec{r} \partial_\beta \vec{r}.
\]

We are now looking for the statistical weight of a membrane configuration. Since the lipid molecules in the membrane are free to move around, the energy, i.e. “Hamiltonian” of the membrane has to be invariant under coordinate transformations. This is achieved by the Canham-Helfrich Hamiltonian \([41, 42]\)

\[
H[\vec{r}] = \int d^2x \sqrt{g(x)} \left[ \tau + \frac{\kappa}{2} (H(x) - H_0)^2 \right].
\]

\( d^2x \sqrt{g} \) is the invariant volume-element of the membrane, \( \tau \) its surface tension, and \( \kappa \) the bending-rigidity, which is coupled to the square of the mean curvature

\[
H = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right),
\]

where \( R_1 \) and \( R_2 \) are the two curvature radii. \( H_0 \) is a spontaneous curvature, present in the case of symmetry breaking between the two sides of the membrane. Physically, rigidity is explained by the finite thickness of the membrane. RG-calculations indicate that bending-rigidity should be irrelevant at large distances \([43, 44]\); this however has recently been criticized in \([45]\).

Experimentally, fluid membranes offer a wide range of interesting and complex phenomena. Let us only mention the budding of a fluid membrane, as given in figure 1 and the appearance of higher genus objects (figure 3).

For a general review about fluid membranes, see \([44, 46, 47, 48]\).

Interestingly, the Hamiltonian (3) with \( \kappa = 0 \) also plays a central role in string theory. Here, one of the inner coordinates \( \vec{x} = (x_1, x_2) \) is identified as \( i \times \text{time} \), and the other one as length on the string. Eq. (3) is

![Figure 4: Model of a fluid membrane: Bilayer of lipid molecules that are composed of a hydrophilic head and two hydrophobic hydrocarbon chains.](image-url)
then the action generating the motion of the string. Further generalizations use a metric $g_{\alpha\beta}$ independent of the imbedding space \[8, 49, 9, 10, 11\]. Strings are considered as one of the most promising candidates for unifying all fundamental interactions.

### 2.2 Tethered (polymerized) membranes

A microscopic model is given by the so-called “spring and bead model” (see figure 5), which consists of balls (beads) which are connected by springs and form a regular lattice. The model membrane is called “self-avoiding” since the beads cannot intersect each other. We will discuss Monte-Carlo simulations of this model in section 2.6.

A simpler situation occurs when self-intersections are allowed (“phantom-membrane”). Simulations as well as renormalization group calculations \[50, 51\] indicate that such a membrane is crumpled for weak bending rigidity, $\kappa < \kappa_c$ and flat for $\kappa > \kappa_c$. At the phase-transition point $\kappa = \kappa_c$, the membrane is in another critical (or more precisely tricritical) state with a fractal dimension $d_f$ in between the dimensions of the crumpled and flat phases. A mean-field treatment of this so-called “crumpling transition” is given in section 2.3. Contrary to intuition, the flat phase is not destroyed by fluctuations. This is demonstrated
in section 2.4, where also the tricritical state at $\kappa = \kappa_c$ is discussed. On the other hand, in the small-rigidity phase, phantom-membranes will have a fractal dimension of infinity. For physical (self-avoiding) membranes which can not intersect themselves, this is clearly impossible, and one expects the physical bound

$$d_f \leq d$$  \hspace{1cm} (6)

induced by self-avoidance.

A continuous model to describe a self-avoiding membrane is

$$\mathcal{H}[\vec{r}] = \int d^Dx \frac{1}{2} (\nabla \vec{r}(x))^2 + \frac{b}{2} \int d^Dx \int d^Dy \delta^{(D)}(\vec{r}(x) - \vec{r}(y)) .$$ \hspace{1cm} (7)

It has first been proposed by Edwards [4] to describe polymers ($D = 1$). In that case, it is equivalent to scalar $\phi^4$-field theory in the limit of $N = 0$ components [7]. In 1986 the model has been generalized to membranes ($D = 2$), independently by Kardar and Nelson [17, 18] and by Aronovitz and Lubensky [19]. They observed that a direct calculation at $D = 2$ is impossible, but that one can make an analytic continuation from $D < 2$.

In contrast to polymers, with their equivalence to scalar field-theory, renormalization is not evident. At leading order, renormalizability has been verified by Duplantier, Hwa and Kardar in 1990 [52]. For the general case, an important step was achieved by David, Duplantier and Guitter [53, 54] who showed renormalizability of the theory

$$\mathcal{H}[\vec{r}] = \int d^Dx \frac{1}{2} (\nabla \vec{r}(x))^2 + g \int d^Dx \delta^{(D)}(\vec{r}(x)) ,$$ \hspace{1cm} (8)

which describes a phantom (non self-avoiding) membrane in interaction with a single point (an impurity). The proof is based on a generalization of the forest algorithm introduced by Zimmermann [55] to $\delta$-like interactions. Their last step was to prove the renormalizability of the full model [56, 57], which we shall describe in section 5.

To extract numerical predictions from the $\varepsilon$-expansion is a tedious task. One of the problems is that since one cannot start from $D = 2$, an analytic continuation has to be performed starting at any point $(D, d)$ on the critical curve, which will be defined in section 3.1. The first calculations which tried to fix the expansion-point via a minimal sensitivity scheme at 1-loop order were performed in [22]. The result of $d_f \approx 3.5$ for membranes in 3 dimensions even violated the geometric bound of 3 discussed above. It became therefore necessary to perform 2-loop calculations, not only to test the renormalization proof, but also to obtain more reliable values for the fractal dimension. This task was accomplished in [25, 26], and we review the main steps in section 6. For membranes in 3 dimensions these calculations predict a fractal dimension of about 2.4, eventually seen in some experiments and numerical simulations, see sections 2.5 and 2.6.

It is interesting to note that the model (7) can also be used to study self-avoiding fractal objects like Sierpinsky gaskets [58]. (But attention: One has to be careful in distinguishing the fractal and the spectral dimensions of the membrane.)

Let us mention still another class of membranes, namely hexatic membranes. They play an intermediate role between tethered and fluid membranes. For a review see [46] and [59, 60, 61].

In the rest of this section, we review some simple arguments for tethered membranes, as well as experiments.
2.3 Crumpling transition, the role of bending rigidity, and some approximations

Let us start by studying the different terms appearing in a mean-field description of membranes. Let

$$\vec{r} : x \in \mathbb{R}^D \longrightarrow \vec{r}(x) \in \mathbb{R}^d$$

be the coordinates of a $D$-dimensional manifold embedded into a $d$-dimensional space. For $D = 1$, this represents a polymer, for $D = 2$ a membrane. Suppose that the underlying lattice is regular and that after integration over the fast degrees of freedom the effective model becomes translationally invariant. An expansion à la Landau then leads to an effective free energy or “Hamiltonian” [62]

$$\mathcal{H}[\vec{r}(x)] = \int d^Dx \left( \frac{\kappa}{2} (\partial_\alpha \partial_\alpha r)^2 + t (\partial_\alpha r)^2 + u (\partial_\alpha r \partial_\beta r)^2 + v (\partial_\alpha r \partial_\alpha r)^2 \right) + b \int d^Dx \int d^Dy \delta^d(\vec{r}(x) - \vec{r}(y)).$$

(10)

The last term, a self-repulsion upon contact, is a non-local interaction in the internal coordinates $x$, but local in the membrane position $\vec{r}(x)$. The local terms are the different contributions to the elastic energy. The coefficients $t$, $u$ and $v$ weight the elastic and inelastic harmonic energies, whereas $\kappa$ measures the bending-rigidity.

The analogy to the usual $\phi^4$-theory becomes apparent upon identifying the tangents $\vec{t}_\alpha := \partial_\alpha \vec{r}$ as order-parameter. However, this analogy is only valid at the mean-field level, and will be destroyed by fluctuations. Mean-field theory suggests a phase-transition at $t = 0$, where the parameter $t$ is equal to $T - T_c$, the difference in temperature $T$ to the critical temperature $T_c$.

At high temperature, $t$ is positive due to entropy and the correlation between the tangential vectors decays exponentially fast. The membrane is in a crumpled phase.

For negative $t$, the terms proportional to $(\partial r)^4$ restore positivity of the action, provided that $u + Dv > 0$ and $u + Dv > 0$. The symmetry is spontaneously broken, and the order-parameter $\vec{t}_\alpha$ has a non-zero expectation value, of the form $\vec{t}_\alpha = \zeta \vec{e}_\alpha$, where $\vec{e}_\alpha$ is a set of orthonormal base vectors. At zero temperature, the membrane is in a flat (ordered) phase, with

$$\zeta = \frac{1}{2} \sqrt{\frac{|t|}{u + Dv}}.$$  

(11)

This resembles the XY-model in 2 dimensions. There, long-range order is destroyed by spin-waves. We shall see in the next section, that fluctuations renormalize the rigidity of the membrane and render it stiffer. This renormalization is sufficient to make the membrane flat. For further discussion of the thermodynamic behavior see [63].

To incorporate self-avoidance, let us use the Flory-approximation. This consists in replacing $\vec{r}(x)$ by the radius of gyration $R_G$ and derivatives with respect to $x$ by $1/L$, as well as the integration over $x$ by $L^D$, where $L$ is the size of the flat membrane. This leads (up to numerical factors) to

$$\mathcal{H} \approx \kappa L^{D-4} R_G^2 + tL^{D-2} R_G^2 + (u + Dv) L^{D-4} R_G^4 + b L^{2D} R_G^{-d}.$$  

(12)

First of all, the bending-rigidity $\kappa$ can always be neglected with respect to $t$ and $u$. For $t < 0$ and in the physical region $(D \leq d)$, the terms proportional to $t$ and $u + Dv$ dominate and minimizing the free energy leads to

$$R_G \sim L.$$  

(13)

Self-avoidance can be neglected at large scale.
Basic properties of membranes

Figure 6: Free energy for $t < 0$ (left) and $t > 0$ (right) in the limit of large membranes.

For $t > 0$, self-avoidance prevents the membrane from collapsing, and balancing the terms of order $t$ and $b$ gives

$$R_G \sim L^{\nu_{\text{Flory}}}$$

with the Flory-exponent

$$\nu_{\text{Flory}} = \frac{2 + D}{2 + d}.$$  \hspace{1cm} (15)

We will show in section 7.5 that Eq. (15) is a reasonable approximation in the crumpled phase. In general we will find

$$R_G \sim L^{\nu^*}$$

with some non-trivial exponent $\nu^*$.

Let us still mention the results for $\nu^*$ in the crumpled phase, obtained by a Gaussian variational approximation. We shall show in section 7.4 that this approximation becomes exact in the limit of $d \to \infty$ with probably exponentially small corrections. The work by Goulian[64], Le Doussal[65] and Guitter and Palmeri[66] predicts:

$$\nu_{\text{var}} = \frac{2D}{d}.$$  \hspace{1cm} (17)

For 2-dimensional membranes ($D = 2$), this differs from the Flory approximation by terms of order $1/d^2$.

2.4 Stability of the flat phase

In the last section, we saw that a simple scaling analysis suggests the existence of a flat phase. This phase could of course be destroyed by fluctuations. We shall show here that this is indeed the case for fluid membranes, but that a non-zero shear-modulus, i.e. a fixed connectivity, stabilizes the membrane in the flat phase [59].

Our presentation is largely inspired by the lecture of Nelson [67], but we will use an $\epsilon$-expansion here instead of a self-consistent approximation.

To describe fluctuations of a membrane with inner coordinates $x = (x_1, x_2)$ around a flat configuration,
it is advantageous to use the representation
\[
\vec{r}(x_1, x_2) = \zeta \left( \begin{array}{c} x_1 + u_1(x_1, x_2) \\ x_2 + u_2(x_1, x_2) \\ h(x_1, x_2) \end{array} \right).
\] (18)

The line-element \(d\vec{r}\) is
\[
d\vec{r} = \zeta \left( \begin{array}{c} (1 + \partial_1 u_1) \, dx_1 + \partial_2 u_1 \, dx_2 \\ \partial_1 u_2 \, dx_1 + (1 + \partial_2 u_2) \, dx_2 \\ \partial_1 h \, dx_1 + \partial_2 h \, dx_2 \end{array} \right).
\] (19)

The deformation of this line-element is described by the deformation-matrix \(u_{\alpha\beta}\) [68]
\[
dr^2 = \zeta^2 \left( d^2x + 2u_{\alpha\beta} \, dx_\alpha \, dx_\beta \right).
\] (20)

With the help of Eq. (19) we find:
\[
u_{\alpha\beta} = \frac{1}{2} \left( \partial_\alpha u_{\beta} + \partial_\beta u_{\alpha} \right) + \frac{1}{2} \left\{ \partial_\alpha h(\partial_\beta h) + \frac{1}{2}(\partial_\alpha u_{\gamma})(\partial_\beta u_{\gamma}) \right\}.
\] (21)

The last term is of higher order in \(u\) and can be neglected in the following. (It has to be included at order \(\varepsilon^2\).) We shall thus use
\[
u_{\alpha\beta} \approx \frac{1}{2} \left( \partial_\alpha u_{\beta} + \partial_\beta u_{\alpha} \right) + \frac{1}{2}(\partial_\alpha h)(\partial_\beta h).
\] (22)

The energy of a nearly flat membrane is the sum of bending-rigidity and deformation energy
\[
\mathcal{H}[u, h] = \int d^2x \frac{k}{2} (\Delta h)^2 + \frac{1}{2} \left\{ 2\bar{\mu}u_{\alpha\beta}^2 + \lambda u_{\gamma\gamma}^2 \right\}.
\] (23)

\(\bar{\mu}\) and \(\lambda\) are the Lamé-coefficients [68]. (We use \(\bar{\mu}\) instead of the usual notation of \(\mu\) [68] to reserve \(\mu\) for the renormalization scale.) \(\bar{\kappa}\), \(\bar{\mu}\) and \(\lambda\) are related to \(\kappa\), \(u\) and \(v\) by \(\bar{\kappa} = \kappa\zeta^2\), \(\bar{\mu} = 4u\zeta^4\) and \(\lambda = 8u\zeta^4\).

In this expression, the displacement vector \(u_\alpha\) appears only quadratic and can thus be eliminated by calculating its path-integral
\[
\mathcal{H}_{\text{eff}}[h] = -k_B T \ln \left[ \int D[u] e^{-\mathcal{H}[u, h]/k_B T} \right].
\] (24)

We separate in \(u_{\alpha\beta}(x)\) the \((q = 0)\)-mode and use for the other modes the Fourier decomposition
\[
u_{\alpha\beta}(x) = u^0_{\alpha\beta} + A^0_{\alpha\beta} + \sum_{q \neq 0} \left( \frac{i}{2} [q_\alpha \bar{u}_\beta(q) + q_\beta \bar{u}_\alpha(q)] + \tilde{A}_{\alpha\beta}(q) \right) e^{iqx},
\] (25)

where
\[
\tilde{u}_\alpha(q) = \int d^2x e^{-iqx} u_\alpha(x)
\] (26)

and \(\tilde{A}_{\alpha\beta}(q)\) is the Fourier transform of \(\bar{A}_{\alpha\beta}(x) = \frac{1}{2} \partial_\alpha h(x) \partial_\beta h(x)\):
\[
\tilde{A}_{\alpha\beta}(q) = \frac{1}{2} \int d^2x e^{-iqx} \partial_\alpha h(x) \partial_\beta h(x).
\] (27)

For \(q \neq 0\), \(\tilde{A}_{\alpha\beta}(q)\) is now decomposed into its longitudinal and transversal parts. (That this is indeed possible is shown in appendix A.3.)
\[
\tilde{A}_{\alpha\beta}(q) = \frac{i}{2} [q_\alpha \bar{\varphi}_\beta(q) + q_\beta \bar{\varphi}_\alpha(q)] + P^\text{T}_{\alpha\beta}(q) \tilde{\Phi}(q),
\] (28)
where
\[ P^T_{\alpha\beta}(q) = \delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \] (29)
is the transversal projector and
\[ \tilde{\Phi}(q) = P^T_{\alpha\beta}(q) \tilde{A}_{\alpha\beta}(q) . \] (30)
We can now absorb the longitudinal part \( \tilde{\varphi}_\beta(q) \) of \( \tilde{A}_{\alpha\beta}(q) \) by shifting the variable \( \tilde{u}_\alpha(q) \):
\[ \tilde{u}_\alpha(q) \longrightarrow \tilde{u}_\alpha(q) - \tilde{\varphi}_\alpha(q) . \] (31)
It remains to integrate over \( \tilde{u}_\alpha(q) \). To this aim expand
\[ T := 2\bar{\mu} \tilde{u}_{\alpha\beta}(q) \tilde{u}_{\alpha\beta}(-q) + \lambda \tilde{u}_{\alpha\alpha}(q) \tilde{u}_{\alpha\alpha}(-q) \] (32)
in the basis of rotational invariants \( q^2, |q\tilde{u}(q)|^2 \) and \( \tilde{u}(q)\tilde{u}(-q) \):
\[ T = \bar{\mu} q^2 |\tilde{u}(q)|^2 + (\bar{\mu} + \lambda)|q\tilde{u}(q)|^2 + (2\bar{\mu} + \lambda)|\tilde{\Phi}(q)|^2 + \lambda \left( iq\tilde{u}(q)\tilde{\Phi}(-q) + c.c. \right) . \] (33)
By a second variable-transformation
\[ \tilde{u}_\alpha(q) \longrightarrow \tilde{u}_\alpha(q) + \frac{\lambda}{2\bar{\mu} + \lambda} \frac{iq_\alpha}{q^2} \tilde{\Phi}(q) \] (34)
terms proportional to \( \tilde{\Phi} \) and \( \tilde{u} \) are decoupled and we obtain
\[ T = \frac{4\bar{\mu}(\bar{\mu} + \lambda)}{2\bar{\mu} + \lambda} \tilde{\Phi}(q)\tilde{\Phi}(-q) + \text{quadratic terms in } \tilde{u} . \] (35)
Up to a constant, the effective Hamiltonian (24) thus becomes
\[ \mathcal{H}_{\text{eff}}[h] = \frac{\kappa}{2} \int d^2 x \left( \Delta h \right)^2 + \frac{K}{2} \int d^2 x \left( P^T_{\alpha\beta} [\partial_\alpha h(x) \partial_\beta h(x)] \right)^2 . \] (36)
The “prime” indicates that the 0-mode is excluded from the integral. The coupling constant \( \bar{K} \) is
\[ \bar{K} = \frac{\bar{\mu}(\bar{\mu} + \lambda)}{2\bar{\mu} + \lambda} . \] (37)
We see that the shear-modulus \( \bar{\mu} \) is responsible for the interaction. For fluid membranes, \( \bar{\mu} = 0 \) and no correction appears, even if \( \lambda \neq 0 \).

We shall now study Eq. (36) in perturbation theory, by using an \( \epsilon = 4 - D \) expansion. A similar technique was employed by Aronovitz and Lubensky in [20], where they study the RG-flow for all fields. A self-consistent method was utilized in [59, 67].

To carry out an \( \epsilon \)-expansion, we rewrite the effective Hamiltonian (36) as
\[ \mathcal{H}_{\text{eff}}[h] = \frac{Z}{2} \int d^D x \left( \Delta h \right)^2 + \frac{K}{2} Z_K \mu^\epsilon \int d^D x \left( P^T_{\alpha\beta} [\partial_\alpha h(x) \partial_\beta h(x)] \right)^2 , \] (38)
where \( \kappa \) has been absorbed into the field-normalizations (\( h \rightarrow h/\sqrt{\kappa} \)) and
\[ K_0 = \frac{\bar{K}}{\kappa^2} = \frac{Z_K}{Z^2} \mu^\epsilon \]
\[ h_0(x) = \sqrt{Z} h(x) . \] (39)
The renormalization factors $Z$ and $Z_k$ absorb the divergences and are fixed by the minimal subtraction scheme. $\mu$ is the renormalization scale, $\epsilon = 4 - D$ the dimension of the bare coupling. Bare quantities are indexed as $0$. The vertex is

$$\frac{K}{2} \times \frac{\delta^D(p_1 + p_2 + q_1 + q_2)}{(2\pi)^D} \prod_{i=1,2} \frac{(p^i q^i)^2 - (p^i)^2 (q^i)^2}{(p^i + q^i)^2}.$$  \hfill (40)

We shall now calculate perturbative corrections. As the 0-mode is excluded from the integration, the contribution to $\kappa$ coming from the “tadpole” is 0:

$$= 0.$$  \hfill (41)

The second contribution to the renormalization of $\kappa$ is:

$$= \int_k \left( \frac{(pk)^2 - p^2 k^2}{(p + k)^2} \right)^2 \frac{1}{k^4}.$$  \hfill (42)

A divergence for $k \to \infty$ is manifest as a pole in $1/\epsilon$ with positive residue $C$ (which needs not be specified):

$$= (p^2)^2 \frac{C}{\epsilon} p^{-\epsilon}.$$  \hfill (43)

The divergence of this diagram is subtracted at scale $\mu$ by choosing

$$Z = 1 - \frac{2C}{\epsilon} K.$$  \hfill (44)

The sign is such that the interaction re-enforces the bending-rigidity. To analyze the renormalization of the vertex, we remark that due to the transversal projector, all three possible diagrams are convergent:

$$, , , .$$  \hfill (45)

This is not evident from power-counting. Hence at 1-loop order

$$Z_K = 1 ,$$  \hfill (46)

and renormalization becomes particularly simple. The function $\beta(K)$ and the full scaling dimension $\zeta(K)$ of the field $h$, the roughness-exponent, are obtained from Eq. (39) as

$$\beta(K) = \mu \frac{\partial}{\partial \mu} \bigg|_0 K = \frac{-\epsilon K}{1 + K \frac{\partial}{\partial K} \ln Z_K - 2 K \frac{\partial}{\partial K} \ln Z}$$  \hfill (47)

$$\zeta(K) = \frac{4 - D}{2} - \frac{1}{2} \mu \frac{\partial}{\partial \mu} \bigg|_0 \ln Z = \frac{4 - D}{2} - \frac{1}{2} \beta(K) \frac{\partial}{\partial K} \ln Z .$$  \hfill (48)
Since $\mathcal{C}$ is positive, the $\beta$-function possesses a positive, IR-stable fixed point at 1-loop order, which we denote $K^*$. Then

$$\zeta^* = \zeta(K^*) = \frac{4 - D}{4} + O(\epsilon^2).$$

(This result could have faster been obtained by using the method of exact exponent identities explained in section 3.9.) In $D = 2$

$$\zeta^* = \frac{1}{2} + O(\epsilon^2).$$

This can be interpreted as an effective $k$-dependent bending-rigidity

$$\kappa_{\text{eff}}(k) \approx \tilde{\kappa} \frac{\mu}{k}.\quad(51)$$

We can now analyze the stability of the flat phase. Following De Gennes and Taupin [69], we estimate the fluctuations of the normal to the surface projected on $x_3$ (the component parallel to $h(x)$):

$$n_3(x) = \frac{1}{\sqrt{1 + (\nabla h(x))^2}}.\quad(52)$$

The first term of the expansion is the mean of $(\nabla h(x))^2$. Without interaction ($K = 0$) it is:

$$\langle (\nabla h(x))^2 \rangle_0 = k_B T \int \frac{d^2 q}{(2\pi)^2} \frac{q^2}{\tilde{\kappa} q^4} \approx \frac{k_B T}{2\pi \tilde{\kappa}} \ln \left( \frac{L}{a} \right),\quad(53)$$

where $L$ and $a$ are IR and UV-cutoffs. As for many two-dimensional systems, the logarithmic divergence at large distances indicates that order is destroyed by fluctuations.
For membranes with non-zero shear-modulus, the estimate (53) is incorrect. One has to take care of the
renormalization of $\kappa$, hence replace $\kappa$ in Eq. (53) by $\kappa_{\text{eff}}(k)$, given by Eq. (51). This yields:

$$\left\langle (\nabla h(x))^2 \right\rangle_{\kappa_{\text{eff}}} = k_BT \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{\kappa_{\text{eff}}(q)} = \text{IR-convergent}.$$  \hspace{1cm} (54)

The normals keep their preferred direction parallel to $x_3$, even for systems with infinite size. The symmetry
is broken and the membrane flat. This seems to be a violation of the Mermin-Wagner theorem: In fact, the
fluctuations in the membrane give rise to long-range interactions, for which the Mermin-Wagner theorem
is not valid.

To conclude: As soon as the membrane is in the phase of high bending-rigidity, i.e. the flat phase, the
in-membrane fluctuations reinforce the bending-rigidity and stabilize the membrane. Stated differently:
The fixed point of the flat phase is attractive.

Nevertheless, the fluctuations in the height $h$ are large and described by a non-trivial roughness exponent $\zeta$

$$\left\langle (h(x) - h(y))^2 \right\rangle \sim |x - y|^{2\zeta}.$$  \hspace{1cm} (55)

This exponent was estimated above to be $\frac{1}{2}$. It can also be calculated by an expansion in $1/d$ [70], $\epsilon = 4 - D$
[50] or within a self-consistent screening approximation [71] and can be compared with experiments [72],
and numerics [16, 73, 74, 75, 76, 77]. This should rule out the value of $\zeta = \frac{1}{2}$, proposed in [78, 79, 80].
This is summarized in figure 7.

We have also mentioned above that the crumpling transition occurs at a critical value of the bending
rigidity. This transition point is a different tri-critical state, accessible to renormalization-group treatments
and numerics. The fractal exponent $\nu^*$ is then 0 in the crumpled phase, 1 in the flat phase, and at the
crumpling transition given by the $1/d$-estimate [70, 51]

$$\nu^*_c = 1 - \frac{1}{d},$$  \hspace{1cm} (56)

which agrees with numerical values in $d = 3$ [81, 82]. See also [46, 50, 63, 84, 85, 86, 87].

Also see [88] for a study of the membrane elasticity at low temperatures and [89] for a stack of mem-
branes.

### 2.5 Experiments on tethered membranes

Few experiments have been realized up to now. The most promising are:

- The spectrin-network of red blood cells forms a natural membrane, easily accessible experimentally
[94, 72]. The inconvenience of this system is the large intrinsic bending-rigidity which first has to
be reduced. No experiment showing a crumpled phase has been done. In the flat phase, one finds an
anomalous roughness exponent $\zeta$ of about $\zeta_{\text{flat}} \approx 0.6$ [72], as discussed at the end of the preceding
subsection.

- 2-dimensional networks of polymers [95] seem to be promising. However, experimental measure-
ments are missing. Recently, Rehage and coworkers have succeeded in producing sufficiently highly
polymerized membranes [96] and experiments to find the fractal phase are planned [97].

- Molybdenum disulfide (MoS$_2$) can be produced in extremely pure form. The experiments which we
know of [98] find it in a strongly folded phase.
Graphite oxide: For this material, experiments have been realized: Graphite is a layered material, and only very weak (van der Waals) forces exist between different layers. One therefore may cut out a piece of such a layer. By an exothermic reaction of graphite with some oxidant (the principle of black powder), one obtains a sample which consists of pieces of a single layer of graphite, decorated with oxygen-atoms at its border. One expects that these membranes have a very small bending-rigidity.

The first experiments undertaken by Hwa et al. [93] have shown such a crumpled phase with a fractal dimension near to the Flory results ($d_f = 2.5$) besides a collapsed and a flat phase. This was achieved by varying the concentration of $\text{H}^+$ of the dispersion. In later experiments by Spector et al. [99] this intermediate phase was no longer observed. The interpretation of these experiments is however
not unambiguous. Extrapolating the light-scattering data of [93] reproduced on figure 10 predicts a fractal dimension of $d_f = 2.4$ whereas the very similar data of [99] lead to $d_f = 2.3$. However, based on a technique, where the sample is frozen ultra-fast, then cut into thin samples and analyzed via transmission electron microscopy, the authors of [99] were unable to see fractal objects and therefore concluded on the absence of a fractal phase. This debate certainly deserves further clarification. For more details see [100].

In summary: The experimental situation is not very transparent.

Let us still mention another very amusing class of experiments. Crunching a thin aluminum foil in the attempt to form a ball [101, 102], also allows to measure a fractal dimension, which turns out to be very close to the Flory-result of Eq. (15). This result is easily reproduced on a table-top experiment with paper, see figure 11. However, since crunching aluminum foil is certainly a non-equilibrium process, this may be a coincidence.

### 2.6 Numerical simulations of self-avoiding membranes

In this section we review existing numerical simulations of tethered membranes. If not stated otherwise, these are membranes ($D = 2$) embedded into 3 dimensions.

The first simulations for self-avoiding membranes were performed for very small systems (121 beads) by Kantor, Kardar and Nelson [14, 15] in 1986. They obtained $\nu^* = 0.80 \pm 0.05$ in agreement with the Flory-approximation. Here, as in most of the simulations, self-avoidance is effective between the beads (of finite size) of the network. There exists thus a maximal angle smaller than $\pi$, by which the membranes can be folded. (For a visualization, see figure 5.)

As we discussed in section 2.4, phantom membranes show a crumpling transition induced by bending-rigidity. Shortly after this had been established numerically [81, 82], an attempt was made to study this transition in the presence of self-avoidance [103, 104, 105, 106, 107]. The transition has completely disappeared and the membranes were always found flat for any (positive) value of the bending-rigidity. A simple explanation due to Abraham and Nelson [16] goes as follows: the simulated model consists out of beads (of finite size) and tethers linking the beads together. The tether-length is chosen such that the beads
Basic properties of membranes

\[ \log_2(1/L) = \frac{1}{2} \text{ (German DinA size)} \]

Figure 11: Result of crunching a sheet of paper of linear size \( L \) to a ball of diameter \( R \). This leads to a fractal dimension of \( d_f = 2.4 \), equivalent to \( \nu^* = 0.82 \).

cannot penetrate through the holes left in-between. Then, the range of possible configurations is restricted and is re-interpreted as an effective bending-rigidity. This bending-rigidity was claimed responsible for the flat phase, following the scenario of the crumpling transition of a phantom membrane, induced by bending-rigidity.

The question therefore arises, whether the flat phase is an artifact of the simulations, or whether it is generic. Let us mention two simulations in this context: The first is due to Kantor and Kremer \[108\]. They studied the usual bead-and-tether model, but restricted self-avoidance on the membrane to a finite distance \( l \). Since now the interaction is local, one can study the crumpling transition induced by the bending-rigidity \( \sigma \). For \( \sigma > \sigma_c \) a flat phase is found, whereas for \( \sigma < \sigma_c \) the membrane is found in a crumpled state. Taking now the limit of large \( l \), the value of the critical bending-rigidity \( \sigma_c \) scales to 0. They then concluded that this indicates that the flat phase persists down to \( \sigma_c = 0 \). It would be nice to have more extensive simulations available than the 169 to 331 beads studied there.

In another simulation, Liu and Plischke \[109\] have found an intermediate fractal phase by adding long-range attraction, and then adjusting the temperature. This intermediate phase was found for some range of temperature and membranes of up to 817 particles. In a similar simulation, Grest and Petsche \[110\] were also able to find this intermediate phase, but only for a specific value of the temperature. This is not surprising from the renormalization-group point of view: Long-range forces are in general relevant operators, such that a fine-tuning is necessary to reach the critical point. Let us also mention another trick used in \[110\]: They rendered the membrane much more flexible by adding additional beads between the nodes of the lattice, forming the membrane.

A similar idea is to dilute the membrane by randomly cutting off links \[111, 112\]. This attempt was not very fruitful: The flat phase persisted up to the percolation threshold.

The best numerical realization of tethered membranes is obtained by imposing self-avoidance not between beads but between the plaquettes forming the membrane. The first such simulation was carried out by Baumgärtner et al. \[113, 114\], who indeed found the fractal phase. Within a very similar simulation, Kroll and Gompper \[115\] were not able to confirm these conclusions. A repetition of these simulations with larger systems as those studied there (up to 496 plaquettes) would be very much welcome to clarify
the situation.

Other interesting simulations are for membranes in a 4-, 5-, 6- and 8-dimensional space. Grest found in [116] that membranes are flat in dimensions \( d = 4 \), but crumpled swollen in larger dimensions. Complementary simulations by Barsky and Plischke [117] confirm this conclusion. These simulations are in agreement with the value of \( \nu^* \) predicted by the Gaussian variational ansatz, \( \nu_{\text{var}} = 2D/d \) (see section 7.4), and larger than the 2-loop results (see figure 20 of page 80).

Remains to mention simulations on a Sierpinsky gasket with fractal dimension of about 1.585 and spectral dimension of about 1.356 [58]. As in the case of polymers, the results for \( d = 3 \) are in agreement with the Flory-approximation Eq. (15).

Also the folding transition of a membrane has been studied numerically [118].

Let us also mention studies of tethered membranes in confined geometries [119, 120, 121], of boundary effects [122], with negative bending-rigidity [123], of dynamics [124], and a couple of short reviews about the simulational aspects of tethered membranes [125, 126].

### 2.7 Membranes with intrinsic disorder

A lot of publications have been devoted to the treatment of tethered (phantom) membranes with intrinsic disorder, including two-dimensional gels [71, 111, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144]. Let us give a brief summary of the main ideas, following the first publications [127, 128, 129, 130, 131]. Two kinds of disorder can be added. Since we are interested in the stability of the flat phase to such disorder, we study the Hamiltonian of a membrane in an expansion about a flat configuration, generalizing Eq. (23). We consider the general case of a \( D \)-dimensional membrane embedded in a \( d \)-dimensional space, such that

\[
\overline{r}(x) = \zeta \left( x_\alpha + u_\alpha(x) \right) \frac{1}{h^j(x)},
\]

where \( u(x) \in \mathbb{R}^D \) describes the \( D \) in-membrane (stretching) modes and \( \mathbf{h}(x) \in \mathbb{R}^{d-D} \) the fluctuations in the \( d-D \) transverse directions. The full Hamiltonian then reads in generalization of Eq. (23)

\[
\mathcal{H}[u, h] = \int d^Dx \frac{\bar{\kappa}}{2} (\Delta \mathbf{h})^2 + \frac{1}{2} \left[ 2\bar{\mu} u_{\alpha\beta}^2 + \lambda u_{\gamma\gamma}^2 \right] + \sigma_{\alpha\beta}(x)u_{\alpha\beta}(x) + \mathbf{c}(x)\Delta \mathbf{h}(x),
\]

where we recall the definition of the deformation-matrix

\[
u_{\alpha\beta}(x) \text{ is a quenched random stress field, or variation of the metric. Microscopically it is due to different tether-lengths in the spring and bead model of figure 5. } \mathbf{c}(x) \text{ is a quenched random curvature field, favoring the mean curvature } \Delta \mathbf{h}(x), \text{ and breaking the reflection symmetry between the two sides of the membrane. It may be caused by a local difference in the chemical composition between the two sides of the membrane. The correlations are short ranged, of the form}

\[
\overline{\sigma_{\alpha\beta}(x)\sigma_{\gamma\delta}(x')} = \frac{\Delta_{\chi\delta} \delta_{\alpha\beta} \delta_{\gamma\delta} + 2\Delta_{\mu}(\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma})}{\mathbf{c}(x)\mathbf{c}(x')} = \Delta_{\kappa\delta} \delta_{ij} \delta^D(x - x') \]
replicas. One can then parallel the calculations of the pure model. The outcome is that at finite temperature, the long-wavelength properties of the membrane are unchanged. New physics emerges at or very near to zero temperature, characterized by a new non-trivial fixed point. Membranes with non-zero random spontaneous curvature are found in a flat phase with non-trivial critical exponents, analogous to the flat phase of the pure model at non-zero temperature \([129, 130, 131]\). This fixed point is accessible within an \(\varepsilon\)-expansion. Membranes with disorder in the metric are more difficult to access, since the fixed point lies outside the perturbatively accessible domain \([127, 128]\).

3 Field theoretic treatment of tethered membranes

3.1 Definition of the model, observables, and perturbation expansion

We start from the continuous model for a \(D\)-dimensional flexible polymerized membrane introduced in \([19, 17]\). This model is a simple extension of the well known Edwards’ model for continuous chains. The membrane fluctuates in \(d\)-dimensional space. Points in the membrane are labeled by coordinates \(x \in \mathbb{R}^D\) and the configuration of the membrane in physical space is described by the field \(r : x \in \mathbb{R}^D \rightarrow r(x) \in \mathbb{R}^d\), i.e. from now on we note \(r\) instead of \(\vec{r}\). In section 2.3 we had discussed that at high temperatures the free energy for a configuration is given by the (properly rescaled) Hamiltonian

\[
\mathcal{H}[r] = \frac{Z}{2-D} \int_x \left( \frac{1}{2} (\nabla r(x))^2 + bZ_b \mu \varepsilon \int \mathcal{D}^d (r(x) - r(y)) \right). \tag{61}
\]

The so-called renormalization-factors \(Z\) and \(Z_b\) have the form \(Z = 1 + O(b)\) and \(Z_b = 1 + O(b)\); they will be explained later. The reader may safely set both to 1 for the moment. The integral \(\int_x\) runs over...
$D$-dimensional space and $\nabla$ is the usual gradient operator. The normalizations are

$$\int_x := \frac{1}{S_D} \int d^D x, \quad S_D = 2 \frac{\pi^{D/2}}{\Gamma(D/2)} \quad (62)$$

and

$$\tilde{\delta}^d(r(x) - r(y)) = (4\pi)^{d/2} \delta^d(r(x) - r(y)). \quad (63)$$

The latter term is normally used in Fourier-representation

$$\tilde{\delta}^d(r(x) - r(y)) = \int_p e^{ip[r(x) - r(y)]}, \quad (64)$$

where the normalization of $\int_p$ is given by

$$\int_p = \pi^{-d/2} \int d^d p \quad (65)$$

to have

$$\int_p e^{-p^2 a} = a^{-d/2}. \quad (66)$$

All normalizations are chosen in order to simplify the calculations, but are unimportant for the general understanding. (They are collected in appendix A.1). $\mu$ is an internal momentum scale, such that $\mu x$ is dimensionless. It is introduced to render the coupling $b$ dimensionless. The first term in the Hamiltonian is a Gaussian elastic energy which is known to describe the free “phantom” surface. The interaction term corresponds (for $b > 0$) to a weak repulsive interaction upon contact. The expectation values of physical observables are obtained by performing the average over all field-configurations $r(x)$ with the Boltzmann weight $e^{-\mathcal{H}[r]}$. This average can not be calculated exactly, but one can expand about the configurations of a phantom, i.e. non-interacting surface.

Such a perturbation theory is constructed by performing the series expansion in powers of the coupling constant $b$. This expansion suffers from ultraviolet (UV) divergences which have to be removed by renormalization and which are treated by dimensional regularization, i.e. analytical continuation in $D$ and $d$. A physical UV-cutoff could be introduced instead, but would render the calculations more complicated. Long-range infrared (IR) divergences also appear. They can be eliminated by using a finite membrane, or by studying translationally invariant observables, whose perturbative expansion is also IR-finite in the thermodynamic limit (infinite membrane). Such observables are “neutral” products of vertex operators

$$\mathcal{O} = \prod_{a=1}^N e^{ik_a r(x_a)}, \quad \sum_{a=1}^N k_a = 0. \quad (67)$$

An example is given at the end of subsection 3.3.

Let us now analyze the theory by power-counting. We use internal units $\mu \sim 1/x$, and note $[x]_x = 1$, and $[\mu]_x = -[\mu]_\mu = -1$. The dimension of the field and of the coupling-constant are:

$$\nu := [r]_x = \frac{2 - D}{2}, \quad \varepsilon := [b \mu^\varepsilon]_\mu = 2D - \nu d. \quad (68)$$

In the sense of Wilson [2] the interaction is relevant for $\varepsilon > 0$, see figure 12. Perturbation theory is then expected to be UV-finite except for subtractions associated to relevant operators. We shall come back to this point later.
For clarity, we represent graphically the different interaction terms which have to be considered. The local operators are

\[ \frac{1}{2} (\nabla r(x))^2 = \mathbf{1} \]  \hspace{0.5cm} (69)

\[ \mathbf{1} = \mathbf{1} \]  \hspace{0.5cm} (70)

The bi-local operator, the dipole, is

\[ \tilde{\delta}^d (r(x) - r(y)) = \cdots \cdots . \] \hspace{0.5cm} (71)

The expectation-value of an observable is

\[ \langle O[r] \rangle_b = \frac{\int D[r] O[r] e^{-\mathcal{H}[r]} \int D[r] e^{-\mathcal{H}[r]} }{\int D[r] e^{-\mathcal{H}[r]} }. \] \hspace{0.5cm} (72)

Perturbatively, all expectation-values are taken with respect to the free theory:

\[ \langle O[r] \rangle_0 = \frac{\int D[r] O[r] e^{-\frac{1}{2} D \int_x \frac{1}{2} (\nabla r(x))^2}}{\int D[r] e^{-\frac{1}{2} D \int_x \frac{1}{2} (\nabla r(x))^2}}. \] \hspace{0.5cm} (73)

A typical term in the expansion of (72) is

\[ (-bZ_b \mu \varepsilon)^n \int \cdots \int \int \langle O \cdots \cdots \cdots \cdots \rangle_0^c, \] \hspace{0.5cm} (74)

where the integral runs over the positions of all dipole-endpoints.

### 3.2 Locality of divergences

In this section, we show that all divergences are short distance divergences. Note that even for massless theories and in the absence of IR-divergences, this is not trivial. Divergences could as well appear, when some of the distances involved become equal, or multiple of each other. A simple counter-example is the integral of \(|a| - |b|\)\(^{-\nu d}\), where \(a\) and \(b\) are two of the distances involved.

That divergences only occur at short distances (i.e. when at least one of the distances involved tends to 0), is a consequence of Schoenbergs theorem [145]. Here, we present an proof, based on the equivalence with electrostatics.

We first state that with our choice of normalizations (see appendix A.1), the free correlation-function \(C(x_1, x_2)\)

\[ C(x_1, x_2) := \frac{1}{d} \left \langle \frac{1}{2} |r(x_1) - r(x_2)|^2 \right \rangle_0 = |x_1 - x_2|^{2-D} \]

\[ \equiv (2 - D) S_D \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \left (1 - e^{ip (x_1 - x_2)} \right) \] \hspace{0.5cm} (75)

is the Coulomb potential in \(D\) dimensions. Furthermore, the interaction part of the Hamiltonian \(\mathcal{H}\) is reminiscent of a dipole, and can be written as

\[ \mathcal{H}_{\text{int}} = bZ_b \mu \varepsilon \int \int \delta^d (r(x_1) - r(x_2)) \]

\[ = bZ_b \mu \varepsilon \int \int e^{ik [r(x_1) - r(x_2)]}, \] \hspace{0.5cm} (76)
where \( k \) may be seen as a \( d \)-component (vector-) charge.

The next step is to analyze the divergences appearing in the perturbative calculation of expectation values of observables. To simplify the calculations, we focus on the normalized partition function

\[
\frac{Z}{Z_0} = \frac{1}{Z_0} \sum_{\text{all configurations}} e^{-\mathcal{H}} = \langle e^{-\mathcal{H}_\text{int}} \rangle_0 . \tag{77}
\]

To exhibit the similarity to Coulomb systems, consider the second order term

\[
\frac{1}{2} \langle \mathcal{H}_{\text{int}}^2 \rangle_0 = \frac{(bZ_b\mu^2)^2}{2} \int \int \int \int \int \int \langle e^{i[p(x_1) - p(x_2)]} e^{i[p(y_1) - p(y_2)]} \rangle_0
\]

\[
= \frac{(bZ_b\mu^2)^2}{2} \int \int \int \int \int \int e^{-E_c} E_c = k^2C(x_1 - x_2) + p^2C(y_1 - y_2)
\]

\[
+ kp [C(x_1 - y_2) + C(x_2 - y_1) - C(x_1 - y_1) - C(x_2 - y_2)] , \tag{78}
\]

where \( E_c \) is the Coulomb-energy of a configuration of dipoles with charges \( \pm k \) and \( \pm p \), respectively. More generally, for any number of dipoles (and even for any Gaussian measure) we have

\[
\langle e^{i \sum k_i r(x_i)} \rangle_0 = e^{-E_c} , \quad E_c = \frac{1}{2} \sum_{i,j} \langle k_i r(x_i) k_j r(x_j) \rangle_0 . \tag{79}
\]

Since \( \sum_i k_i = 0 \), the latter can be rewritten with the help of the usual correlation function \( C(x - y) = \frac{1}{2d} \langle |r(x) - r(y)|^2 \rangle_0 \) as

\[
E_c = -\frac{1}{4d} \sum_{i,j} k_i k_j \langle |r(x_i) - r(x_j)|^2 \rangle_0 . \tag{80}
\]

As for any configuration of dipoles, specified by their coordinates and charges, the total charge is zero, the Coulomb-energy is bounded from below, i.e.

\[
E_c \geq 0 . \tag{81}
\]

Formally, this is proven by the following line of equalities (remember that \( D < 2 \))

\[
E_c = \frac{1}{2} \sum_{i,j} \langle k_i r(x_i) k_j r(x_j) \rangle_0
\]

\[
= \frac{(2 - D)S_D}{2} \int \frac{d^D p}{(2\pi)^D} \sum_{i,j} k_i k_j \frac{1}{p^2} e^{ip(x_i - x_j)}
\]

\[
= \frac{(2 - D)S_D}{2} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \left| \sum_i k_i e^{ipx_i} \right|^2 \geq 0 . \tag{82}
\]

The last inequality is again due to the global charge neutrality, which ensures convergence of the integral for small \( p \). Hence, \( E_c \) vanishes, if and only if the charge density vanishes everywhere. This implies that

\[
e^{-E_c} \leq 1 , \tag{83}
\]

and the equality is obtained for vanishing charge density. Noting \( E_c = \sum_{i,j} k_i k_j Q_{ij} \), Eq. (82) even states that as long as \( x_i \neq x_j \) for all \( i \neq j \), \( Q_{ij} \) is a non-degenerate form on the space of \( k_i \) with \( \sum_i k_i = 0 \).
implies that integrating $e^{-E_c}$ as in Eq. (78) over all $k_i$ with $\sum_i k_i = 0$ gives a finite result, as long as not some of the $x_i$ coalesce. Consequently, divergences in the integration over $x_i$ can only appear when at least some of the distances vanish, as stated above.

This does of course not rule out IR-divergences. We will see later that they are absent in translationally invariant observables. An explicit example is given at the end of the next section; for a proof see [57].

### 3.3 More about perturbation theory

Let us apply the above observation to evaluating the integrals in Eq. (78); this will give an intuitive idea of the kind of counter-terms needed to cancel the UV-divergences, as will be made formal later. The basic idea is to look for classes of configurations which are similar. The integral over the parameter which indexes such configurations is the product of a divergent factor, and a “representative” operator. For the case of two dipoles, one with charge $k$ and the other with charge $p - k$, and approaching its endpoints (as indicated by the dashed lines below), one only sees a single dipole with charge $p$ from far away, i.e.

$$k \begin{array}{c} \vdots \end{array} \begin{array}{c} \vdots \end{array} -k \approx p \begin{array}{c} \vdots \end{array} -p \times e^{-k^2(|s|^2-D+|t|^2-D)}. \quad (84)$$

The second factor on the r.h.s. contains the dominant part of the Coulomb energy $E_c = k^2(|s|^2-D+|t|^2-D)$ of the interaction between the two dipoles; $s$ and $t$ are the distances between the contracted (approached) ends. The integral over $k$ is now factorized, and we obtain

$$\int_k e^{-k^2(|s|^2-D+|t|^2-D)} = (|s|^2-D+|t|^2-D)^{-d/2}. \quad (85)$$

Finally integrating over $p$ in Eq. (84) gives back the $\delta$-interaction $\quad$ multiplied with $\begin{array}{c} \vdots \end{array} | \begin{array}{c} \vdots \end{array} |$$, where we define the coefficient as

$$\begin{array}{c} \vdots \end{array} | \begin{array}{c} \vdots \end{array} | = (|s|^2-D+|t|^2-D)^{-d/2}. \quad (86)$$

The notation, which will be explained later, reminds of a scalar product or projection of a singular configuration of two dipoles onto a single dipole. Eq. (86) contains the dominant UV-divergence upon approaching the endpoints; this will be made formal later.

As an example of an expectation value, use in Eq. (67) the observable $O = e^{ik[r(s)-r(t)]]}$, which is the generating function for the moments of $[r(s) - r(t)]$; the series up to first order in $b$ reads (remind $Z_b = 1 + O(b)$)

$$\langle O \rangle_b = e^{-k^2C(s-t)} \times \left\{ 1 + b\mu^2 \int_x \int_y \left[ 1 - \exp \left( \frac{k^2}{4} C(s-x)+C(t-y)-C(s-y)-C(x-y) \right)^2 \right] C(x-y)^{-d/2} \right\} + O(b^2). \quad (87)$$

Note that the integral over $x$ and $y$ is IR-convergent, but UV-divergent at $\varepsilon \leq 0$: There is a singularity for $|x - y| \rightarrow 0$. This is a general feature of such expectation values. The purpose of the rest of this section is to introduce the basic tools to handle these divergences. On the example of Eq. (87), this is verified in exercise E.6, see page 164.
3.4 Operator product expansion (OPE), a pedagogical example

Throughout this review, we will use the techniques of normal-ordering and operator product expansion to analyze the short distance behavior of the theory. Since their technical simplicity is as little recognized as their 1 to 1 correspondence to standard Feynman-graphs, we shall give here a pedagogical derivation of the 2-loop result for the exponent $\eta$ in standard scalar $\phi^4$ theory, before discussing the case of a membrane in the next section. Complementary material can be found in [146]. Readers familiar with the procedure can continue with section 3.5.

Define the renormalized $\phi^4$-Hamiltonian as

$$\mathcal{H} = \frac{Z}{d-2} \int_x \frac{1}{2}(\nabla \phi(x))^2 + bZ_b\mu^\epsilon \int_x :\phi^4(x):. \quad (88)$$

The integration measure is normalized as

$$\int_x = \frac{1}{S_d} \int d^d x, \quad S_d = 2 \frac{\pi^{d/2}}{\Gamma(d/2)}, \quad (89)$$

where $S_d$ is the surface of the $d$-dimensional unit sphere. This is done in order to obtain for the free expectation values (denoted by subscript $0$)

$$C(x-y) := \langle \phi(x)\phi(y) \rangle_0 = |x-y|^{2-d}. \quad (90)$$

Note the similarity and difference between the definitions in Eq. (75) and Eq. (90); the difference results from the 0-mode, which has to be subtracted in the case of polymers and membranes ($D < 2$), but not of the $\phi^4$-model ($d > 2$).

The dimensional regularization parameter $\epsilon$ is

$$\epsilon = 4 - d, \quad (91)$$

and $\mu$ is the renormalization (subtraction) scale. Note the difference to Eq. (68), where we use $\varepsilon$ instead of $\epsilon$. The renormalization $Z$-factors, introduced to render the theory finite, start with 1, and higher order terms in $b$ will be added to cancel the divergences.

The dots “:” indicate the normal-order procedure. We define the normal order of an operator $\mathcal{O}$ as

$$:\mathcal{O} : = \mathcal{O} - \text{all tadpole-like diagrams constructed from } \mathcal{O}. \quad (92)$$

In other words: By normal-ordering an operator, we just subtract all self-contractions. Let us give some examples

$$:\phi^2(x): = \phi^2(x) - C(0) \cdot 1 \quad \phi^4(x): = \phi^4(x) - 6C(0) \cdot \phi^2(x) : -3C^2(0) \cdot 1 \quad (93)$$

Note that on the right-hand side all subtracted terms are normal-ordered. One can of course recursively replace them, which for $:\phi^4(x):$ e.g. leads to

$$:\phi^4(x): = \phi^4(x) - 6C(0)\phi^2(x) + 3C^2(0) \cdot 1 \quad (94)$$

In the dimensional regularization scheme, these relations are much simplified through the rule that $C(0) \equiv 0$. Note also that the normal-order prescription is associative.
Normal ordering is a powerful tool to organize the perturbation expansion. Let us show this by proceeding to the real calculation. We want to study the short-distance behavior of two operators $\phi^4(x)$ and $\phi^4(y)$ in an OPE. To this aim we first normal-order the product of the two interactions:

\[
\phi^4(x) \phi^4(y) = \phi^4(x) \phi^4(y) + 16 \phi^3(x) \phi^3(y) C(x - y) + 72 \phi^2(x) \phi^2(y) C^2(x - y) + 96 \phi(x) \phi(y) C^3(x - y) + 24 \phi^4(x - y).
\] (95)

It is now essential that the normal-ordered product of two operators is free of divergences when these operators are approached; the divergences are contained in the factors of powers of $C(x - y)$. E.g. at leading order, the first term in Eq. (95) becomes

\[
\phi^4(x) \phi^4(y) = \phi^8(z) + \ldots,
\] (96)

where $z = \frac{x + y}{2}$. Let us now consider the perturbation expansion of the expectation value of an observable $O$

\[
\langle O \rangle_b := \frac{1}{Z} \int D[\phi] e^{-\mathcal{H}} O = \langle e^{-bZ_0 \mu^\epsilon} \int \phi^4(x) : O : \rangle_0^{\text{conn}},
\] (97)

where $\langle \ldots \rangle_0$ denotes the free expectation value, and we retain only diagrams that are connected to points in the observable $O$. The term quadratic in $b$ contains (setting all $Z$-factors equal to 1 for the moment)

\[
\frac{b^2 \mu^{2\epsilon}}{2} \int_x \int_y : \phi^4(x) : \phi^4(y) : O ,
\] (98)

Observe now that

\[
\int_x \int_y : \phi^4(x) : \phi^4(y) :
\] (99)

possesses short-distance divergences according to Eq. (95). More explicitly, the first two terms, $\phi^4(x) \phi^4(y)$ and $16 \phi^3(x) \phi^3(y) C(x - y)$ are free of divergences when $|x - y| \to 0$. The third one is upon integration over $x$ and $y$

\[
72 \int_x \int_y : \phi^2(x) \phi^2(y) : C^2(x - y) = 72 A \int_z : \phi^4(z) : + \text{finite} ,
\] (100)

where

\[
A = \int_t C^2(t) = \int_0^{\mu^{-1}} \frac{dt}{t} t^d \times t^{2(2-d)} = \frac{1}{\epsilon} \mu^{-\epsilon} .
\] (101)

It is very important to note that the integral over $C^2(x - y)$ is localized at $x - y = 0$. This means that for any smooth function $f(x, y)$

\[
\int_x \int_y C^2(x - y) f(x, y) = \frac{\mu^{-\epsilon}}{\epsilon} \int_z f(z, z) + O(\epsilon^0) ,
\] (102)

or more formally that $C^2(x - y)$ becomes in the limit of $\epsilon \to 0$ a distribution

\[
C^2(x - y) = \frac{\mu^{-\epsilon}}{\epsilon} S_d \delta^d(x - y) + O(\epsilon^0) .
\] (103)
This explains why in Eq. (100) we could simply replace $\phi^2(x)\phi^2(y)$: by $\phi^4(z)$. It is now easy to see that after introduction of a renormalization factor

$$Z_b = 1 + 36 \frac{b}{\epsilon}$$

(104)
a second term of order $b^2$ will appear in the perturbation expansion, namely

$$-36 \frac{b^2 \mu_0}{\epsilon} \int \phi^4(z) : \mathcal{O},$$

(105)
which will cancel the divergence. This is the only renormalization necessary at 1-loop order. Especially, no counter-term for $\int x^{1/2} : (\nabla \phi(x))^2$: is necessary at leading order in $b$. However, it demands a renormalization at second order, arising form the term

$$\phi(x)\phi(y): C^a(x - y).$$

(106)
As above, we now have to analyze the integral ($t = x - y$)

$$\int C^3(t) : \phi(x)\phi(y): .$$

(107)
Noting that

$$\int C^3(t) = \int \frac{dt}{t} t^{d-2} = \int \frac{dt}{t} t^{2\epsilon - 2}$$

(108)
the leading term is a relevant (quadratic) divergence. We therefore have to expand $\phi(x)$ and $\phi(y)$ up to second order

$$\phi(x) = \phi(z) + \frac{x - y}{2} \nabla \phi(z) + \frac{1}{2} \left( \frac{y - x}{2} \nabla \right)^2 \phi(z) + O \left( (x - y)^3 \right)$$

(109)
to obtain

$$\phi(x)\phi(y) = \left[ \phi(z) + \frac{x - y}{2} \nabla \phi(z) + \frac{1}{2} \left( \frac{x - y}{2} \nabla \right)^2 \phi(z) + O \left( (x - y)^3 \right) \right] \times \left[ \phi(z) + \frac{y - x}{2} \nabla \phi(z) + \frac{1}{2} \left( \frac{y - x}{2} \nabla \right)^2 \phi(z) + O \left( (x - y)^3 \right) \right]$$

$$= \phi(z)^2 - \frac{1}{4} \left( (x - y) \nabla \phi(z) \right)^2 + \frac{1}{4} \phi(z) \left( (x - y) \nabla \right)^2 \phi(z) : + O \left( (x - y)^3 \right).$$

(110)

With the help of Eq. (108), Eq. (107) becomes

$$\int_0^{\mu^{-1}} \frac{dt}{t} t^{2\epsilon - 2} \left[ \phi(z)^2 - \frac{t^2}{4d} : (\nabla \phi(z))^2 : + \frac{t^2}{4d} \phi(z) \Delta \phi(z) : + O \left( (t)^3 \right) \right]$$

$$= \frac{\mu^{2 - 2\epsilon}}{2 + 2\epsilon} \phi(z)^2 - \frac{\mu^{2 - 2\epsilon}}{2\epsilon} \left[ (\nabla \phi(z))^2 : - \phi(z) \Delta \phi(z) : \right] + \text{finite}.$$

(111)
The first term does not come with a pole in $1/\epsilon$ and in addition scales to 0 in the large $L = 1/\mu$ limit. It will thus be neglected. The remaining two terms are equivalent up to a total derivative, and thus Eq. (98) yields another divergent term

$$\frac{24b^2}{d} \int \frac{1}{2} : (\nabla \phi(z))^2 : \mathcal{O}.$$

(112)
This is renormalized (canceled) by setting

\[
Z = 1 - \frac{24(d-2)b^2}{d} \epsilon = 1 - 12\frac{b^2}{\epsilon} + \text{finite}.
\] (113)

The last step is as usual to calculate the renormalization group functions \(\beta(b)\) and \(\eta(b)\), quantifying the flow of the coupling \(b\) and the field \(\phi\) upon changing \(\mu\) [147]. The result is

\[
\beta(b) := \mu \frac{\partial}{\partial \mu} b = -e b + 36 b^2 + O(b^3)
\] (114)

\[
\eta(b) := \mu \frac{\partial}{\partial \mu} \ln Z = 24 b^2 + O(b^3).
\] (115)

Note that the \(\beta\)-function has a non-trivial IR-stable fixed point \((\beta(b^*) = 0)\) at \(b^* = \epsilon/36\) and that this is sufficient to get the exponent \(\eta\) up to order \(\epsilon^2\):

\[
\eta = \eta(b^*) = \frac{\epsilon^2}{54}.
\] (116)

Finally, let us still note the equivalence of the OPE with standard Feynman-diagrams. The first integral was

\[
: \phi^2(x)\phi^2(y) : \int_{x-y} C^2(x-y) = \ldots
\] (117)

Usually, this is written in momentum space as

\[
\frac{1}{(k+p)^2} \frac{1}{k^2} \int \ldots
\] (118)

The other diagram was

\[
: \phi(x)\phi(y) : \int_{x-y} C^3(x-y) = \ldots
\] (119)

Note that if we parameterize the latter by the momentum \(p\) which is running through, then

\[
\frac{1}{q_1 q_2} \frac{1}{(q_1 + q_2 + p)^2} \sim \frac{1}{\epsilon p^2 - 2\epsilon}.
\] (120)

The factor of \(p^2\) is the equivalent of the derivatives appearing in Eq. (111).

---

1For membranes, a derivation of the renormalization group functions is given in appendix A.4
3.5 Multilocal operator product expansion (MOPE)

In section 3.2, we showed that for self-avoiding membranes divergences only occur at short distances. The situation is thus similar to local field-theories for which we discussed in the last section how the techniques of operator product expansion can be used to analyze the divergences. Our aim is now, to generalize these techniques to the multilocal case \[56, 57\]. Intuitively, in the context of multilocal theories – by which we mean that the interaction depends on more than one point – we also expect multilocal operators to appear in such an operator product expansion, which therefore will be called “multi-local operator product expansion” (MOPE). Its precise definition is the aim of this section, whereas we shall calculate some examples in the following one.

We start our analysis by recalling the general form of a (local) operator product expansion of two scaling-operators \( \Phi_A(z + \lambda x) \) and \( \Phi_B(z + \lambda y) \) in a massless theory in the limit of \( \lambda \to 0 \):

\[
\Phi_A(z + \lambda x)\Phi_B(z + \lambda y) = \sum_i C_i(z, \lambda x, \lambda y) \Phi_i(z),
\]

(121)

where \( C_i(z, \lambda x, \lambda y) \) are homogeneous functions of \( \lambda \)

\[
C_i(z, \lambda x, \lambda y) = \lambda^{[\Phi_A]_x + [\Phi_B]_x - [\Phi_i]_x} C_i(z, x, y).
\]

(122)

Here \( [\Phi]_x \) is the canonical dimension of the operator \( \Phi \) in space-units such that \( [x]_x = 1 \), as obtained by naive power-counting. If the theory is translationally invariant, \( C_i(z, x, y) \) is also independent of \( z \), and we will suppose that this is the case, if not stated otherwise\(^2\). Also recall that this relation is to be understood as an operator identity, i.e. it holds inserted into any expectation value, as long as none of the other operators sits at the point \( z \), to which the contraction is performed.

An example for the multilocal theory is

\[
\begin{align*}
\text{\includegraphics[width=\textwidth]{example.png}}
\end{align*}
\]

(123)

Let us explain the formula. We consider \( n \) dipoles (here \( n = 5 \)) and we separate the \( 2n \) end-points into \( m \) subsets (here \( m = 3 \)) delimited by the dashed lines. The MOPE describes how the product of these \( n \) dipoles behaves when the points inside each of the \( m \) subsets are contracted towards a single point \( z_j \). The result is a sum over multilocal operators \( \Phi_i(z_1, \ldots, z_m) \), depending on the \( m \) points \( z_1, \ldots, z_m \), of the form

\[
\sum_i C_i(x_1 - z_1, \ldots) \Phi_i(z_1, z_2, \ldots, z_m),
\]

(124)

where the MOPE-coefficients \( C_i(x_1 - z_1, \ldots) \) depend only on the distances \( x_l - z_j \) inside each subset. This expansion is again valid as an operator-identity, i.e. inserted into any expectation value and in the limit of small distances between contracted points. Again, no other operator should appear at the points

\(^2\)Translation invariance is e.g. broken when regarding systems with boundaries or initial time problems, see section 8.4 and [148] for a review. It is also broken when the underlying metric is not constant, see [57, 149].
$z_1, \ldots, z_m$, towards which the operators are contracted. As the Hamiltonian (61) does not contain a mass-scale, the MOPE-coefficients are as in Eq. (122) homogeneous functions of the relative positions between the contracted points, with the degree of homogeneity given by simple dimensional analysis. In the case considered here, where $n$ dipoles are contracted to an operator $\Phi_i$, this degree is simply $-n\nu d - [\Phi_i]_x$. This means that

$$C_i(\lambda(x_1 - z_1), \ldots) = \lambda^{-n\frac{d-D}{2} d - [\Phi_i]_x} C_i(x_1 - z_1, \ldots),$$

where $[\Phi_i]_x$ is the canonical dimension of the operator $\Phi_i$ and $-d(2-D)/2$ is simply the canonical dimension of the dipole.

In order to evaluate the associated singularity, one finally has to integrate over all relative distances inside each subset. This gives an additional scale factor with degree $D(2n - m)$. A singular configuration, such as in Eq. (123), will be UV-divergent if this degree of divergence

$$D(2n - m) - \frac{2 - D}{2} d - [\Phi_i]_x,$$

is negative. It is superficially divergent if the degree is zero and convergent otherwise. The idea of renormalization, formalized in section 3.8 and proven to work in section 5, is to remove exactly these superficially divergent contributions recursively.

### 3.6 Evaluation of the MOPE-coefficients

The MOPE therefore gives a convenient and powerful tool to calculate the dominant and all subdominant contributions from singular configurations. In this section, we explain how to calculate the MOPE-coefficients on some explicit examples. These examples will turn out to be the necessary diagrams at 1-loop order.

In the following we shall use the notion of normal-ordering introduced in section 3.4. The first thing, which we use, is that

$$\langle : e^{ikr}(x) : e^{ipr}(y) : \rangle_0 = e^{kpC(x-y)} : e^{ikr}(x) e^{ipr}(y) : .$$

Explicitly, tadpole-like contributions which are powers of

$$\int \! d^D p \frac{1}{p^2}$$

are omitted. This is done via a finite part prescription (analytic continuation, dimensional regularization), valid for infinite membranes, for which the normal-order prescription is defined. Let us stress that this is a pure technical trick, which is not really necessary. However, adopting this notation, the derivation of the MOPE-coefficients is much simplified, and we will henceforth stick to this convention. The suspicious reader may always check that the same results are obtained without this procedure. This is clear from the uniqueness of the finite-part prescription.

The key-formula for all further manipulations is

$$\langle : e^{ikr}(x) : e^{ipr}(y) : \rangle_0 = e^{kpC(x-y)} : e^{ikr}(x) e^{ipr}(y) : .$$

This can be proven as follows: Consider the (free) expectation value of any observable $\mathcal{O}$ times the operators of Eq. (129). Then the the left- and right-hand sides of the above equation read

$$\mathcal{L} = \langle \mathcal{O} : e^{ikr}(x) : e^{ipr}(y) : \rangle_0,$n

$$\mathcal{R} = e^{kpC(x-y)} \langle \mathcal{O} : e^{ikr}(x) e^{ipr}(y) : \rangle_0 .$$
First of all, for $O = 1$, the desired equality of $L = R$ holds, because $\langle :e^{ikr(x)} :e^{ipr(y)} :\rangle_0 = 1$ and $\langle :e^{ikr(x)} :e^{ipr(y)} :\rangle_0 = e^{kpC(x-y)}$. Now consider a non-trivial observable $O$, and contract all its fields $r$ with $e^{ikr(x)}$ or $e^{ipr(y)}$, before contracting any of the fields $r(x)$ with $r(y)$. The result is a product of correlation-functions between the points in $O$ and $x$ or $y$, and these are equivalent for both $L$ and $R$. However, contracting an arbitrary number of times $e^{ikr(x)}$, leaves the exponential $e^{ikr(x)}$ invariant. Completing the contractions for $L$ therefore yields a factor of $e^{kpC(x-y)}$, and the latter one also appears in $R$. Thus, the equality of $L$ and $R$ holds for all $O$ and this proves Eq. (129).

Now proceed to the first explicit example, the contraction of a single dipole with endpoints $x$ and $y$.

$$\langle \mathcal{O} \rangle = \int \frac{dk}{(2\pi)^d} \left| e^{ikr(x)} :e^{-ikr(y)} :\right| = \int \frac{dk}{(2\pi)^d} \left| e^{ikr(x)} :e^{-ikr(y)} :\right| = e^{kpC(x-y)} \cdot \left| e^{ikr(x)} :e^{-ikr(y)} :\right| = e^{kpC(x-y)} \cdot \left| e^{ikr(x)} :e^{-ikr(y)} :\right|,$$

This configuration may have divergences when $x$ and $y$ come close together. Let us stress that in contrast to $\phi^4$-theory, these divergences are not obtained as a finite sum of products of correlators: Since $C(x-y) = |x-y|^{2-D}$, the latter is always well-behaved at $x = y$. The singularity only appears when summing an infinite series of diagrams as we will do now. To this purpose, we first normal-order the two exponentials using Eq. (129)

$$\int \frac{dk}{(2\pi)^d} \left| e^{ikr(x)} :e^{-ikr(y)} :\right| = e^{-k^2|x-y|^{2\nu}}.$$

Note that the operators $e^{ikr(x)}$ and $e^{-ikr(y)}$ are free of divergences upon approaching each other, since no more contractions can be made. The divergence is captured in the factor $e^{-k^2|x-y|^{2\nu}}$. Therefore, we can expand the exponential $e^{ikr(x)}$ for small $x-y$ and consequently in powers of $|r(x) - r(y)|$. This expansion is

$$\int \frac{dk}{(2\pi)^d} \left| 1 + ik |r(x) - r(y)| - \frac{1}{2} \left( k |r(x) - r(y)| \right)^2 + \ldots \right| e^{-k^2|x-y|^{2\nu}}.$$

We truncated the expansion after the third term. It will turn out later that this is sufficient, since subsequent terms in the expansion are proportional to irrelevant operators for which the integral over the MOPE-coefficient is UV-convergent.

Due to the symmetry of the integration over $k$ the term linear in $k$ vanishes. Also due to symmetry, the next term can be simplified with the result

$$\int \frac{dk}{(2\pi)^d} \left| 1 - \frac{k^2}{2d} : |r(x) - r(y)|^2 : + \ldots \right| e^{-k^2|x-y|^{2\nu}}.$$

Finally, the integration over $k$ can be performed. Recall that normalizations were chosen such that $\int_k e^{-sk^2} = s^{-d/2}$ to obtain

$$|x-y|^{-\nu d} \cdot \left| 1 - \frac{1}{4} \left( (x-y) \nabla r \left( \frac{x+y}{2} \right) \right)^2 \cdot |x-y|^{-(d+2)} + \ldots \right|.$$

The second operator has a tensorial structure, which has to be taken into account in order to construct the subtraction operator. Using the short-hand notation $\alpha+\beta = \frac{1}{2} (\partial_\alpha r)(\partial_\beta r)$, we can write this symbolically as

$$\langle \mathcal{O} \rangle = \left( \mathcal{O} \right) + \left( \mathcal{O} \right) e^{kpC(x-y)} e^{kpC(x-y)} + \ldots ,$$
with the MOPE-coefficients (reminding Feynman’s bra-ket notation)

\[
\begin{align*}
\langle \mathbf{O} | 1 \rangle &= |x - y|^{-\nu_d} \\
\langle \mathbf{O} | \alpha + \beta \rangle &= -\frac{1}{2} (x - y)_\alpha (x - y)_\beta |x - y|^{-\nu(d+2)}.
\end{align*}
\]

(136) (137)

As long as the angular average is taken (and this will be the case when integrating the MOPE-coefficient to obtain the divergence), we can replace in Eq. (135) \(\alpha + \beta\) by \(\mp \equiv \frac{1}{2}(\nabla r)^2\) and Eq. (137) by

\[
\langle \mathbf{O} | \mp \rangle = -\frac{1}{2D}|x - y|^{D - \nu_d}.
\]

(138)

Next consider a real multi-local example of an operator-product expansion, namely the contraction of two dipoles towards a single dipole:

\[
x + u/2 \rightarrow x - u/2, \quad y + v/2 \rightarrow y - u/2 = \int_k \int_p e^{ik[r(x+u/2)-r(y+u/2)]} e^{ip[r(x-u/2)-r(y-v/2)]}.
\]

(139)

This has to be analyzed for small \(u\) and \(v\), in order to control the divergences in the latter distances. As above, we normal-order operators which are approached, yielding

\[
e^{ikr(x+u/2)} e^{ipr(x-u/2)} = :e^{ikr(x+u/2)} :e^{ipr(x-u/2)} = :e^{ikr(x+u/2)} e^{ipr(x-u/2)} :e^{kpC(u)}.
\]

(140)

A similar formula holds when approaching \(e^{-ikr(y+v/2)}\) and \(e^{-ipr(y-v/2)}\)

\[
e^{-ikr(y+v/2)} e^{-ipr(y-v/2)} = :e^{-ikr(y+v/2)} :e^{-ipr(y-v/2)} = :e^{-ikr(y+v/2)} e^{-ipr(y-v/2)} :e^{kpC(v)}.
\]

(141)

Eq. (139) then becomes

\[
\int_k \int_p :e^{ikr(x+u/2)+ipr(x-u/2)} :e^{-ikr(y+v/2)-ipr(y-v/2)} e^{kp(C(u)+C(v))}.
\]

(142)

In order to keep things as simple as possible, let us first extract the leading contribution before analyzing subleading corrections. This leading contribution is obtained when expanding the exponential operators (here exemplified for the second one) as

\[
:e^{-ikr(y+v/2)} e^{-ipr(y-v/2)} = :e^{-i(k+p)r(y)} (1 + O(\nabla r)):
\]

(143)

and dropping terms of order \(\nabla r\). This simplifies Eq. (142) to

\[
\int_k \int_p :e^{i(k+p)r(x)} :e^{-i(k+p)r(y)} e^{kp(C(u)+C(v))}.
\]

(144)

In the next step, first \(k\) and second \(p\) are shifted

\[
k \rightarrow k - p, \quad \text{then} \quad p \rightarrow p + \frac{k}{2}.
\]

(145)

The result is (dropping the normal-ordering according to Eq. (127))

\[
\int_k e^{ik[r(x)-r(y)]} \int_p e^{i(\frac{1}{2}k^2-p^2)[C(u)+C(v)]}.
\]

(146)
The factor of $\int_k e^{ik[r(x)-r(y)]}$ is again a $\delta$-distribution, and the leading term of the short distance expansion of Eq. (146). Derivatives of the $\delta$-distribution appear when expanding $e^{(\frac{1}{2}k^2-p^2)[C(u)+C(v)]}$ in $k^2$; these are less relevant and only the first sub-leading term will be displayed for illustration:

$$\int_k e^{ik[r(x)-r(y)]} \int_p e^{-p^2[C(u)+C(v)]} \left( 1 + \frac{k^2}{4} [C(u)+C(v)] + \ldots \right)$$

$$= \left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) - \left( \begin{array}{c} \circ \cr \bullet \cr \bullet \cr \circ \end{array} \right) + \ldots ,$$

(147)

where in analogy to Eqs. (135) and (137)

$$\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) = [C(u)+C(v)]^{-d/2} ,$$

$$\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) = \frac{1}{4} [C(u)+C(v)]^{1-d/2}$$

and

$$\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) = \delta^d(r(x) - r(y)) , \quad \left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) = (-\Delta_r)\delta^d(r(x) - r(y)).$$

(148)

Let us already mention that the leading contribution proportional to the $\delta$-distribution will renormalize the coupling-constant, and that the next-to-leading term is irrelevant and can be neglected. The same holds true for the additional term proportional to $\nabla r$ which was dropped in Eq. (143).

There is one more possible divergent contribution at the 1-loop level, namely $\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right)$. We now show that the leading term of its expansion, which is expected to be proportional to $\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right)$, is trivial. To this aim consider

$$\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) = \int_{k,p} \, e^{ikr(u)} \, e^{-ikr(x)} \, e^{ipr(y)} \, e^{-ipr(z)} \, e^{kp[C(x-z)-C(x-y)]} .$$

(150)

We want to study the contraction of $x$, $y$, and $z$, and look for all contributions which are proportional to

$$\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) = \int_k \, e^{ikr(u)} \, e^{-ikr((x+y+z)/3)} .$$

(151)

The key-observation is that in Eq. (150) the leading term is obtained by approximating $e^{kp[C(x-z)-C(x-y)]} \approx 1$. All subsequent terms yield factors of $k$, which after integration over $k$ give derivatives of the $\delta^d$-distribution. The result is that

$$\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right) - \left( \begin{array}{c} \circ \cr \bullet \cr \bullet \cr \circ \end{array} \right) = 0 .$$

(152)

This means that divergences of $\left( \begin{array}{c} \bullet \cr \circ \cr \circ \cr \bullet \end{array} \right)$ are already taken into account by a proper treatment of the divergences in $\left( \begin{array}{c} \circ \cr \bullet \cr \circ \cr \bullet \end{array} \right)$, analyzed in Eq. (135).
3.7 Strategy of renormalization

In the last two sections, we discussed how divergences occur, how their general structure is obtained by the MOPE, and how the MOPE-coefficients are calculated. In the next step, the theory shall be renormalized. The basic idea is to identify the divergences through the MOPE, and then to introduce counter-terms which subtract these divergences. These counter-terms are nothing else than integrals over the MOPE-coefficients, properly regularized, i.e. cut off.

In order to properly understand this point, let us recall the two main strategies employed in renormalization: The first one subtracts divergences in correlation-functions or equivalently vertex-functions. This amounts to adding counter-terms to the Hamiltonian which can be interpreted as a change of the parameters in this Hamiltonian. Calculating observables with this modified Hamiltonian leads to finite physical expectation values, but it is not evident that the integrals appearing in these calculations are convergent.

The other procedure is inspired by ideas employed in a formal proof of renormalizability, or more precisely when applying the $R$-operation to the perturbation expansion, as will be discussed in the next section 5. It consists in adding to the Hamiltonian counter-terms which are integrals, such that each integrand which appears in the perturbative expansion becomes an integrable function, and as a consequence the integrals and thus the perturbation expansion are finite. Of course, to finally obtain the critical exponents, the integral counter-terms have to be reduced to numbers. However, we really want to think of them as integrals in the intermediate steps. The reason is the following: It is extremely difficult to calculate observables. However, this is not really necessary as long as one is only interested in renormalization. The above-mentioned procedure is then sufficient to ensure finiteness of any observable as long as there is no additional divergence when the dipole is contracted towards this observable. The latter situation would require a new counter-term, which is a proper renormalization of the observable itself. The procedure of considering whole integrals as counter-terms is in the heart of our renormalization procedure, and the reader should bear this idea in mind throughout this review.

3.8 Renormalization at 1-loop order

Let us continue on the concrete example of the 1-loop divergences, from which are obtained the scaling exponents to first order in the dimensional regularization parameter $\epsilon$. Explicitly, the model shall be renormalized through two renormalization group factors $Z$ (renormalizing the field $r$) and $Z_b$ (renormalizing the coupling $b$). Recalling Eq. (61), this is

$$\mathcal{H}[r] = \frac{Z}{2-D} \int_x \left( \frac{1}{2} (\nabla r(x))^2 + bZ_b \mu^\epsilon \int_y \tilde{\delta}^d (r(x) - r(y)) \right),$$

(153)

where $r$ and $b$ are the renormalized field and renormalized dimensionless coupling constant, and $\mu = L^{-1}$ is the renormalization momentum scale.

Let us start to eliminate the divergences in the case, where the end-points $(x, y)$ of a single dipole are contracted towards a point (taken here to be the center-of-mass $z = (x + y)/2$). The MOPE is

$$\begin{array}{l}
\left. \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
\scriptstyle x \quad \scriptstyle y
\end{array}
\end{array} \right| = \left. \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
\scriptstyle x \quad \scriptstyle y
\end{array} \right|_1 + \left. \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
\scriptstyle x \quad \scriptstyle y
\end{array} \right|_\alpha \beta + \ldots .
\end{array}
\end{array}
\right.$$  

(154)

The MOPE-coefficients were obtained in the last section as

$$\left. \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
\scriptstyle x \quad \scriptstyle y
\end{array} \right|_1 = |x - y|^{-\nu d},$$

(155)

$$\left. \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
\scriptstyle x \quad \scriptstyle y \quad \alpha \beta
\end{array} \right| = -\frac{1}{2} (x - y)_\alpha (x - y)_\beta |x - y|^{-\nu(d+2)}.$$

(156)
We now have to distinguish between counter-terms for relevant operators and those for marginal operators. The former can be defined by analytic continuation, while the latter require a subtraction scale. Indeed, the divergence proportional to 1 is given by the integral
\[
\int_{\Lambda^{-1} < |x-y| < L} \left( \begin{array}{c} x \bullet y \\ \alpha \beta \end{array} \right) = \int_{\Lambda^{-1}} \frac{dx}{x} x^{D-\nu d} = \frac{1}{D-\varepsilon} \left( \Lambda^{D-\varepsilon} - L^{\varepsilon-D} \right),
\]
where \( \Lambda \) is a high-momentum UV-regulator and \( L \) a large distance regulator. For \( \varepsilon \approx 0 \) this is UV-divergent but IR-convergent. The simplest way to subtract this divergence is therefore to replace the dipole operator by
\[
\left( x \bullet y \right) \rightarrow \left( x \bullet y - x \bullet y \right),
\]
where \( x \bullet y = |x - y|^{-\nu d} \). This amounts to adding to the bare Hamiltonian (61) the UV-divergent counter-term
\[
\Delta H_1 = -b Z b \mu \varepsilon \int_x^y |x - y|^{-\nu d},
\]
which is a pure number and thus does not change the expectation-value of any physical observable.

We next consider marginal operators: In the MOPE of Eq. (154), the integral over the relative distance of \( \int_{x-y} \left( \begin{array}{c} x \bullet y \\ \alpha \beta \end{array} \right) \) is logarithmically divergent at \( \varepsilon = 0 \). In order to find the appropriate counter-term, we use dimensional regularization, i.e. set \( \varepsilon > 0 \). An IR-cutoff \( L \), or equivalently a subtraction momentum scale \( \mu = L^{-1} \), has to be introduced in order to define the subtraction operation. As a general rule, let us integrate over all distances appearing in the MOPE-coefficient, bounded by the subtraction scale \( L = \mu^{-1} \).

Defining
\[
\left\langle \begin{array}{c} \bullet \bullet \bullet \\ \alpha \beta \end{array} \right\rangle_L := \int_{|x-y| < L} \left( \begin{array}{c} x \bullet y \\ \alpha \beta \end{array} \right)
\]
we need the following counter-term in the Hamiltonian
\[
\Delta H_+ = -b Z b \mu \varepsilon \int_x^y \left\langle \begin{array}{c} \bullet \bullet \bullet \\ \alpha \beta \end{array} \right\rangle_L \int_x^y \alpha \beta \mu x,
\]
subtracting explicitly the divergence in the integrals, as discussed in the last section. The reader is invited to verify this explicitly in exercise E.6 (see page 164) on the example of the expectation value of \( \mathcal{O} = e^{ik[r(s) - r(t)]} \), as given in Eq. (87).

Since the angular integration in Eq. (160) reduces \( \alpha \beta \) to \( \mu \), we can replace Eq. (161) by the equivalent expression
\[
\Delta H_+ = -b \mu \varepsilon \left\langle \begin{array}{c} \bullet \bullet \bullet \\ \mu \end{array} \right\rangle_L \int_x \mu x,
\]
for which the numerical value of the diagram is calculated as
\[
\left\langle \begin{array}{c} \bullet \bullet \bullet \\ \mu \end{array} \right\rangle_L = \int_{|x-y| < L} \left( \begin{array}{c} x \bullet y \\ \mu \end{array} \right) = -\frac{1}{2D} \int_0^L \frac{dx}{x} x^{2D-\nu d} = -\frac{1}{2D} \frac{L^\varepsilon}{\varepsilon}. \quad (163)
\]

We can now subtract this term in a minimal subtraction scheme (MS). The internal dimension of the membrane \( D \) is kept fixed and (163) is expanded as a Laurent series in \( \varepsilon \), which here starts at \( \varepsilon^{-1} \). Denoting by
\[ \langle | \rangle_{\varepsilon} \text{ the term of order } \varepsilon^p \text{ of the Laurent expansion of } \langle | \rangle_{L} \text{ for } L = 1, \text{ the residue of the pole in Eq. (163)} \]
\[ \text{is found to be} \]
\[ \langle \bullet \bullet | + \rangle_{\varepsilon = 1} = - \frac{1}{2D} \frac{1}{\varepsilon}. \] (164)

We shall also frequently employ the notation for the residue
\[ \langle \bullet \bullet | + \rangle_{\varepsilon} = - \frac{1}{2D}. \] (165)

It is this pole that is subtracted in the MS-scheme by adding to the Hamiltonian a counter-term
\[ \Delta \mathcal{H}_+ = - b \langle \bullet \bullet | + \rangle_{\varepsilon = 1} \int_{x} x. \] (166)

Note that by going from Eq. (161) to Eq. (166), we have reduced the integral counter-term to a number. We recall our initial remark that if one wants to check that this counter-term renders the theory finite, one should think of it as its defining integral (161), and verify that in the resulting perturbation theory, the first-order divergence is absent.

Similarly, the divergence arising from the contraction of two dipoles to a single dipole is subtracted by a counter-term
\[ \Delta \mathcal{H}_{\bullet} = b^2 \mu^2 \langle \bullet \bullet \bullet | \bullet \bullet \rangle_{L} \int_{x} \int_{y} x y, \] (167)

with
\[ \langle \bullet \bullet \bullet | \bullet \bullet \rangle_{L} = \int_{|x|<L} \int_{|y|<L} \langle \bullet \bullet \bullet | \bullet \bullet \rangle. \] (168)

Reducing this integral counter-term to a number, we subtract the residue of the single pole of
\[ \langle \bullet \bullet \bullet | \bullet \bullet \rangle_{L} = \int_{|x|<L} \int_{|y|<L} \langle \bullet \bullet \bullet | \bullet \bullet \rangle = \int_{|x|<L} \int_{|y|<L} \left( |x|^{2\nu} + |y|^{2\nu} \right)^{-d/2}. \] (169)

Note that the regulator \( L \) cuts off both integrations. One can now either utilize some simple algebra or show by the methods of conformal mapping (see section 4.3) that the residue is obtained by fixing one distance to equal 1 and by freely integrating over the remaining one
\[ \langle \bullet \bullet \bullet | \bullet \bullet \rangle_{\varepsilon} = \int_{0}^{\infty} \frac{dx}{x} x^D \left( 1 + x^{2-D} \right)^{-2D/(2-D)}. \] (170)

(Recall that \( d/2 = 2D/(2 - D) + O(\varepsilon) \).) The above is easily related to Euler’s B-function and reads
\[ \langle \bullet \bullet \bullet | \bullet \bullet \rangle_{\varepsilon} = \frac{1}{2 - D} \frac{\Gamma(D/2 - D)^2}{\Gamma(D/2 - D)}. \] (171)

As a result, the model is UV-finite at 1-loop order, if we use in the renormalized Hamiltonian (153) the renormalization factors \( Z \) and \( Z_b \)
\[ Z = 1 - (2 - D) \langle \bullet \bullet | + \rangle_{\varepsilon = 1} b + O(b^2) \] (172)
\[ Z_b = 1 + \langle \bullet \bullet \bullet | \bullet \bullet \rangle_{\varepsilon = 1} \frac{b}{\varepsilon} + O(b^2). \] (173)
Note that due to Eq. (152) no counter-term for is necessary.

The renormalized field and coupling are re-expressed in terms of their bare counterparts through

\[ r_0(x) = Z^{1/2} r(x), \quad b_0 = b Z_b Z^{d/2} \mu^\epsilon. \]  \hspace{1cm} (174)

Finally, the renormalization group functions are obtained from the variation of the coupling constant and the field with respect to the renormalization scale \( \mu \), keeping the bare coupling fixed. (For a derivation, see appendix A.4). The flow of the coupling is written in terms of \( Z \) and \( Z_b \) as

\[ \beta(b) := \mu \frac{\partial}{\partial \mu} \bigg|_{b_0} = \frac{-\varepsilon b}{1 + b \frac{d}{d_b} \ln Z_b + \frac{d b}{2 b} \frac{d}{d_b} \ln Z} \]

\[ = -\varepsilon b + \left( \frac{1}{2 - D} \Gamma \left( \frac{D}{2 - D} \right)^2 + \frac{(2 - D) d}{4 D} \right) b^2 + O(b^3). \]  \hspace{1cm} (175)

Similarly, the full dimension of the field (the exponent entering into the correlation function) is obtained as

\[ \nu(b) := \frac{2 - D}{2} - \frac{1}{2} \mu \frac{\partial}{\partial \mu} \bigg|_{b_0} \ln Z = \frac{2 - D}{2} - \frac{1}{2} \beta(b) \frac{\partial}{\partial b} \ln Z \]

\[ = \frac{2 - D}{2} \left[ 1 - b \left( \frac{D}{2 - D} \right) \right] + O(b^2) \]

\[ = \frac{2 - D}{2} \left[ 1 + b \frac{1}{2D} \right] + O(b^2). \]  \hspace{1cm} (176)

Note that minimal subtraction is used on the level of counter-terms or equivalently \( Z \)-factors. Since \( Z \) enters as \( Z^d \) into the \( \beta \)-function, the latter also contains a factor of \( d \) in the 1-loop approximation, i.e. \( Z^d \) is not minimally renormalized. In order to calculate the leading order in \( \varepsilon \), the factor of \( d \) can be replaced by \( d_c = \frac{4D}{2 - D} \).

The \( \beta \)-function has a non-trivial fixed-point with \( \beta(b^*) = 0 \), which has positive slope and thus describes the behavior of the model at large distances:

\[ b^* = \frac{\varepsilon}{1 + \frac{1}{2 - D} \Gamma \left( \frac{D}{2 - D} \right)^2} + O(\varepsilon^2). \]  \hspace{1cm} (177)

The anomalous dimension \( \nu^* := \nu(b^*) \) becomes to first order in \( \varepsilon \)

\[ \nu^* = \frac{2 - D}{2} \left[ 1 + \frac{\varepsilon}{2D} \frac{1}{1 + \frac{1}{2 - D} \Gamma \left( \frac{D}{2 - D} \right)^2} \right] + O(\varepsilon^2). \]  \hspace{1cm} (178)

For polymers, this result reduces to the well-known formula

\[ \nu^*(D = 1) = \frac{1}{2} + \frac{4 - d}{16} + O((4 - d)^2). \]  \hspace{1cm} (179)
### 3.9 Non-renormalization of long-range interactions

Long-range interactions are in general not renormalized \([56]\). This is very useful, as it immediately enforces scaling relations among the critical exponents, which in some cases are already sufficient to determine these exponents. Let us explain the non-renormalization by analyzing the long-range interaction \((\alpha > 0)\)

\[
\int \frac{|k|^{-\alpha} e^{ik|x-y|}}{|x-y|^{\alpha-d}} \sim |x-y|^{\alpha-d}
\]

(180)

Then the most simple singular configurations which give rise to a renormalization of the interaction are those for which two interactions are contracted to a single one, as we have discussed in 3.5. We claim that their multilocal operator product expansion (MOPE), \(\otimes \), does not contain a contribution proportional to \(\otimes \), but that the leading term is proportional to the short-range interaction \(\otimes \). This is a consequence of the analytic structure of the long-range interaction: The contraction \(\otimes \) is in complete analogy to Eq. (147) and with the same notations as there

\[
\int \frac{|k|^{-\alpha} |p|^{-\alpha} e^{i(k+p)|x-y|} e^{kp[C(u)+C(v)]}}{\text{subdominant terms}}
\]

(181)

In order to obtain a long-range term, a singularity at \(k + p = 0\) is necessary. However, expression (181) is analytic at \(k + p = 0\), and no long-range term is generated. This is easily generalized to any contraction towards \(\otimes \) and hence to any order in perturbation theory.

Let us now analyze the consequences. We want to study tethered membranes with long-range interactions, generalizing Eq. (61) or Eq. (153) to

\[
\mathcal{H}_{LR} = \frac{Z}{2 - D} \int_x + b \mu^\delta \int_x \int_y \otimes .
\]

(182)

Note that since in contrast to Eq. (61) the interaction is not renormalized, there is only one \(Z\)-factor in Eq. (182), namely for elasticity. This does, however, not mean that the \(\beta\)-function is trivial. In analogy to Eq. (174), the relation between bare and renormalized coupling is

\[
b_0 = b Z^{(d-\alpha)/2} \mu^\delta ,
\]

(183)

where \(Z\) is as in Eq. (174) the renormalization of the field, and

\[
\delta = 2D - \nu(d - \alpha) .
\]

(184)

The \(\beta\)-function now reads

\[
\beta(b) = \mu \left. \frac{\partial}{\partial \mu} \right|_{b_0} b = \left[ -\delta + \frac{\alpha - d}{2} \mu \frac{\partial}{\partial \mu} \ln Z \right] b .
\]

(185)

Using the fact that \(\mu \frac{\partial}{\partial \mu} \ln Z\) is nothing but \((-2)\) times the anomalous dimension of the field, see Eq. (176), we make the replacement

\[
\mu \frac{\partial}{\partial \mu} \ln Z = 2(\nu - \nu(b))
\]

(186)
in Eq. (185). The result is
\[ \beta(b) = -[2D - (d - \alpha)\nu(b)]b. \] (187)

This \( \beta \)-function has to zeros: For \( \delta < 0 \), the fixed point at \( b^* = 0 \) is attractive. For \( \delta > 0 \) the non-trivial zero and fixed point of \( \beta(b) \) is at \( b^* > 0 \), implying the exponent-identity
\[ \nu^* = \nu(b^*) = \frac{2D}{d - \alpha}. \] (188)

Non-renormalization of the coupling thus allows to obtain \( \nu^* \) without calculating any diagram. Since this observation is quite generally useful, let us give a heuristic derivation of Eq. (188). We may then consider the formal derivation given above as a proof of the heuristic argument, and employ the latter confidently throughout this review.

“Power counting” for the dimension \( D \) of the interaction at a fixed point yields
\[ D = 2D - \nu^*(d - \alpha), \] (189)

and this power-counting gives the correct dimension of the operator, since the latter has no proper renormalization. Three different scenarios are now possible: If \( D < 0 \), then the associated coupling scales to 0, and the operator plays no role in the large scale limit. If \( D > 0 \), then the associated coupling grows under renormalization and we are not at an IR-fixed point; by definition this is not the situation considered here. The last possibility is that we are at an IR-fixed point, and this is (at least for one coupling) equivalent to \( D = 0 \). It again follows the exponent identity
\[ \nu^* = \nu(b^*) = \frac{2D}{d - \alpha}. \] (190)

Also the crossover from short-range to long-range self-avoidance in a model with both couplings can be discussed in this framework. Following the line of arguments given above, long-range self-avoidance will scale to 0 and the short-range fixed point is completely attractive as long as \( D \), Eq. (189), evaluated with \( \nu^* \) as obtained from short-range self-avoidance only, is negative. As a consequence always that interaction wins, which yields the larger value for \( \nu^* \).

Physically, long-range forces play an important role for charged membranes, as discussed in [150].

4 Some useful tools and relation to polymer theory

4.1 Equation of motion and redundant operators

The equation of motion reflects the invariance of the functional integral under a global rescaling of the field \( r \). This has important consequences. Consider the expectation value of an observable \( \mathcal{O} \) in the free theory:
\[ \langle \mathcal{O} \rangle_0 = \frac{\int \mathcal{D}[r] \mathcal{O} e^{-\frac{1}{2}D \int x \mathcal{L}[r]}}{\int \mathcal{D}[r] e^{-\frac{1}{2}D \int x \mathcal{L}[r]}}. \] (191)

We now perform a global rescaling of \( r(x) \):
\[ r(x) \rightarrow (1 + \kappa) r(x). \] (192)
The expectation value of $O$, Eq. (191) remains unchanged. Expanding up to first order in $\kappa$ yields

$$\langle O \rangle_0 = \frac{\int D[r] \mathcal{O} (1 + \kappa [\mathcal{O}]_r) \left( 1 - \frac{2\kappa}{2 - D} \int_x \right) e^{-\frac{1}{2} \int_x f_x} +}{\int D[r] \left( 1 - \frac{2\kappa}{2 - D} \int_x \right) e^{-\frac{1}{2} \int_x f_x} +}$$

(193)

where $[\mathcal{O}]_r$ is the canonical dimension of the operator $\mathcal{O}$, measured in units of $r$, i.e. such that $[r]_r = 1$. Calculating the difference of Eqs. (191) and (193) gives:

$$\langle \mathcal{O} \int_0 \rangle^{\text{conn}}_0 = \nu [\mathcal{O}]_r \langle O \rangle_0 ,$$

(194)

where $^{\text{conn}}$ denotes the connected expectation value. For several operators we have

$$\langle \mathcal{O}_1 \mathcal{O}_2 \int_0 \rangle^{\text{conn}}_0 = \nu ([\mathcal{O}_1]_r + [\mathcal{O}_2]_r) \langle \mathcal{O}_1 \mathcal{O}_2 \rangle_0 .$$

(195)

A specific example is

$$\langle \bullet \int_0 \rangle^{\text{conn}}_0 = -\nu d \langle \bullet \rangle_0 .$$

(196)

In the case of infinitely large membranes, these relations are equivalently valid for non-connected expectation values. (To prove this, remark that $\langle \bullet \rangle_0 = 0$ by analytic continuation.)

Let us try to understand Eq. (194) perturbatively. For this purpose, it will turn out to be convenient to integrate by parts the free Hamiltonian as

$$\frac{1}{2} \int_x (\nabla r(x))^2 = \frac{1}{2} \int_x r(x)(-\Delta) r(x) .$$

(197)

For simplicity, we consider infinite membranes, such that connected expectation values can be replaced by standard ones. For computational convenience further suppose that $O[r]$ is a function of $r(y)$, $O[r] = O(r(y))$. Then

$$\langle O(r(y)) \int_x \rangle_0 = \langle O(r(y)) \frac{1}{2} \int_x r(x)(-\Delta) r(x) \rangle_0 .$$

(198)

We now proceed according to the following strategy: First contract the field $r(x)$ which is preceded by ($-\Delta$) with any field in $O(r(y))$. This yields (for normalizations and conventions see appendix A.1)

$$\frac{1}{2} \int_x \left< r(x)(-\Delta_x) C(x-y) \frac{\partial O(r(y))}{\partial r(y)} \right>_0 = \frac{2 - D}{2} \int_x \left< r(x) \delta^D (x-y) \frac{\partial O(r(y))}{\partial r(y)} \right>_0$$

$$= \frac{2 - D}{2} \left< r(y) \frac{\partial O(r(y))}{\partial r(y)} \right>_0 .$$

(199)

Since $O$ is a homogeneous function in $r$, then

$$r(y) \frac{\partial O(r(y))}{\partial r(y)} = [O]_r O(r(y)) ,$$

the operator is reproduced and we recover the equation of motion (194). Note that in this argumentation, it is irrelevant how the second field $r$ of $+$ is finally contracted.
In the case of several operators, the field $r(x)$ which is preceded by $(-\Delta)$ can be contracted with any of these operators, and one recovers Eq. (195).

Note also that without partially integrating the free action, no $\hat{\delta}^D$-distribution is obtained and it is impossible to assign $+$ to one of the points with which its fields are contracted: The integral is delocalized. This is a subtle point which was ingeniously avoided up to now. To understand this point remember that the renormalization of the coupling and by this means the $\beta$-function (175), not only contains the direct term $\langle | \rangle$, but also a term $-\frac{2}{2-D}d\langle | + \rangle$. In exercise E.3 (page 161) the reader can show that the above arguments can be used to obtain this term directly.

We now turn to another concept, which is also a consequence of reparametrization invariance, namely redundant operators, as introduced by Wegner [151].

Consider the path-integral
\[
\int D[r] e^{-\mathcal{H}[r]}
\]
with the Hamiltonian of the interacting theory
\[
\mathcal{H}[r] = \frac{Z}{2-D} \int_x + bZ_b \mu \int_x \int_y \int_x \int_y
\]
and make a change of variables
\[
r(x) \longrightarrow r(x) + \kappa(x) \mathcal{F}[r].
\]
$\mathcal{F}[r]$ is an arbitrary function of $r(x)$, but may also involve fields $r(y)$ at different points. (Explicitly, we think of $\mathcal{F}[r] = f_1(r(x))$ or $\mathcal{F}[r] = f_2(r(x), r(y))$ with $x \neq y$, where both $f_1(r)$ and $f_2(r, r')$ are functions of $r$ and $r, r'$ respectively. More general expressions for $\mathcal{F}[r]$ including derivatives of $r$ are possible, but shall not be considered here.) Of course, since this is a simple variable-transformation, the path-integral itself remains unchanged, even-though formally new terms are generated\(^3\). These newly generated terms contain no physical information, and are thus called redundant operators [151]. They are useful in relating apparently different operators.

Let us extract the terms linear in $\kappa(u) \sim \delta^D(u - x)$. Two contributions have to be taken into account: First, from the expansion of the exponential, one obtains a term
\[
-F[r] \frac{\delta \mathcal{H}[r]}{\delta r(x)}.
\]
Second, the integration measure is changed, resulting in a term
\[
\frac{\delta \mathcal{F}[r]}{\delta r(x)}.
\]
(Note that in the cases of $\mathcal{F}[r] = f_1(r(x))$ or $\mathcal{F}[r] = f_2(r(x), r(y))$ with functions $f_1$ and $f_2$, this is equivalent to $\frac{\partial \mathcal{F}[r]}{\partial r(x)} \hat{\delta}^D(0)$.) Combining both of them, we obtain the redundant operator
\[
\mathcal{R} = \mathcal{F}[r] \frac{\delta \mathcal{H}[r]}{\delta r(x)} - \frac{\delta \mathcal{F}[r]}{\delta r(x)}.
\]
\(^3\)Also note that the inclusion of an observable $\mathcal{O}[r(z)]$ is possible in the path-integral, but leads to additional contact-terms for $z = x$, and at other points on which $\mathcal{F}[r]$ depends.
The second term just subtracts the contraction of \( F[r] \) with the variation of the (free) quadratic part of the Hamiltonian \( \frac{\delta H_0[r]}{\delta r(x)} = -(2-D)^{-1} \Delta r(x) \) at the same point. (This is the same structure as encountered within different discretization prescriptions in dynamic theories, see e.g. [152].) Since \( \frac{\delta F[r]}{\delta r(x)} \propto \delta^D(0) \equiv \int d^Dp \) is zero by analytic continuation, we will drop it in the following.

Let us now explore some of the consequences of the above construction. First set \( F[r] := 1 \), yielding the redundant operator – or Dyson-Schwinger equation of motion [153] in the terminology of elementary particle physics: \( R = 0 \) with

\[
R = \frac{\delta H[r]}{\delta r(x)} = \frac{Z}{2-D}(-\Delta)r(x) + bZ_\mu \varepsilon \int_y \int_k (2ik)e^{i[kr(x) - r(y)]}. \tag{206}
\]

Another example is obtained by choosing \( F[r] := r(x) \), yielding the redundant operator

\[
R = \frac{Z}{2-D} r(x)(-\Delta)r(x) + bZ_\mu \varepsilon \int_y \int_k [2ikr(x)]e^{i[kr(x) - r(y)]}. \tag{207}
\]

### 4.2 Analytic continuation of the measure

We now define the explicit form for the integration measure in non-integer dimension \( D \) using the general formalism of distance geometry [154, 54].

The general problem is to integrate a function \( f(x_1, \ldots, x_N) \), which is invariant under Euclidean displacements (and therefore depends only on the \( N(N-1)/2 \) relative distances \( |x_i - x_j| \) between these points) over the \( N-1 \) first points (the last point is fixed, using translational invariance) in \( \mathbb{R}^D \) for non-integer \( D \). In order to define the integration, let us take \( D \geq N - 1 \) and integer. For \( i < N \) we denote by \( y_i = x_i - x_N \) the \( i \)’th distance-vector and by \( y_i^a \) its \( a \)’th component (\( a = 1, \ldots, D \)).

The integral over \( y_1 \) is simple: Using rotation invariance, we fix \( y_1 \) to have only the first (\( a = 1 \)) component non-zero. The measure becomes with the normalizations as listed in appendix A.1

\[
\int_{y_1} = \frac{1}{S_D} \int d^D y_1 = \int_0^\infty dy_1^1 (y_1^1)^{D-1}, \quad y_1 = (y_1^1, 0, \ldots, 0). \tag{208}
\]

We now fix \( y_2 \) to have only \( a = 1 \) and \( a = 2 \) as non-zero components. The integral over \( y_2 \) consists of the integration along the direction fixed by \( y_1 \) and the integration in the orthogonal space \( \mathbb{R}^{D-1} \):

\[
\int_{y_2} = \frac{1}{S_D} \int d^D y_2 = \frac{S_{D-1}}{S_D} \int_{-\infty}^\infty dy_2^1 \int_0^\infty dy_2^2 (y_2^2)^{D-2}, \quad y_2 = (y_2^1, y_2^2, 0, \ldots, 0). \tag{209}
\]

For the \( j \)’th point, one proceeds recursively to integrate first over the hyper-plane defined by \( y_1, \ldots, y_{j-1} \) and then the orthogonal complement:

\[
\int_{y_j} = \frac{1}{S_D} \int d^D y_j = \frac{S_{D-j+1}}{S_D} \prod_{a<j} \int_{-\infty}^\infty dy_j^a \int_0^\infty dy_j^j (y_j^j)^{D-j}, \quad y_j = (y_j^1, \ldots, y_j^j, 0, \ldots, 0). \tag{210}
\]

The final result for an integral over all configurations of \( N \) points is

\[
\prod_{j=1}^{N-1} \int_{y_j} = \frac{S_{D-1} \cdots S_{D-N+2}}{S_D^{N-2}} \prod_{j=1}^{N-1} \left( \prod_{a=1}^{j-1} \int_{-\infty}^\infty dy_j^a \int_0^\infty dy_j^j (y_j^j)^{D-j} \right). \tag{211}
\]

This expression for the measure, now written in terms of the \( N(N-1) \) variables \( y_j^a \), can be analytically continued to non-integer \( D \). For \( D \leq N - 2 \) this measure is not integrable when some of the \( y_j^a \to 0 \). For
not integer, the integration is defined through the standard finite-part prescription. This means that the measure (211) becomes a distribution.

Let us made this explicit on the example of \( N = 3 \) points. The measure is then

\[
\frac{S_{D-1}}{S_D} \int_0^\infty dy_1 (y_1^{1})^{D-1} \int_0^{+\infty} dy_2 \int_0^{\infty} dy_2' (y_2^{2})^{D-2}.
\]

It is well defined and integrable for \( D > 1 \). For \( D = 1 \) the integral over \( y_2^{2} \) diverges logarithmically at \( y_2^{2} \to 0 \), but this singularity is canceled by the zero of \( S_{D-1} \) and the measure becomes

\[
\frac{1}{2} \int_0^\infty dy_1 \int_0^{+\infty} dy_2 \int_0^{\infty} dy_2' \delta(y_2^{2}) = \frac{1}{4} \int_0^\infty dy_1 \int_0^\infty dy_2,
\]

thus it reduces to the measure for two points on a line. (The factor of \( \frac{1}{4} = \left(\frac{1}{2}\right)^2 \) is due to our definition of the measure (713).) For \( 0 < D < 1 \) the integral over \( y_2^{2} \) diverges at \( y_2^{2} \to 0 \), but this divergence is treated by a finite part prescription.

For integrals over \( N > 3 \) points, a finite part prescription is already necessary for \( D < 2 \). This is the case of the 2-loop calculations, see section 6.

### 4.3 IR-regulator, conformal mapping, extraction of the residue, and its universality

For the simple case of the MOPE-coefficients Eq. (163) and Eq. (169), the residues could easily be calculated directly. In more complicated situations however, it is useful to employ a more formal procedure to extract the residue, which is presented now; first on the example of the 1-loop counter-terms, then in a more formal setting.

Note from (163) that

\[
\text{lim}_{\varepsilon \to 0} \left[ \frac{L}{\partial L} \langle \quad \bullet \quad \rangle_L \right] = -\frac{1}{2D} \int_{x < L} x^{D-\nu d} = -\frac{1}{2D} L^{\varepsilon}.
\]

(For the normalization of the measure, see appendix A.1). The residue can most easily be extracted by applying \( \frac{L}{\partial L} \) to (214). This yields

\[
\left[ \frac{L}{\partial L} \langle \quad \bullet \quad \rangle_L \right] = -\frac{1}{2D} L \int_x x^{D-\nu d} \delta(x - L) = -\frac{1}{2D} L^{\varepsilon}.
\]

So the residue of Eq. (214) is

\[
\langle \quad \bullet \quad \rangle^{\varepsilon} = \lim_{\varepsilon \to 0} \left[ \frac{L}{\partial L} \langle \quad \bullet \quad \rangle_L \right] = -\frac{1}{2D}.
\]

We can apply this recipe to the second 1-loop counter-term:

\[
\langle \quad \bullet \quad \rangle_L = \int_{x < L} \int_{y < L} \langle \quad \bullet \quad \rangle_L \quad \bullet \quad \rangle,
\]

\[\text{This section may be skipped upon first reading.}\]
since it is also proportional to \( L^{\varepsilon} \). We thus have to calculate
\[
L \frac{\partial}{\partial L} \langle \Phi(x,y) \rangle_L = L \left[ \int_{x<y=L} + \int_{y<x=L} \right] \left( x^{2\nu} + y^{2\nu} \right)^{-d/2}.
\] (218)

We now introduce a general method which is very useful to manipulate and simplify such integrals. It relies on (global) conformal transformations in position space and is called conformal mapping of sectors. It has first been introduced in [27], where a geometric interpretation can be found. We will explain the method on a concrete example and then state the general result.

Let us consider the second integral on the r.h.s. of Eq. (218)
\[
L \int_{y<x=L} \left( x^{2\nu} + y^{2\nu} \right)^{-d/2} = L \int_0^\infty \frac{dx}{x^D} \int_0^\infty \frac{dy}{y^D} \left( x^{2\nu} + y^{2\nu} \right)^{-d/2} \delta(x-L)\Theta(y<x).
\] (219)

Now two changes of variables are performed: The first one
\[
x \longrightarrow \tilde{x}, \quad x = \tilde{x}yL^{-1}
\] (220)
leads to
\[
L^{-1-D+\nu} \int_0^\infty \frac{d\tilde{x}}{\tilde{x}^{d/2}} \int_0^\infty \frac{d\tilde{y}}{\tilde{y}^{d-D+\nu}} \left( \tilde{x}^{2\nu} + \tilde{y}^{2\nu} + L^{2\nu} \right)^{-d/2} \delta(\tilde{x}yL^{-1} - L)\Theta(L < \tilde{x}).
\] (221)
The second one
\[
y \longrightarrow \tilde{y}, \quad y = \tilde{y}\tilde{x}^{-1}L
\] (222)
yields
\[
L^{1+D} \int_0^\infty \frac{d\tilde{x}}{\tilde{x}^{d-\varepsilon}} \int_0^\infty \frac{d\tilde{y}}{\tilde{y}^{d-\varepsilon}} \left( \tilde{x}^{2\nu} + L^{2\nu} \right)^{-d/2} \delta(\tilde{y} - L)\Theta(L < \tilde{x}),
\] (223)
which using \( \tilde{y} = L \) finally gives
\[
L^{1+\varepsilon} \int_0^\infty \frac{d\tilde{x}}{\tilde{x}^{d-\varepsilon}} \int_0^\infty \frac{d\tilde{y}}{\tilde{y}^{d-\varepsilon}} \left( \tilde{x}^{2\nu} + \tilde{y}^{2\nu} \right)^{-d/2} \delta(\tilde{y} - L)\Theta(\tilde{y} < \tilde{x}).
\] (224)
Replacing the second integral on the r.h.s. of Eq. (218) by Eq. (224) gives
\[
L \frac{\partial}{\partial L} \langle \Phi(x,y) \rangle_L = L^{1+\varepsilon} \int_0^\infty \frac{dx}{x^{D}} \int_0^\infty \frac{dy}{y^{D}} \left( x^{2\nu} + y^{2\nu} \right)^{-d/2} \max(x,y)^{-\varepsilon} \delta(y-L).
\] (225)

Now one distance (here \( y \)) is fixed, whereas the integral over the other distance (here \( x \)) runs from 0 to \( \infty \). The former constraint \( \max(x,y) = L \) has been transformed into the factor \( \max(x,y)^{-\varepsilon} \) times the constraint \( y = L \).

Before generalizing this formula, we shall show how it can be used in practice. The residue in \( \frac{1}{\varepsilon} \) (which determines the corresponding 1-loop counter-term) is given by the simple formula with \( d_c(D) = 4D/(2 - D) \):
\[
\langle \Phi(x,y) \rangle_L = \int_0^\infty \frac{dx}{x^{D}} \left( x^{2\nu} + 1 \right)^{-d_c(D)/2} = \frac{1}{2 - D} \Gamma \left( \frac{2D}{2 - D} \right).
\] (226)
The subleading term can analogously be calculated by expanding \( (x^{2\nu} + 1)^{-\varepsilon(d-d_c(D))/2} \) and \( \max(x,y)^{-\varepsilon} \) in \( \varepsilon \). We obtain the convergent integral representation
\[
\langle \Phi(x,y) \rangle_{\varepsilon} = \int_0^\infty \frac{dx}{x^{2\nu}} \left( x^{2\nu} + 1 \right)^{-d_c(D)/2} \left( \frac{1}{2 - D} \ln(x^{2\nu} + 1) - \ln(\max(x,1)) \right).
\] (227)
This method extends to the integrals which appear in the counter-terms associated to the contraction of any number of points. In general, we have to compute integrals over $N(N-1)$ distances $x, y, \ldots$, of the form

$$I(\varepsilon) = \int_{\max(x,y,\ldots) \leq L} f(x,y,\ldots)$$

with a homogeneous function $f$ such that the integral has a conformal weight (dimension in $L$) $\kappa$: $I(\varepsilon) \sim L^\kappa$. For the integrals which appear in $n$-loop diagrams, this weight is simply

$$\kappa = n \varepsilon.$$  \hfill (229)

The integral over the distances is defined by the $D$-dimensional measure (211). The residue is extracted from the dimensionless integral

$$J(\varepsilon) = \kappa L^{-\kappa} I(\varepsilon) = L^{-\kappa} L \frac{\partial}{\partial L} I(\varepsilon) = L \int_{\max(x,y,\ldots) = L} f(x,y,\ldots) \max(x,y,\ldots)^{-\kappa}.$$  \hfill (230)

The domain of integration can be decomposed into “sectors”, for instance

$$\{ \ldots < y < x = L \}, \{ \ldots < x < y = L \},$$  \hfill (231)

and we can map these different sectors onto each other by global conformal transformations. For instance we can rewrite the integral (230) as

$$J(\varepsilon) \equiv L \int_{x=L;y,\ldots} f(x,y,\ldots) \max(x,y,\ldots)^{-\kappa} \equiv L \int_{y=L;x,\ldots} f(x,y,\ldots) \max(x,y,\ldots)^{-\kappa}.$$  \hfill (232)

The constraint on the maximum of the distances is replaced by the constraint on an arbitrarily chosen distance.

This mapping of sectors is one of the basic tools used in calculating more complicated diagrams, e.g. at the 2-loop order or for the disorder-dynamics.

Also note that it implies the universality of the leading pole in $1/\varepsilon$: Fixing the longest, the shortest or one of the intermediate distances always give the same leading pole. Finally, starting from these regularizations, any other one can be constructed.

### 4.4 Factorization for $D = 1$, the Laplace De Gennes transformation

The methods described so far also apply to polymers. For polymers however, some simplifications are valid. Let us first give a simple (perturbative) example before stating the general result, namely the equivalence of polymers with the limit of zero components for a $\phi^4$-model.

In perturbation theory, we may consider expectation values with two dipoles inserted. The simplification for a polymer is, that its 1-dimensional nature enforces an ordering of the intervening points, resulting into topologically different diagrams, as exemplified by the equation below:

$$D \rightarrow 1 \quad + \quad + \quad \Rightarrow \quad \text{Diagram}.$$  \hfill (233)
It turns out that topologically different diagrams are also different analytically, whereas for membranes, a single diagram contains all these contributions. This is due to the additive nature of the polymer correlation-function, which reads for ordered points \(x_1 < x_2 < x_3\):

\[
C(x_1 - x_2) + C(x_3 - x_2) = C(x_3 - x_1). \tag{234}
\]

This property is used to simplify the perturbation expansion.

On a more formal level, there exists an integral-relation between 2-point functions for polymers and a local scalar field theory, which was first discovered by De Gennes [7] and which we discuss now, following the derivation in [155].

Consider the discretized version of the Edwards-model (suppressing all indices \(0\) for bare quantities)

\[
G(r, r_0, L) = \prod_{s=1}^{L} \int \frac{d^dr_s}{(4\pi\lambda)^{d/2}} \exp \left( -\sum_{t=0}^{L-1} \frac{(r_t - r_{t+1})^2}{4\lambda} - \frac{b\lambda^2}{4} \sum_{t=0}^{L} \sum_{u=0}^{L} \delta^d(r_t - r_u) \right),
\]

where the first and last monomer are fixed by \(r(0) = r_0, r(L) = r\). \(L\) is the number of monomers, \(\lambda\) their length, and the integration measure is normalized such that \(\int \frac{d^dr_s}{(4\pi\lambda)^{d/2}} \exp(-r^2/4\lambda) = 1\). With this normalization and setting \(b = 0\), \(G\) is the probability conserving diffusion propagator at time \(t = \lambda L\).

The self-avoidance interaction can be disentangled through an auxiliary field \(\Psi(r)\)

\[
G(r, r_0, L) = \int \mathcal{D}[\Psi] \prod_{s=1}^{L} \frac{d^dr_s}{(4\pi\lambda)^{d/2}} \exp \left( -\sum_{t=0}^{L-1} \frac{(r_t - r_{t+1})^2}{4\lambda} + \sum_{t=0}^{L} i\lambda\Psi(r_t) \right) \exp \left( -\int d^dr \frac{\Psi(r)^2}{b} \right),
\]

where a suitable normalization factor is absorbed into the integration measure \(\mathcal{D}[\Psi]\). One then sees, that the first part, namely the partition function of the polymer in the potential \(\Psi(r)\)

\[
G(r, r_0, L; \Psi) := \prod_{s=1}^{L} \int \frac{d^dr_s}{(4\pi\lambda)^{d/2}} \exp \left( -\sum_{t=0}^{L-1} \frac{(r_t - r_{t+1})^2}{4\lambda} + \sum_{t=0}^{L} i\lambda\Psi(r_t) \right)
\]

satisfies the equation

\[
G(r, r_0, L + 1; \Psi) = \int \frac{d^dr'}{(4\pi\lambda)^{d/2}} e^{-(r' - r)^2/4\lambda} e^{i\lambda\Psi(r')} G(r', r_0, L; \Psi). \tag{238}
\]

In the limit of a continuous chain, \(\lambda\) becomes small and the r.h.s. can be expanded in \(\lambda\), with the result

\[
G(r, r_0, L + 1; \Psi) = \left( 1 + i\lambda\Psi(r) + \lambda\Delta_r + \mathcal{O}(\lambda^2) \right) G(r, r_0, L; \Psi). \tag{239}
\]

This can also be written as \((\ell := \lambda L)\)

\[
\frac{\partial}{\partial \ell} G(r, r_0, \ell; \Psi) = (i\Psi(r) + \Delta_r) G(r, r_0, \ell; \Psi). \tag{240}
\]

Using a notation inspired from quantum mechanics, the solution to this equation is

\[
G(r, r_0, \ell; \Psi) = \left< r \left| e^{-\ell(-i\Psi(r) - \Delta_r)} \right| r_0 \right>. \tag{241}
\]

The usual method, to solve this equation in quantum mechanics, consists in going from the “time-dependent” Schrödinger-equation to the “time-independent” one. In statistical mechanics one equivalently writes down the Laplace-transform

\[
\tilde{G}(r, r_0, t; \Psi) := \int_0^{\infty} dt \ e^{-\ell t} G(r, r_0, \ell; \Psi), \tag{242}
\]
which gives
\[
\bar{G}(r, r_0, t; \Psi) = \left\langle r \left| \frac{1}{t - i\Psi(r) - \Delta r} \right| r_0 \rightangle.
\] (243)

This is up to a factor of \(Z^{-n}\) the correlation-function of a \(n\)-component scalar field theory,
\[
\bar{G}(r, r_0, t; \Psi) = Z^{-n} \int \mathcal{D}[\phi] \phi_1(r) \phi_1(r_0) e^{-\frac{1}{2} \int r \bar{\phi}(r)(t - i\Psi(r) - \Delta r)\phi(r)} ,
\] (244)

where \(Z\) is the partition function of the 1-component version,
\[
Z = \int \mathcal{D}[\phi] e^{-\frac{1}{2} \int r \bar{\phi}(r)(t - i\Psi(r) - \Delta r)\phi(r)} .
\] (245)

Formally, the factor of \(Z^{-n}\) is easily eliminated by setting \(n = 0\). Combining Eqs. (236), (242), (243) and the latter statement, one obtains that the Laplace-transformed polymer correlation-function
\[
\bar{G}(r, r_0, t) := \int_0^\infty dt e^{-\frac{\lambda t}{b}} G(r, r_0, \ell) ,
\] (246)

where \(G(r, r_0, \ell)\) is the continuum version of \(G(r, r_0, L)\), equals
\[
\bar{G}(r, r_0, t) = \lim_{n \to 0} \int \mathcal{D}[\Psi] \mathcal{D}[\bar{\phi}] \phi_1(r) \phi_1(r_0) e^{-\frac{1}{2} \int r \bar{\phi}(r)(t - i\Psi(r) - \Delta r)\phi(r)} e^{-\int r \Psi(r)^2/b} .
\] (247)

The path-integral over \(\Psi\) can still be performed to obtain the final result
\[
\bar{G}(r, r_0, t) = \lim_{n \to 0} \int \mathcal{D}[\phi] \phi_1(r) \phi_1(r_0) e^{-\mathcal{H}_{\phi^4}[\phi]}
\]
\[
\mathcal{H}_{\phi^4}[\phi] = \int d^4r \left( \frac{t}{2} \bar{\phi}^2(r) + \frac{1}{2} \left[ \nabla \bar{\phi}(r) \right]^2 + \frac{b}{16} \left[ \bar{\phi}^2(r) \right]^2 \right) .
\] (248)

This is the path-integral representation of a correlation function in the \(n\)-component \(\phi^4\)-model, after taking the limit of \(n \to 0\).

This remarkable result, first discovered by De Gennes [7], allows for two seemingly unrelated methods to calculate the same physical quantities.

The derivation given above allows for some straightforward generalizations. Consider as in Eq. (235)
\[
G(r, r_0, L) = \prod_{s=1}^{L-1} \int \frac{d^d r_s}{(4\pi \lambda)^{d/2}} \exp\left( -\mathcal{H}_{\text{polymer}}^{\text{gen}}[\rho] \right) ,
\] (249)

with
\[
\mathcal{H}_{\text{polymer}}^{\text{gen}}[\rho] = \sum_{t=0}^{L-1} \frac{(r_t - r_{t+1})^2}{4\lambda} + \int d^d r' \mathcal{F}[\rho(r')] .
\] (250)

\(\rho(r')\) is the polymer-density
\[
\rho(r') = \lambda \sum_{t=0}^{L} \delta^d(r_t - r') ,
\] (251)

and \(\mathcal{F}[\rho(r')]\) any functional of \(\rho(r')\). Then, following the same lines as above
\[
\bar{G}(r, r_0, t) = \lim_{n \to 0} \int \mathcal{D}[\phi] \phi_1(r) \phi_1(r_0) e^{-\mathcal{H}_{\mathcal{F}}[\phi]}
\]
\[
\mathcal{H}_{\mathcal{F}}[\phi] = \int d^d r \left( \frac{t}{2} \bar{\phi}^2(r) + \frac{1}{2} \left[ \nabla \bar{\phi}(r) \right]^2 + \mathcal{F} \left[ \frac{1}{2} \bar{\phi}^2(r) \right] \right) .
\] (252)
Some examples are (in continuous notation)

\[
\int dx \, \delta^d(r(x)) \leftrightarrow \int d^d r \, \frac{1}{2} \phi^2(r) \tag{253}
\]

\[
\int dx \int dy \, \delta^d(r(x) - r(y)) \leftrightarrow \int d^d r \, \left[ \frac{1}{2} \phi^2(r) \right]^2 \tag{254}
\]

\[
\int dx \int dy \int dz \, \delta^d(r(x) - r(y)) \delta^d(r(x) - r(z)) \leftrightarrow \int d^d r \, \left[ \frac{1}{2} \phi^2(r) \right]^3 \tag{255}
\]

\[
\int dx \int dy \, (-\Delta_r) \delta^d(r(x) - r(y)) \leftrightarrow \int d^d r \, \left[ \frac{1}{2} \phi^2(r) \right] (-\Delta_r) \left[ \frac{1}{2} \phi^2(r) \right] \tag{256}
\]

These relations are exploited when studying subdominant operators in section [28] and in the generalization to membranes of a \(N\)-component \(\phi^4\)-model in section 13.

The derivation also shows, and this is the reason why we have discussed it in detail, that a similar procedure is not amenable for membranes. Trying to generalize the above methods to membranes leads to the problem of solving the diffusion-equation of a \((D - 1)\)-dimensional membrane in a random potential, which is a formidable task. It seems that no local field-theory, which is equivalent to self-avoiding membranes can be constructed, even though this is hard to really prove.

Another hint comes from the following observation: Suppose, one wants to construct a scalar field theory with upper critical dimension \(d_c = \frac{4D}{2 - D}\), i.e. the same upper critical dimension as for \(D\)-dimensional polymerized tethered membranes. In addition, this theory shall have two non-trivial renormalizations, as otherwise the critical exponent \(\nu^*\) is a simple algebraic function in \(D\) and \(d\) – which certainly is wrong.

Now suppose that the Hamiltonian has the form \(\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}\), where \(\mathcal{H}_0\) is the free Hamiltonian and \(\mathcal{H}_{\text{int}}\) the interacting part. \(\mathcal{H}_0\) is

\[
\mathcal{H}_0 = \int_{r,r'} \phi(r) K(r - r') \phi(r') , \tag{257}
\]

with some kernel \(K(r - r')\). First, in order to have a non-trivial renormalization, \(K(r - r')\) has to be local, for reasons similar to what we discussed in section 3.9. The first possibility, \(K(r - r') = \delta^d(r - r')\) gives a trivial theory, so the simplest non-trivial choice is \(K(r - r') = \delta^d(r - r')(-\Delta)^n\), with \(n \geq 1\) integer, leading for the field \(\phi\) to a canonical dimension of \([\phi]_r = \frac{2n-d}{2}\). (There may of course be an additional massive term.)

The interaction can also be chosen local or non-local, where we do not consider \(\delta\)-like interactions, by assumption. Non-local interactions of the form \(\int_{r,r'} \phi^m(r) K'(r - r') \phi^m(r')\) with a suitable kernel can be rendered local through an auxiliary field. However since only local interactions renormalize, this fixes the interaction to have the form

\[
\mathcal{H}_{\text{int}} := \int_r \phi^m(r) , \quad \text{with} \quad m = \frac{2d_c}{d_c - 2n} . \tag{258}
\]

Only for special values of \(d_c, m\) will be integer and a non-trivial renormalization is possible.

Thus it seems already impossible to construct a non-trivial scalar field-theory with the correct upper critical dimension.
5 Proof of perturbative renormalizability

5.1 Introduction

In this section, we discuss the proof of perturbative renormalizability to all orders in perturbation theory as given by David, Duplantier and Guitter [53, 54, 56, 57]. (A pedagogic presentation of some of the ideas can also be found in a course by Guitter [156]; in French.) The proof itself is lengthy and rather involved. It is impossible to give more than a condensed version here. Since the proof stands in the context of methods developed for local field theory, as the $\phi^4$-model, we start our discussion there. The first to construct a general theory of renormalization were Bogoliubov and Parasiuk [157]. They introduced what since then is called a R-operation, which subtracts the divergences from a given Feynman-diagram. However it still took more than ten years, before Hepp [158] could prove rigorously, that this operation indeed renders Feynman-integrals not only finite, but even absolutely convergent. (This means that the integrals involved have no more non-integrable divergences and can e.g. be performed by numerical integration, supposed that one is looking at an Euclidean theory.) This was done by considering each ordering of the distances in the Feynman-integrals, the since then so-called Hepp-sectors, separately. Renormalization however demands more: A theory is said to be renormalizable if and only if divergences can be absorbed into a finite number of “renormalized” (i.e. redefined) quantities, which are in general proportional to the original quantities, with a proportionality factor of $Z$, which may also be a matrix $Z_{ij}$ to allow for mixing of the operators. It remains to show that the R-operation can indeed be interpreted as a multiplicative renormalization, i.e. to introducing $Z$-factors. This was most clearly demonstrated by Zimmerman [55], who reformulated the R-operation in terms of forests, i.e. mutually disjoint or included sets. A lot of material can be found in [159, 160, 161, 162].

An equivalent formulation, which in some respects is technically more convenient uses nests. It is this formulation of the proof of perturbative renormalizability, introduced by Bergère and Lam [163], which finally has been generalized by David, Duplantier and Guitter [53, 54, 56, 57] to polymerized tethered membranes. In order to do so, they had to overcome three major difficulties: First, the proof of [163] works in the Schwinger-proper-time formulation ($\alpha$-parameter representation) after elimination of the momentum-integrals. In the language of membranes, this is the special case of polymers, for which a general membrane-diagram splits into topologically disjoint polymer-diagrams. This allows the simplification, that a Hepp-sector is just an ordering of nearest neighbor distances. Clearly, such an ordering is no longer possible for tethered membranes, leading to a more involved definition of Hepp-sectors. Bergère and Lam [163] then construct what they call a “tableau” in order to show the absolute convergence of the subtracted Feynman-integrals in each sector separately. Also this construction gets more involved in the case of membranes as will be discussed below. The second difficulty was that membrane integrals over relative distances are distributions instead of simple integrals as in the case considered in [163]. These two problems were first clarified in the context of a simplified model, namely the interaction of a phantom membrane with an impurity, i.e. a single $\delta$-interaction [53, 54]. Finally, for self-avoiding membranes, also bilocal (or more generally multilocal) counter-terms as e.g. the interaction $\tilde{\delta}^d(r(x) - r(y))$ had to be incorporated into the R-operation. This was achieved by introducing the multi-local operator product expansion [56, 57], discussed in the preceding sections.

Let us now proceed to the proof. In order to do so, we have to introduce some notations and definitions.

5.2 Proof
Proof of perturbative renormalizability

Figure 13: Connected components (dashed lines) and δ-interactions, solid lines. The resulting connected molecules are shaded in gray.

1. Definition: $X, A, I, I^{\text{conn}}$
Consider a general term in the perturbation-expansion of the observable $O(z_1, \ldots, z_m)$. The interaction is δ-like and $l$ is the number of points on which it depends, i.e. e.g. $l = 2$ for the usual self-avoidance. Perturbation theory consists of $N$-fold integrals over the positions $A = \{\vec{x}_1, \ldots, \vec{x}_N\}$ of all of the $N$ involved vertices, $x_1, \ldots, x_N$, denoted by $\mathcal{X} := \{x_1, \ldots, x_N\}$. Note that we carefully distinguish the vertex $x_i$ from its position $\vec{x}_i$. Explicitly, the term has the form $I(\vec{x}_1, \ldots, \vec{x}_N) = \langle O(z_1, \ldots, z_m) \Phi(\vec{x}_1, \ldots) \cdots \Phi(\ldots, \vec{x}_N) \rangle_0$.

We also consider connected expectation values $I^{\text{conn}}(\vec{x}_1, \ldots, \vec{x}_N) = \langle O(z_1, \ldots, z_m) \Phi(\vec{x}_1, \ldots) \cdots \Phi(\ldots, \vec{x}_N) \rangle_0^{\text{conn}}$. $\Phi$ may be the $(l = 2)$-point $\tilde{\delta}^d$-interaction, but the same formulation applies to $(l = 1)$-, $(l = 3)$- or higher point interactions and is unnecessary to be specified for the following.

2. Definition: Diagram, connected components
A diagram is a partition of any subset of $\mathcal{X}$ into disjoint subsets $C_i$, which we denote by $\mathcal{P} = \{C_1, \ldots, C_n\}$. The elements of $\mathcal{P}$ are called connected components.

3. Definition: Inclusion, intersection, and union of diagrams
A diagram $\mathcal{P}_1$ is included in a diagram $\mathcal{P}_2$, $\mathcal{P}_1 \prec \mathcal{P}_2$, if every connected component of $\mathcal{P}_1$ is subset of a connected component of $\mathcal{P}_2$. Note that inclusion defines a half-ordering on the set of diagrams.

The intersection $\mathcal{P}_1 \cap \mathcal{P}_2$ of two diagrams $\mathcal{P}_1$ and $\mathcal{P}_2$ is the unique maximal diagram $\mathcal{P}$ included in both $\mathcal{P}_1$ and $\mathcal{P}_2$. (I.e. $\forall \mathcal{P} \prec \mathcal{P}_1$ and $\mathcal{P} \prec \mathcal{P}_2 \Rightarrow \mathcal{P} \prec \mathcal{P}_1 \cap \mathcal{P}_2$.)

The union $\mathcal{P}_1 \cup \mathcal{P}_2$ of two diagrams $\mathcal{P}_1$ and $\mathcal{P}_2$ is the unique minimal diagram $\mathcal{P}$ which contains both $\mathcal{P}_1$ and $\mathcal{P}_2$. (I.e. $\forall \mathcal{P}$ with $\mathcal{P}_1 \prec \mathcal{P}$ and $\mathcal{P}_2 \prec \mathcal{P} \Rightarrow \mathcal{P}_1 \cup \mathcal{P}_2 \prec \mathcal{P}$.)

4. Definition: Root
A set $C$ of points is rooted by choosing one of its elements $\omega$ as root, and is denoted $C_\omega = \{C, \omega\}$. Similar, a rooted diagram $\mathcal{P}_\omega$ is a diagram, where each connected components $C_i$ is rooted $\mathcal{P}_\omega = \{\{C_i, \omega_i\}\}$.

5. Definition: Connected Molecule
Consider an arbitrary diagram $\mathcal{P}$ with connected components $C_i$. Grouping together those connected components, which are linked together by δ-interactions (see figure 13) yields a partition of the diagram $\mathcal{P}$ into subdiagrams $\mathcal{P}_i$. These subdiagrams $\mathcal{P}_i$ are called connected molecules.

6. Definition: Forest
A forest is a set $\mathfrak{F} = \{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$ of pairwise distinct connected molecules which each consist of at
least one diagram with at least two points \(|P_i| \geq 2\), and which are either disjoint or included, i.e. for all \(i \neq j\): \(P_i \cap P_j = \{\}\), or \(P_i \subset P_j\) or \(P_i \supset P_j\). This inclusion defines a half-ordering for the diagrams in \(\mathcal{F}\). The empty set \(\{\}\) is also considered a forest.

It is important to realize that this definition depends on the theory considered via the \(\delta\)-interactions; and may turn the analysis difficult. This is the reason, why later a different formulation in terms of nests will be introduced which is independent of the underlying interaction. For an example cf. example 62.

7. Definition: (Compatibly) rooted forest

A forest \(\mathcal{F}\) is rooted, if its elements are rooted. It is called compatibly rooted and denoted \(\mathcal{F}_\oplus\), if for any two elements \(\{P_i, \omega_i\}, \{P_j, \omega_j\}\) which are included, i.e. \(P_i \subset P_j\), either \(\omega_j = \omega_i\) or \(\omega_j \notin P_i\). Note that any forest can be compatibly rooted.

8. Definition: Weight of a compatibly rooted forest

The weight of a compatibly rooted forest is defined as the inverse of the product over all roots of the number of points in the largest connected component, containing the root: \(W(\mathcal{F}_\oplus) := \prod_{\omega_i \in \mathcal{F}_\oplus} [\max(|C_j|, \omega_i | C_j \in C)]^{-1}\). This definition is chosen such that summing over all compatible rootings of a given (unrooted) forest equals unity, \(\sum_{\text{compatible rootings}} W(\mathcal{F}_\oplus) = 1\).

9. Definition: Characteristic function of a diagram

The characteristic function \(\chi(C)\) of the connected component \(C \in P\) is defined as \(\chi(C) := \prod_{x_i, x_m \in C} \Theta((L - |\vec{x}_i - \vec{x}_m|))\). The characteristic function of a diagram \(P\) is defined as the product of the characteristic functions of its connected components: \(\chi(P) := \prod_{C_i \in P} \chi(C_i)\).

10. Definition: Dilation- and Taylor-operator

The Taylor-operator \(T_{P_\omega}\) of a diagram \(P_\omega\) is defined as the sum over all divergent contributions in the short-distance expansion, when contracting the connected components in \(P_\omega\) towards their roots, as given by the MOPE. If the divergence is logarithmic at the upper critical dimension, the latter is multiplied by the characteristic function \(\chi(P_\omega)\). (Recall that only for logarithmic divergences, a cut-off is needed.) More specifically, we define a dilation-operator \(D_{P_\omega}\), which contracts points in the same connected component towards their common root by

\[
D_{P_\omega} : \vec{x}_m \rightarrow \vec{x}_m(\lambda) := \begin{cases} 
\vec{x}_{\omega_i} + \lambda(\vec{x}_m - \vec{x}_{\omega_i}) & \text{if } x_m \in C_{i, \omega_i} \\
\vec{x}_m & \text{if not } \exists_i, x_m \in C_i
\end{cases}
\]

The MOPE then yields the short-distance behavior

\[
I(\vec{x}_1(\lambda), \ldots, \vec{x}_N(\lambda)) \xrightarrow{\lambda \rightarrow 0} \sum_\sigma \lambda^\sigma \left(T_{P_\omega}^{(\sigma)}I\right)(\vec{x}_1, \ldots, \vec{x}_N),
\]

which defines \(T_{P_\omega}^{(\sigma)}\), as discussed in section 3.5, see also the examples in 3.6. All terms \(T_{P_\omega}^{(\sigma)}I\) which are superficially UV-divergent when integrating over the points in \(C_{i, \omega_i}\), but \(\omega_i\) have to be subtracted. To this end, define \(\sigma_{\text{max}}\) as \(-\tilde{D}\) times the number of integrations \(\sigma_{\text{max}} := -\tilde{D} \sum_i (|C_i| - 1)\). The Taylor-operator \(T_{P_\omega}\) of a rooted diagram \(P_\omega\) applied to \(I\) is then defined as (setting \(\varepsilon = 0\))

\[
T_{P_\omega} I := \sum_{\sigma < \sigma_{\text{max}}} T_{P_\omega}^{(\sigma)}I + \chi(P_\omega) T_{P_\omega}^{(\sigma_{\text{max}})}I.
\quad (259)
\]

The last term gives rise to a logarithmic divergence at \(\varepsilon = 0\), thus demanding the introduction of an IR-cutoff (regularization scale). Note that it may, as well as the other terms, be absent from \(T_{P_\omega}\).

We also use the rule that a Taylor-operator does not act on the IR-cutoff. This is important when applying several Taylor-operators successively.

There is now the following important
11. **Lemma:** *Factorization of Taylor-operators in a forest*

Taylor-operators for diagrams that belong to the same forest factorize.

12. **Proof:**

Let $\mathcal{P}_1, \mathcal{P}_2 \in \tilde{\mathcal{F}}$ be elements of the same forest. Then either $\mathcal{P}_1 \wedge \mathcal{P}_2 = \{\}$ or without loss of generality $\mathcal{P}_1 \prec \mathcal{P}_2$. In the first case, this is a consequence of the MOPE. In the second case, this is a consequence of the MOPE and the fact that a Taylor-operator does not act on the IR-cutoff.

We are now in a position to define the subtraction operation.

13. **Definition:** *R-operation*

The subtraction operator $R$ is defined as the sum over all compatibly rooted forests $\tilde{\mathcal{F}}_{\oplus}$ of the Taylor-operator defined by this forest:

$$ R := \sum_{\tilde{\mathcal{F}}_{\oplus}} W(\tilde{\mathcal{F}}_{\oplus}) \prod_{P_{\omega} \in \tilde{\mathcal{F}}_{\oplus}} (-T_{P_{\omega}}) . $$

(260)

Since the empty set is also considered a forest, it contributes the identity. $R$ could also be defined as the sum over all forests, then choosing an arbitrary compatible rooting. Also note, that Taylor-operators of forests that contain only superficially UV-convergent diagrams, vanish identically, and can be excluded from the sum.

We now state the central theorem of David, Duplantier and Guitter [57].

14. **Theorem:** *Renormalizability*

(i) The renormalized integral

$$ \int_{\vec{x}_1, \ldots, \vec{x}_N} R I(\vec{x}_1, \ldots, \vec{x}_N) $$

is UV-finite at $\varepsilon = 0$.

(ii) The renormalized integral, which contributes to the connected expectation value of the observable $O$ at $n$-th order,

$$ O_R^{(n)}(\vec{z}_1, \ldots, \vec{z}_m) := \int_{\vec{x}_1, \ldots, \vec{x}_N} R I_{O_{\text{conn}}}^{\text{conn}}(\vec{x}_1, \ldots, \vec{x}_N) , $$

with $N = \ln$, is UV-finite and IR-finite at $\varepsilon = 0$.

(iii) In perturbation theory, the renormalized expectation value of an observable is given by

$$ O_R(\vec{z}_1, \ldots, \vec{z}_m) := \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} O_R^{(n)}(\vec{z}_1, \ldots, \vec{z}_m) . $$

(iv) The subtraction operation $R$ is equivalent to multiplicative renormalization, i.e. to introducing $Z$-factors in the standard way.

15. **Proof:** *Multiplicative Renormalizability*

Part (iv) of the above theorem is an immediate consequence of the forest structure of the diagrams.

16. To prove parts (i) to (iii) of the theorem, we first rewrite the subtraction operation $R$ in terms of nests instead of forests. Then, by analyzing the reformulated subtraction-operation in “Hepp-sectors”, i.e. within a given ordering of the distances, finiteness of the integrals will be proven.

To do so, we need some more definitions.

17. **Definition:** *Complete diagram, completion of a diagram*

A complete diagram is a diagram, that contains all points of $\mathcal{X}$. In other words, it is a partition of $\mathcal{X}$. An arbitrary diagram $\mathcal{P}$ can be completed by adding for any of the not already included points $x_i$ the set $\{x_i\}$. We shall sometimes use the same notation $\mathcal{P}$ for this completed diagram, whenever confusion is impossible. The completion of the empty set is the union of all sets containing one of the points $x_i$, and is (in the sloppy notation mentioned above) denoted $\mathcal{P}() = \{\{x_1\}, \ldots, \{x_N\}\}$. We also note $\mathcal{P}_{\mathcal{X}} = \{\{x_1, \ldots, x_N\}\} \equiv \{\mathcal{X}\}$. 
18. Definition: Nest

A nest $\mathcal{N}$ is a set of $l + 1$ (with $l < N$) complete diagrams, that are strictly ordered by inclusion, and that contain $\mathcal{P}_\{\}$: $\mathcal{N} = \{ \mathcal{P}_0 = \mathcal{P}_\{\} \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots \prec \mathcal{P}_l \}$. The smallest possible nest thus is $\{ \mathcal{P}_\{\} \}$ and not $\{ \}$. Note that sometimes nests are defined as strictly ordered sets of diagrams instead of complete diagrams as in [54], necessitating the notion of complete nests for our definition 18. In our treatment all elements of nests will be thought of as completed by definition 17.

19. Definition: Compatibly rooted nest, its weight

A compatibly rooted nest is a nest, that is compatibly rooted by the same definition 7 as for forests. Also the weight of a compatibly rooted nest is the same as for a compatibly rooted forest. Summing the weight over all compatible rootings as in definition 8 yields unity.

We now reformulate the subtraction operation $R$ as sum over nests instead of a sum over forests.

20. Lemma: $R$-operation in terms of nests

The $R$-operation defined in Eq. (260) can equivalently be written as the sum over all compatibly rooted nests,

$$ R := - \sum_{\mathcal{N}_\oplus} W(\mathcal{N}_\oplus) \prod_{\mathcal{P}_\omega \in \mathcal{N}_\oplus} (-T_{\mathcal{P}_\omega}) . $$

(261)

21. Proof:

For every nest $\mathcal{N}$, one can define the associated forest $\mathcal{F}(\mathcal{N})$ as the union of all connected molecules with at least two elements, that build up the elements of $\mathcal{N}$. If the nest is compatibly rooted, also the forest will be compatibly rooted, and is denoted $\mathcal{F}_\oplus(\mathcal{N}_\oplus)$. This defines equivalence classes on the set of all nests. (An explicit example is given in example 62.) One then shows that for a given forest $\mathcal{F}_\oplus$

$$ \sum_{\mathcal{N}_\oplus, \mathcal{F}_\oplus(\mathcal{N}_\oplus) = \mathcal{F}_\oplus} \prod_{\mathcal{P}_\omega \in \mathcal{N}_\oplus} (-T_{\mathcal{P}_\omega}) = \prod_{\mathcal{P}_\omega \in \mathcal{F}_\oplus} (-T_{\mathcal{P}_\omega}) $$

and that the weight factors inside the class $\mathcal{F}_\oplus(\mathcal{N}_\oplus)$ coincide. The final step is to remark that the sum over nests also contains the trivial diagram $\mathcal{P}_\{\}$, which accounts for the global minus sign in Eq. (261).

22. The analysis and proof of finiteness of the subtracted integrand $R I$ is performed separately for each ordering of the distances in $I$, the “Hepp-sectors”. This construction avoids problems with “overlapping divergences” known from standard scalar field-theories and is a generalization of the construction in terms of “tableaus” in [163], which reduces to the latter in the case of $D = 1$. We have to introduce some more definitions.

23. Definition: Hepp-Sector, spanning tree

The spanning tree of $\mathcal{X} = \{ x_1, \ldots, x_N \}$ is defined as follows: (i) Link the two vertices $x_i, x_j \in \mathcal{X}$ together that have the smallest mutual distance. This yields $\lambda_1 = \pm(\vec{x}_i - \vec{x}_j)$, and fuses $x_i$ and $x_j$ into a compound. (ii) Repeat this procedure recursively, without forming a closed loops, i.e. link together points $x_i, x_j \in \mathcal{X}$ that belong to different compounds, and that have the shortest mutual difference. This yields $\lambda_k, k = 1 \ldots N - 1$.

Further denote by $T$ the incidence matrix of the above construction, i.e. the information which vertices are linked together by which $\lambda_k$. The Hepp-sector $\mathcal{H}^T$ associated to $T$ is the part of the domain of integration $\mathcal{A}$ for which the above construction leads to the same incidence matrix $T$. 

24. Definition: Saturated nest
A saturated nest is a maximal nest of \( \mathcal{X} \), i.e., a nest \( \mathcal{S} \), that consists of \( N \) ordered complete diagrams and contains \( \mathcal{P}_I \) and \( \mathcal{P}_X \).
\[
\mathcal{S} = \left\{ S^0 = \mathcal{P}_I \prec S^1 \prec \ldots \prec S^{N-2} \prec S^{N-1} = \mathcal{P}_X \right\}. \quad (\text{The upper indices are introduced for later convenience.})
\]

25. Definition: Extended Hepp-sector
To each saturated nest \( \mathcal{S} \) is associated an extended Hepp-sector \( \mathcal{H}^{\mathcal{S}} \). Define for any diagram \( S \) the distance between its connected components \( C_i \) and \( C_j \) as \( d_{ij} := \min(\{ |\vec{x} - \vec{y}|, x \in C_i, y \in C_j \}) \). The minimal distance in a diagram \( S \), \( d_{\min}(S) \) is then defined as the minimal distance between its connected components, \( d_{\min}(S) := \min(d_{ij}, \text{with } C_i, C_j \in S \text{ and } C_i \neq C_j) \). If \( S \) has only one connected component, we define \( d_{\min}(S) := \infty \). The extended Hepp sector, or shortly sector, \( \mathcal{H}^{\mathcal{S}} \), is finally defined as the set of coordinates \( \vec{x}_i \), such that \( d_{\min}(\mathcal{P}_I) < d_{\min}(S^1) < \ldots < d_{\min}(S^I) < \ldots < d_{\min}(\mathcal{P}_X) = \infty \). It is the disjoint union of Hepp-sectors \( \mathcal{H}^{T_i} \) spanned by trees \( T_i \). Trees \( T_i \), that yield the same extended Hepp-sector \( \mathcal{S} = \mathcal{S}(T_i) \) form an equivalence-class. Furthermore, the extended Hepp-sectors themselves form a partition of the domain of integration \( \mathcal{A} \): If \( \mathcal{S} \neq \mathcal{S}' \), then \( \mathcal{H}^{\mathcal{S}} \cap \mathcal{H}^{\mathcal{S}'} = \{ \} \), and \( \bigcup_{\mathcal{S}} \mathcal{H}^{\mathcal{S}} = \mathcal{A} \).

26. Definition: Tree associated to a saturated rooted nest
A saturated rooted nest \( \mathcal{N}_{\omega} = \left\{ N^0_{\omega_0} = \mathcal{P}_I \prec N^1_{\omega_1} \prec \ldots \prec N^I_{\omega_I} \prec \ldots \prec N^{N-1}_{\omega_{N-1}} \right\} \) naturally defines a spanning tree \( T \) as follows: When going from \( N^I_{\omega_I} \) to \( N^{I+1}_{\omega_{I+1}} \), two rooted connected components \( C^I_{\omega_I} \) and \( C^{I+2}_{\omega_{I+2}} \) of \( N^I_{\omega_I} \) are fused into a single connected component. For the spanning tree choose \( \bar{\lambda}_I := \pm(\bar{\omega}_{I,1} - \bar{\omega}_{I,2}) \).

27. Definition: Rooted union and subtraction of diagrams
The rooted union of a diagram \( T \) and a rooted diagram \( \mathcal{R}_\omega \) with roots \( \omega \) is defined as the unrooted diagram \( T \cup_\omega \mathcal{R} := [T \backslash (\mathcal{R}_\omega)] \cup \mathcal{R} \). The most intuitive way to understand this definition is to consider the connected components of \( T \), and to modify them by moving the points of each connected component \( \mathcal{R}_{i,\omega_i} \) of \( \mathcal{R} \) into the connected component of \( T \), where its root \( \omega_i \) lies, or leave it unchanged, if its root is not in any connected component of \( T \). This gives the connected components of \( T \cup_\omega \mathcal{R} \). (Empty sets are eventually to be disregarded.)

We also define the rooted subtraction \( T \setminus_\omega \mathcal{R} \) as \( T \setminus_\omega \mathcal{R} := T \backslash (\mathcal{R}_\omega) \). It is an unrooted diagram, which is obtained by replacing in \( T \) all points of \( \mathcal{R} \) by their corresponding root. (This can be seen as the set-theoretical action of the subtraction operator \( T_{\mathcal{R}_\omega} \).)

28. We now introduce “tableaus”. This is a general construction to group terms together, which cancel in a given Hepp-sector. The result of this construction will thus crucially depend on the Hepp-sector. Some illustrating examples are given in the next section 5.3, see examples 63 – 65.

29. Definition: Tableau
Let \( \mathcal{S} = \left\{ S^0 = \mathcal{P}_I \prec S^1 \prec \ldots \prec S^{N-1} = \mathcal{P}_X \right\} \) be a saturated nest and \( \mathcal{H}^{\mathcal{S}} \) the associated generalized Hepp-sector. Let further \( \mathcal{N}_{\oplus} = \left\{ \mathcal{P}_0 = \mathcal{P}_I \prec \mathcal{P}_1 \prec \ldots \prec \mathcal{P}_I \right\} \) a compatibly rooted nest with roots \( \{ \omega_0, \ldots, \omega_I \} \) (defining the Taylor-operator \( T_{\mathcal{N}_{\oplus}} \) in the subtraction operation \( \mathcal{R} \)) to which for convenience we add (the unrooted set) \( \mathcal{P}_{T+1} = \mathcal{P}_X \) (even if it might already be present in \( \mathcal{N}_{\oplus} \); for \( \mathcal{P}_I \) and \( \mathcal{P}_X \) see definition 17). We then define the tableau \( T^f_j \) as
\[
T^f_j := (S^I \cup_{\omega_j} \mathcal{P}_j) \cap \mathcal{P}_{J+1}. \quad (262)
\]
We think of the tableau as a large matrix made out of diagrams, where the columns are numbered by the
Figure 14: Rooted union $\mathcal{T} \vee_\omega \mathcal{R}$ (thick lines) of $\mathcal{T}$ (dashed lines) and $\mathcal{R}_\omega$ (thin solid lines) with roots $\omega$ (squares).

\[
\begin{pmatrix}
T_0^0 & T_0^1 & \ldots & T_0^I & \ldots & T_0^{N-1} \\
T_1^0 & T_1^1 & \ldots & T_1^I & \ldots & T_1^{N-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
T_I^0 & T_I^1 & \ldots & T_I^I & \ldots & T_I^{N-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
T_T^0 & T_T^1 & \ldots & T_T^I & \ldots & T_T^{N-1} 
\end{pmatrix}
\]

Figure 15: Tableau

subtracting diagram $\mathcal{P}_J$, $J = 0, \ldots, T$ and the rows by the sector element $S^I$, $I = 0, \ldots, N - 1$. (See figure 15.)

30. **Lemma: Ordering of the tableau**

The tableau defined in Eq. (262) is ordered, when reading it like this article (from left to right and top to bottom). Formally, $T_J^I < T_J^{I+1}$ and $T_J^{N-1} < T_J^{0+1}$.

31. **Proof:**

The first of these inclusions follows directly from the definition. For the second one, we have $T_J^{N-1} = (\mathcal{P}_X \vee_\omega \mathcal{P}_J) \land \mathcal{P}_{J+1} = \mathcal{P}_X \land \mathcal{P}_{J+1} = \mathcal{P}_{J+1} \vee_\omega \mathcal{P}_J = T_{J+1}^0$, proving not only the inclusion, but even the equivalence.

32. This shows that the elements of the tableau form an (unrooted) nest $\tilde{\mathfrak{N}}(\mathfrak{S}, \mathfrak{N}_0)$ which contains $\mathfrak{N}$ and which has many identical elements. We now look for all nests $\mathfrak{N}$, that under the above construction yield the same $\tilde{\mathfrak{N}}(\mathfrak{S}, \mathfrak{N}_0)$. The key-idea for that construction is that complete lines may be removed from the tableau, if two vertically adjacent diagrams are equivalent; since then all elements of the tableau in between them are also equivalent, and thus redundant. One then shows that this is equivalent to eliminating
one element from $\mathcal{N}_\oplus$.

We need the following

33. Lemma: Reduction of the tableau

If $T'_I = T'_{J+1}$, then eliminating all elements between these two members of the tableau (in the above discussed natural reading ordering) including one of them, is equivalent to constructing the tableau from

$$\mathcal{N}_\oplus := \{ p_0 = p_1 < \ldots < p_{J-1} < p_{J+1} < \ldots < p_T \},$$

which is obtained from $\mathcal{N}_\oplus$ by omitting the element $p_J$.

34. Proof:

This is proven in appendix E of [54].

35. This procedure can be repeated, until no longer two vertically adjacent elements of the tableau coincide. There is another important

36. Lemma: Commutativity of the reduction-operation

The reduction procedure as defined above is commutative, i.e. independent of the ordering. It therefore defines a unique rooted minimal nest $\mathcal{N}_0\oplus$, such that $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_0\oplus) = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus)$. Note that $\mathcal{N}_0\oplus$ never contains $p_X$, since the latter is added as element $p_{T+1}$ to $\mathcal{N}_\oplus$.

37. Proof:

This is also proven in [54], p. 617 ff.

We can now define equivalence-classes of nests:

38. Definition: Equivalence-class of nests

The reduction procedure prescribed above defines equivalence classes on the set of all compatibly rooted nests (i.e. those appearing in the $R$-operation in Eq. (261)) by $C_\oplus(\mathcal{N}_\oplus) := \{ \mathcal{N}_\oplus', \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus') = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus) \}$.

39. Lemma: Characterization of $C_\oplus(\mathcal{N}_\oplus)$

Given a compatibly rooted nest $\mathcal{N}_\oplus$, then its equivalence-class $C_\oplus(\mathcal{N}_\oplus)$ is the set of all compatibly rooted nests $\mathcal{N}_\oplus'$ such that $\mathcal{N}_0\oplus \subset \mathcal{N}_\oplus' \subset \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_0\oplus)$.

40. Proof:

(i) $\mathcal{N}_\oplus' \in C_\oplus(\mathcal{N}_\oplus) \Rightarrow \mathcal{N}_0\oplus \subset \mathcal{N}_\oplus' \subset \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_0\oplus)$, has been shown above, and the first one follows from the uniqueness of the reduction procedure to the minimal nest $\mathcal{N}_0\oplus$.

(ii) $\mathcal{N}_\oplus \subset \mathcal{N}_\oplus' \subset \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_0\oplus)$ \Rightarrow $\mathcal{N}_\oplus' \in C_\oplus(\mathcal{N}_\oplus)$. This is proven in appendix F of [54].

We now need the following combinatorial

41. Lemma: Sum over weights

Let $\mathcal{N}_\oplus$ a compatibly rooted nest and $\mathcal{M}$ an unrooted nest which contains $\mathcal{N}_\oplus$. Then denote by $\mathcal{M}_\oplus \supset \mathcal{N}_\oplus$ all rootings of $\mathcal{M}$ that are compatible with $\mathcal{N}_\oplus$. For the sum over these roots, we have

$$\sum_{\mathcal{M}_\oplus \supset \mathcal{N}_\oplus, \mathcal{M} \text{ fixed}} W(\mathcal{M}_\oplus) = W(\mathcal{N}_\oplus). \quad (263)$$

42. Proof:

The idea of the proof is to construct all possible rootings of a nest $\mathcal{N}$ as follows: Enumerate the vertices. In a given diagram, the root is always assigned to the vertex with the smallest number. Then check that the weight of such a configuration is given by the number of possible enumerations of the vertices that yield the same rooting, divided by $|\mathcal{N}|!$, where $|\mathcal{N}|$ is the number of vertices in $\mathcal{N}$. The proof of the above lemma is then straightforward; applying the construction to $M$. (Details are given in appendix G of [54].)
43. As announced above, this allows to rewrite the $R$-operation as sum over equivalence-classes, and to reduce the analysis of the finiteness to that of the subtracted integrand within each equivalence-class. To this aim start with

$$R = \sum_{\mathcal{N}_\oplus^0 \text{minimal w.r.t. } \mathcal{S}} R_{C_\mathcal{S}(\mathcal{N}_\oplus^0)}, \quad R_{C_\mathcal{S}(\mathcal{N}_\oplus^0)} := - \sum_{\mathcal{R}_\oplus \in C_\mathcal{S}(\mathcal{N}_\oplus^0)} W(\mathcal{N}_\oplus^0) \prod_{P_\oplus \in \mathcal{N}_\oplus^0} (-T_{P_\oplus})$$

To rewrite $R_{C_\mathcal{S}(\mathcal{N}_\oplus^0)}$ in a factorized form, we first insert Eq. (263) with $\mathcal{M} = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus^0)$, and then use the characterization of $C_\mathcal{S}(\mathcal{N}_\oplus^0)$ to get

$$R_{C_\mathcal{S}(\mathcal{N}_\oplus^0)} = - \sum_{\mathcal{R}_\oplus \in C_\mathcal{S}(\mathcal{N}_\oplus^0)} \mathcal{W}(\mathcal{N}_\oplus^0) \prod_{P_\oplus \in \mathcal{N}_\oplus^0} (-T_{P_\oplus})$$

This can still be rewritten as

$$R_{C_\mathcal{S}(\mathcal{N}_\oplus^0)} = - \sum_{\mathcal{M}_\oplus^0 \supseteq \mathcal{N}_\oplus^0, \mathcal{M} = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus^0)} \mathcal{W}(\mathcal{N}_\oplus^0) \prod_{P_\oplus \in \mathcal{N}_\oplus^0} (-T_{P_\oplus}) \prod_{P_\oplus \in \mathcal{M}_\oplus^0 \setminus \mathcal{N}_\oplus^0} (1 - T_{P_\oplus}),$$

where we used the fact that the subtraction-operation for diagrams in $\mathcal{N}_\oplus^0$ is always performed, whereas a diagram in $\mathcal{M}_\oplus^0 \setminus \mathcal{N}_\oplus^0$ can either be included or not, leading to the factor of $(1 - T_{P_\oplus})$.

Since the equivalence class $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus^0)$ can equivalently be characterized by a maximal nest $\mathcal{M}_\oplus^0$ with $\mathcal{N} = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{M}_\oplus^0) = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus^0)$, we obtain the final result

44. **Theorem:** *Formulation of the R-operation as sum over equivalence-classes of nests*

The $R$-operation can be written as

$$R = \sum_{\mathcal{M}_\oplus \text{maximal w.r.t. } \mathcal{S}} \mathcal{W}(\mathcal{M}_\oplus) R_{\mathcal{M}_\oplus}^0,$$

$$R_{\mathcal{M}_\oplus^0} := - \prod_{P_\oplus^0 \in \mathcal{M}_\oplus^0} (-T_{P_\oplus^0}) \prod_{P_\oplus \in \mathcal{M}_\oplus \setminus \mathcal{M}_\oplus^0} (1 - T_{P_\oplus}), \quad (264)$$

where $\mathcal{M}_\oplus^0$ and $\mathcal{M}_\oplus$ are minimal and maximal elements from the same equivalence-class $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{M}_\oplus) = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus^0)$.

45. Since any term in the perturbative expansion is a sum over all sectors, using the above theorem, it is sufficient to show that $R_{\mathcal{M}_\oplus^0} I(x_1, \ldots, x_N)$, is finite in the sector $\mathcal{S}$ for all $\mathcal{M}_\oplus$ maximal w.r.t. $\mathcal{S}$. This is done in three steps: First, one applies $\prod_{P_\oplus^0 \in \mathcal{M}_\oplus^0} (-T_{P_\oplus^0})$ to $I$. This leads to a factorization of $I$ into subdiagrams. One then shows that the remaining factors of $(1 - T_{P_\oplus})$ act independently on these subdiagrams. This is, as in lemma 11, a consequence of the MOPE. (For details, see [54].) Finally, one introduces tree variables which are appropriate for the sector; finiteness is proven in terms of these variables.

46. **Definition:** *Construction of a spanning tree*

First of all, note that in general a tree associated with the sector $\mathcal{S}$ does not take into account the factorization introduced through $\prod_{P_\oplus^0 \in \mathcal{M}_\oplus^0} (-T_{P_\oplus^0})$. To construct such a tree, we proceed as follows: Consider the tableau, which is constructed from the sector $\mathcal{S}$ and the minimal nest $\mathcal{N}_\oplus^0$, and thus can no further be reduced. Each of its elements is contained in $\mathcal{M}_\oplus$, thus the rooting of $\mathcal{M}_\oplus$ can be transferred to the tableau.
We have also seen above in proof 31 that the first element of each line, $T^0_j \in \mathcal{G}^0_m$. Each line thus encodes what happens inside one of the $|\mathcal{G}^0_m|$ factorized (rooted) subdiagrams $\tilde{P}_j := T^0_j \land T^0_{j-1}$. A spanning tree for the connected components $P_{I,j}$ of $\tilde{P}_j$ is constructed as follows: Consider $\tilde{P}_{I,j} := S^I \land \tilde{P}_{I,j}$. The set $\mathcal{G}_{I,j} := \bigcup_{I,J,j,I,j = 0}^{\mathcal{G}_{I,j}} \tilde{P}^I_{I,j}$ is a saturated nest for the points in $\tilde{P}_{I,j}$, with a rooting induced by $\mathcal{M}^0_m$, and thus defines a spanning tree on $\tilde{P}_{I,j}$, using definition 26. Its elements are denoted by $\lambda^I_{I,j}$; the tree is ordered with increasing indices $I$. Also denote the union of the trees for $\tilde{P}_j$ as $\mathcal{G}_j := \bigcup_j \mathcal{G}_{I,j}$. The spanning tree defined by $\lambda^I_{I,j}$ will be used for the integration in the following. Note that this tree contains exactly $N - 1$ elements, i.e. not all combinations of $I$ and $J, j$ contribute; there may be elements $\lambda^I_{I,j}$ with the same sector-index $I$, but then, necessarily the diagram $J$ or the connected component $j$ are different.

47. Lemma: Inequality for the distances in the spanning tree

The following inequalities hold for the length of the elements of the spanning tree

\[
\frac{1}{I} \leq \left| \frac{\lambda^I_{I,j}}{\lambda^I_{I,j'}} \right| \leq I \quad \text{and} \quad \frac{1}{I'} \leq \left| \frac{\lambda^I_{I,j}}{\lambda^I_{I,j'}} \right| \leq I \quad \forall I < I'.
\]

48. Proof:

Consider a tree $T$, and the corresponding generalized sector $\mathcal{G}(T) = \{S^0, \ldots, S^{N-1}\}$. Then the distance between the $I$ elements of $S^{I-1}$ is smaller than the shortest distance between any element of $S^{I-1}$ and the additional element of $S^I$. Denote by $\lambda^I_i = 1 \ldots I - 1$ the elements of a spanning tree for $S^{I-1}$ and by $\lambda^I_J$ the shortest junction between a point in $S^{I-1}$ and the additional point of $S^I$. Then a vector connecting an arbitrarily chosen point in $S^{I-1}$ with the additional element of $S^I$ can be written as $\tilde{\lambda} = \lambda_i^I + \sum_i \bar{\lambda}_i$, where the sum goes over at least $I - 1$ elements. Using the triangle inequality, $\lambda$ is bounded by $|\bar{\lambda}| \leq |\lambda| + \sum_i |\bar{\lambda}_i| \leq I |\bar{\lambda}_i|$.

Since any $\lambda^I_{I,j}$ can be realized as restriction of a spanning tree for the sector $\mathcal{G}$, the first couple of inequalities is proven. The second one then follows immediately from the nested structure of $\mathcal{G}$.

49. Definition: Index-sets

The index-set $\text{Ind}^I := \{I, \exists J,j \text{ with } \lambda^I_{I,j} \in \mathcal{G}_{I,j}\} = \{I_1 < I_2 < \ldots < I_E\}$ is defined as the set of all indices $I$, such that there is a $\lambda^I_{I,j}$ in the spanning tree. We also define the index-set of the subdiagram $\tilde{P}_j$ as $\text{Ind}(\tilde{P}_j) := \{I, \exists j \text{ with } \lambda^I_{I,j} \in \mathcal{G}_{I,j}\}$.

50. Definition: Integration variables

The tree-variables defined in 46 are now used to perform the integration. To give explicit bounds, they are still reformulated as follows: First, choose for all $I \in \text{Ind}$ a representative $\lambda^I := \lambda^I_{I_0,j_0}$. Second, for all $\lambda^I_{I,j}$ with $(J,j) \neq (J_0,j_0)$ define $\chi^I_{I,j} := |\tilde{\lambda}^I_{I,j}|/|\tilde{\lambda}^I|$. According to lemma 47, $\chi^I_{I,j}$ is bounded: $1/I \leq \chi^I_{I,j} \leq I$. Third define $\beta^I := |\tilde{\lambda}^I|/|\tilde{\lambda}^{I+1}|$, and set $\lambda^{I+1} := \lambda^I$, if $\lambda^{I+1}$ would be undefined otherwise. (This happens since $\text{Ind}$ does not contain all indices $I < N$.) $\beta^I$ thus measures the ratio of successive representatives $\lambda^I$ with $I \in \text{Ind}$. According to lemma 47, $\beta^I$ is bounded by $0 \leq \beta^I \leq I$.

51. Lemma: Integration-measure

The integration-measure is

\[
\prod_{I \in \text{Ind}} \prod_{J=0}^{\mathcal{G}^0_m-1} \prod_{\tilde{P}_{I,j} \in \tilde{P}_j} \int \tilde{d}^{D} \tilde{\lambda}^I_{I,j}.
\]

This will be rewritten in terms of reduced variables $\beta^I$, $\chi^I_{I,j}$ and angles $\Omega^I_{I,j}$. We still need the following notation:
52. Definition: Cumulated level

The cumulated level of a subdiagram $\tilde{\mathcal{P}}_J$, $\Sigma(I, \tilde{\mathcal{P}}_J)$ is defined as the number of $\lambda^I_{J,j}$ with $I' \leq I$, and $J$ fixed. It counts the number of integrations over variables with an index smaller or equal to $I$ in the subdiagram $\tilde{\mathcal{P}}_J$. Replacing recursively the integration variables $\beta^I$ by $\beta^I$, the cumulated level $\Sigma(I, \tilde{\mathcal{P}}_J)$ yields the power of $\beta^I$ from the integration measure as $(\beta^I)_{\Sigma(I, \tilde{\mathcal{P}}_J)}$. We also define the completed cumulated level of a diagram $\tilde{\mathcal{P}}_J$, $\overline{\Sigma}(I, \tilde{\mathcal{P}}_J)$ as the number of integration variables $\lambda^I_{J,j'}$ with $I' \leq I$, where $J'$ is either $J$ or $\tilde{\mathcal{P}}_J$ is a proper subdiagram of $\tilde{\mathcal{P}}_J$.

53. Lemma: Reformulation of the measure

The measure in 51 can equivalently be written as (with $\Delta(I, J, j)$ to be discussed later)

\[
\left[ \prod_{I \in \text{Ind}} \int_0^I \frac{d\beta^I}{\beta^I} \right]^{|\eta_0| - 1} \prod_{I \in \text{Ind}(\tilde{\mathcal{P}}_J)} \prod_{J \in \text{Ind}(\tilde{\mathcal{P}}_J)} (\beta^I)^{\Delta(I, J, j)} \prod_{\lambda^I_{J,j} \in \Theta_J} \int d\Omega_{I,J} \prod_{\lambda^I_{J,j} \in \Theta_J, \lambda^I_{J,j} \neq \lambda^I} \int_0^1 d\chi^I_{J,j} \left( \chi^I_{J,j} \right)^{\Delta(I, J, j)}.
\]

Note that the limits for the integration over $\chi^I_{J,j}$ and $\beta^I$ are only bounds, and not the whole of it contributes. Due to these bounds, the integral over $\chi^I_{J,j}$ is always finite. The same remark applies to the integration over $\beta^I$, with the important difference that we have to check the limit of small $\beta^I$. This is the purpose of the following.

54. Integration over angular and relative variables

The angular integration $\int d\Omega_{I,J}$ does not induce any divergence, as long as $D \geq N - 1$. Since short-distance divergences (which are physical) and small-angle divergences (which come from the measure) are disentangled, we do not expect the latter to destroy the proof of perturbative renormalizability. A proof strictu sensu has not been given in the case of multilocal theories. It has however been checked at second order [25]. To avoid unnecessary complications, we will not specify the angular integrations any further and simply denote it for each subdiagram by $f_{\Omega_J}$. Also the integration over $\chi^I_{J,j}$ is always finite, and the power $\Delta(I, J, j)$ need not be specified. Analogously to the angular integration, we shall denote this finite integral, including the factors of $\left( \chi^I_{J,j} \right)^{\Delta(I, J, j)}$, as $f_{\chi_J}$.

We can now give bounds on each subdiagram separately:

55. Lemma: Bound for a subdiagram

Using theorem 44, we can write the renormalized manifold-integral as product over the subdiagrams $\tilde{\mathcal{P}}_J$ in the form

\[
\prod_{I \in \text{Ind}} \int_0^I \frac{d\beta^I}{\beta^I} \prod_{J = 0}^{\overline{\Sigma}(I, \tilde{\mathcal{P}}_J)} I^C_{\tilde{\mathcal{P}}_J}(\beta^I_1, \ldots, \beta^I_{I_E}),
\]

(265)

where the subdiagram associated to $\tilde{\mathcal{P}}_J$ and C is

\[
I^C_{\tilde{\mathcal{P}}_J}(\beta^I_1, \ldots, \beta^I_{I_E}) := \int_{\Omega_J} \int_{\chi_J} \prod_{P_0 \in \mathcal{M}_J \setminus \eta_0} \left( 1 - T_{P_0} \right) \left[ \prod_{I \in \text{Ind}(\tilde{\mathcal{P}}_J)} (\beta^I)^{\Delta(I, J, j)} \right] J^C_{\tilde{\mathcal{P}}_J}(\tilde{\chi}^I_{J,j}(\beta^I, \chi^I_{J,j}, \Omega^I_{J,j})).
\]

(266)

$I^C_{\tilde{\mathcal{P}}_J}(\tilde{\chi}^I_{J,j})$ is the MOPE-coefficient, when contracting the points inside $\tilde{\mathcal{P}}_J$ and extracting from the MOPE the term proportional to C. Eventually $T_{\mathcal{P}_J}$ yields more than one counter-term proportional to a relevant or
Proof of perturbative renormalizability

Marginal operator $C$. In that case, we have to sum over these contributions. Since the following arguments work for each term separately, we do not explicitly write down this summation. The operators $B_i$ are insertions into $\tilde{P}_J$ from subdiagrams. $[B_i]$ is their naive scaling dimension; relevant operators have a positive dimension, marginal operators dimension zero, and only such terms are included in the subtraction prescription. As for $C$, we only consider the insertion of a single operator from each subdiagram. $\ell'$ is a topological number, the number of “loops” or equivalently the number of interactions contracted in the subdiagrams of $\tilde{P}_J$, and thus always positive.

If we denote $I_{\max}(\tilde{P}_J) := \max\left(I \in \text{Ind}(\tilde{P}_J)\right)$, then $\Gamma^{C}_{\tilde{P}_J}(\beta^I, \ldots, \beta^{IE})$ is bounded by $(\beta^{I_{\max}(\tilde{P}_J)})^{\varepsilon - |C|}$. $\ell$ is the total number of “loops” in $\tilde{P}_J$ and its subdiagrams, so $\ell \geq \ell'$. Since this term is positive and vanishes in the limit of $\varepsilon \to 0$, we shall not specify $\ell$.

For $I < I_{\max}(\tilde{P}_J)$, $\Gamma^{C}_{\tilde{P}_J}(\beta^I, \ldots, \beta^{IE})$ is regular and bounded by $(\beta^I)^{\delta}$, with $\delta > 0$.

56. Proof:

First consider a subdiagram $\tilde{P}_J$ with no proper subdiagrams, and $I < I_{\max}$. Then define $I^{C}_{\tilde{P}_J}(\beta^I, \chi^I_{J,j}, \Omega^I_{J,j}) := \prod_{I \in \text{Ind}(\tilde{P}_J)}(\beta^I)^{D \Sigma(I, \tilde{P}_J) + \varepsilon - \sum_{[B_i]} I^{C}_{\tilde{P}_J}(\chi^I_{J,j}, \chi^I_{J,j}, \Omega^I_{J,j})}$.

57. Theorem: Finiteness of the renormalized integrals

The manifold-integral, given by Eq. (265) is finite.

58. Proof:

First, lemma 55 is applied to each factor $\Gamma^{C}_{\tilde{P}_J}(\beta^I, \ldots, \beta^{IE})$. We only have to check that the product of all powers of $\beta^I$ is positive. First of all, the powers $(\beta^{I_{\max}(\tilde{P}_J)})^{-|C|}$ are canceled at each level, since the operator $C$ is at the same level $I = I_{\max}(\tilde{P}_J)$ inserted into another diagram, thus there appears a factor $(\beta^{I_{\max}(\tilde{P}_J)})^{+|B_i|}$ with $B_i = C$. The remaining factors have the form $(\beta^I)^{\ell \varepsilon}$ or even $(\beta^I)^{\delta}$ (with $\ell$ and $\delta$ as defined in 55 and 56), which are finite for $\varepsilon > 0$. In the limit of $\varepsilon \to 0$, convergence is guaranteed, as long as there remains at least one factor of $(\beta^I)^{\delta}$. However this is guaranteed, since we have seen above that the spanning tree, when restricted to a subdiagram, comes along with a factor of $(1 - T_{\tilde{P}_J})$. This completes the proof.

We can finally state the most general version of the above theorem.

59. Theorem: General criterion for renormalizability

A statistical field theory is perturbatively renormalizable, if

(i) the theory is renormalizable by power-counting,

(ii) divergences are short-range, i.e. no divergences appear at finite distances,
(iii) the dilation operators defined above commute,
(iv) there exist a multilocal operator product expansion, which describes these divergences,
(v) the divergences of the multilocal operator product expansion must not have an accumulation point
at dimension 0. Especially, after subtracting them, the integrand has to be convergent when the
distances are contracted.

60. Remark: Absence of an accumulation point at dimension 0
Note that in proof 58 we have seen that a factor of $(\beta I)^{\delta}$ with $\delta > 0$ is necessary to ensure UV-
convergence. This is not the case if (v) of the above theorem is violated.

61. Remark: Observables which demand a proper renormalization
The above considerations have to be modified in the case of observables $\mathcal{O}$ which demand a proper
renormalization. In that case, the observable points have to be added to $\mathcal{X}$, and the subtraction operator
contains all diagrams which can be constructed from this enlarged set of points, with the exception of those
contributions, that involve contractions of the observable points themselves. Note that in the case when $\mathcal{O}$
does not demand a proper renormalization, the MOPE-coefficients of the contraction towards points of $\mathcal{O}$
factorize, and the subtraction operator contains no new terms.

5.3 Some examples
In this section we give some illustrative examples of the abstract construction presented in the last section,
such that the reader can convince himself that the prescription actually works.

62. Example: Forest construction
Let us consider, as an example for the forest-construction, the divergence when contracting two 3-point
interactions (see section 9) as

This contraction has several subdivergences, which have to be subtracted. One of these is described by the
forest

$$\mathcal{F} = \left\{ \bullet \cdot \cdot \cdot , \bullet \cdot \cdot \cdot , \bullet \cdot \cdot \cdot , \bullet \cdot \cdot \cdot \right\} .$$

(267)

It is important to note that $\bullet \cdot \cdot \cdot$ could not be added to the forest, since it consists of one and only one
connected molecule (see definition 5), and its intersection with $\bullet \cdot \cdot \cdot$ is non-empty, nor is it included or
includes the latter one.

Note, that in the case of

splits into two connected components $\bullet$ and $\bullet$ which appear as individual elements. A possible
forest would be

$$\mathcal{F} = \left\{ \bullet \cdot \cdot \cdot \right\} .$$
Let us finally construct the equivalence-class of nests, as used in proof 21, for $\mathfrak{F}$ of Eq. (267). It consists of three elements, namely

$$\mathcal{N}_1 = \{ \begin{array}{ccc} \bullet & \prec & \bullet \\ \bullet & \prec & \bullet \end{array} \}, \quad \mathcal{N}_2 = \{ \begin{array}{ccc} \bullet & \prec & \bullet \\ \bullet & \prec & \bullet \end{array} \},$$

$$\mathcal{N}_3 = \{ \begin{array}{ccc} \bullet & \prec & \bullet \\ \bullet & \prec & \bullet \end{array} \}$$

Since in the $R$-operation $\mathcal{N}_3$ comes with a relative minus sign with respect to $\mathcal{N}_1$ and $\mathcal{N}_2$, the contribution to $R$ of $\mathcal{N}_1$ to $\mathcal{N}_3$ is the same as the contribution of $\mathfrak{F}$.  

63. Example: \textit{Tableau-construction, convergent subdiagrams}

As an example consider the sector $S := \{ \begin{array}{ccc} \bullet & \prec & \bullet, \\ \bullet & \prec & \bullet \end{array} \}$. The subtraction shall be given by the nest $\mathcal{N}_\oplus := \{ \begin{array}{ccc} \times & \prec & \times, \\ \times & \prec & \times \end{array} \}$, which we have chosen maximal. Roots are marked by crosses. The tableau defined in definition 262 is

$$\text{sector} \quad \quad \text{diagram} \quad \quad \text{reduction}$$

$$\begin{pmatrix}
\begin{array}{ccc}
\bullet & \prec & \bullet \\
\bullet & \prec & \bullet
\end{array} & \ldots
\end{pmatrix} \quad \quad \quad \quad \quad \begin{pmatrix}
\begin{array}{ccc}
\bullet & \prec & \bullet \\
\bullet & \prec & \bullet
\end{array} & \ldots
\end{pmatrix}$$

The only reducible element is $\begin{array}{ccc} \times & \prec & \times \\ \times & \prec & \times \end{array}$, corresponding to the global subtraction. Note that this element always has to be reducible, since in any sector the global divergence has to be subtracted. $\mathcal{N}(S, \mathcal{N}_\oplus) = \mathcal{N}$ and the equivalence-class $C_{\mathcal{N}_\oplus}(\mathcal{N}_\oplus)$ consists of three elements, namely

$$C_{\mathcal{N}_\oplus}(\mathcal{N}_\oplus) = \{ \{ \begin{array}{ccc} \times & \prec & \times, \\ \times & \prec & \times \end{array} \}, \{ \begin{array}{ccc} \times & \prec & \times, \\ \times & \prec & \times \end{array} \}, \{ \begin{array}{ccc} \times & \prec & \times, \\ \times & \prec & \times \end{array} \} \}$$

Its minimal generating nest is $\mathcal{N}_0 = \{ \begin{array}{ccc} \times & \prec & \times, \\ \times & \prec & \times \end{array} \}$.  

Concerning the subtraction-operator $R$, this is the situation, where the counter-terms (given by $\mathcal{N}_\oplus$) belong to sectors not present in $S$, and are therefore finite in $S$. This leads in theorem 44 to two terms

$$R_{\{ \begin{array}{ccc} \times & \prec & \times, \\ \times & \prec & \times \end{array} \}} = \left( 1 - T \begin{array}{ccc} \times & \prec & \times \\ \times & \prec & \times \end{array} \right) T \begin{array}{ccc} \times & \prec & \times \\ \times & \prec & \times \end{array}$$

which are just distinguished by their roots. 

64. Example: \textit{Tableau-construction, (maximally) divergent subdiagrams}

In this example, we study the opposite case, where all (but the trivial diagram) are reducible. Let
\( S := \{ \bullet \bullet \bullet \bullet \} \) and \( \mathfrak{M} := \{ \bullet \bullet \bullet \bullet \} \). Then the tableau is

\[
\begin{pmatrix}
\text{sector} \\
\text{diagram} \\
\text{reduction}
\end{pmatrix}
\]

It is (maximally) reducible, such that the minimal generating nest is \( \mathfrak{N}_0 = \{ \bullet \bullet \} \), \( \mathfrak{N}(S, \mathfrak{M}) = \mathfrak{N} \) and \( C_S(\mathfrak{M}) \) is given by (see classification lemma 39): \( \{ \bullet \bullet \} \subset \mathfrak{N}_0 \subset \{ \bullet \bullet \bullet \bullet \} \). It has \( 2^3 = 8 \) elements, as long as the roots are not specified.

Concerning the subtraction operator \( R \), this is the situation, where all subdiagrams in \( \mathfrak{N} \) belong to a subdivergence of \( S \). In theorem 44 the contributions for different rootings belong to different maximal nests. In the case of \( \mathfrak{N} = \{ \bullet \bullet \bullet \} \), the subtraction operator \( R_{\mathfrak{N}} \) is

\[
R_{\mathfrak{N}} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) .
\]

Note that the global minus sign in Eq. (264) has been canceled against the factor \( (-T_{\bullet \bullet}) = -1 \), and that there are seven other possible maximal nests, and factorized subtraction operations.

65. Example: Tableau-construction, single subdivergence

In this example, we study the case of a single subdivergence. Let as before \( S := \{ \bullet \bullet \bullet \bullet \} \), but \( \mathfrak{M} := \{ \bullet \bullet \bullet \bullet \} \). Then the tableau is

\[
\begin{pmatrix}
\text{sector} \\
\text{diagram} \\
\text{reduction}
\end{pmatrix}
\]

The minimal generating nest is \( \mathfrak{N}_0 = \{ \bullet \bullet \} \), and \( \mathfrak{N}(S, \mathfrak{M}) = \mathfrak{N} \). Let us explicitly write down the six elements of \( C_S(\mathfrak{M}) \):

\[
C_S(\mathfrak{M}) = \{ \{ \bullet \bullet \bullet \bullet \} \}, \{ \bullet \bullet \bullet \bullet \} \}, \{ \bullet \bullet \bullet \bullet \} \}, \{ \bullet \bullet \bullet \bullet \} \}, \{ \bullet \bullet \bullet \bullet \} \}, \{ \bullet \bullet \bullet \bullet \} \} .
\]

The simple subdivergence occurs, when in the subdiagram \( \bullet \bullet \bullet \bullet \) the distances \( \bullet \bullet \bullet \bullet \) are contracted. The last four elements of \( C_S(\mathfrak{M}) \) subtract the global divergence, the first and second two of them are distinguished.
by their rooting. In theorem 44, the corresponding subtraction-operators are written as

\[ R \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) T_{\bullet \bullet} \]

\[ R \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) T_{\bullet \bullet} \]

**66. Example: Tableau-construction, a small change**

In this example, we give a variant of example 65. We will finally obtain a maximal nest, of which some elements contain the minimal nest of example 65, but whose minimal nest is different.

Let as before \( S := \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} \), and \( \mathcal{N}_{\oplus} := \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} \). Then the tableau is

\[
\xymatrix{ & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots \\
\} \quad \xymatrix{ \text{diagram} \ar[r] & \text{reduction} } \]

The minimal generating nest of the tableau is \( \mathcal{N}_{\oplus}^0 = \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} \), and \( \tilde{\mathcal{N}}(S, \mathcal{N}_{\oplus}) = \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} \).

Let us explicitly write down the seven elements of \( \mathcal{C}_{\mathcal{S}}(\mathcal{N}_{\oplus}) \):

\[
\mathcal{C}_{\mathcal{S}}(\mathcal{N}_{\oplus}) = \left\{ \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\}, \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\}, \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\}, \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\}, \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\}, \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\}, \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} \right\} .
\]

In theorem 44, there are four contributions to the subtraction operator, corresponding to four different maximal nests \( \mathcal{M}_{\oplus} \):

\[ R \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) T_{\bullet \bullet} \]

\[ R \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) T_{\bullet \bullet} \]

\[ R \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) T_{\bullet \bullet} \]

\[ R \left\{ \begin{array}{ccc}
\bullet & \bullet & \cdots \\
\bullet & \bullet & \cdots \\
\end{array} \right\} = \left( 1 - T_{\bullet \bullet} \right) \left( 1 - T_{\bullet \bullet} \right) T_{\bullet \bullet} \]

6 Calculations at 2-loop order

In the introduction, we had mentioned that the first calculations using some criterium to fix the expansion-point at 1-loop order were performed in [22]. The result of \( d_f \approx 3.5 \) for membranes in 3 dimensions even violated the geometric bound of 3 discussed above. It became therefore necessary to perform 2-loop calculations, not only to test the proof of perturbative renormalizability, but also to obtain more reliable values for the fractal dimension. This is the aim of this section.
6.1 The 2-loop counter-terms in the MS scheme

In this section we apply the formalism explained in sections 3 and 4 to determine the counter-terms which renormalize the theory at second order [25, 26]. If we consider the bare theory, given by the Hamiltonian (61) when setting $Z = Z_b = 1$, power counting gives three UV-divergent diagrams (together with their weight)

\[
\begin{align*}
(a) &= \frac{1}{2} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right), \\
(b) &= \frac{2}{3} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right), \\
(c) &= 2 \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right),
\end{align*}
\]

which give short distance singularities when the points inside the subsets are contracted to a single point. These singularities give double and single poles at $\varepsilon = 0$. There are two other potentially dangerous diagrams

\[
\begin{align*}
\text{Diagram} \\
\text{Diagram}
\end{align*}
\]

These diagrams do not give new poles at $\varepsilon = 0$ for reasons similar to what happens with diagram (152). Now one has to remember that the model is already renormalized at 1-loop order, i.e. that we use the renormalized Hamiltonian (153), with the counter-terms (172) and (173). As a consequence there are five additional divergent diagrams, which come from the insertion of the 1-loop counter-terms

\[
\begin{align*}
(d) &= -2 \mu^{-\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}, \\
(e) &= -\mu^{-\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}, \\
(f) &= -\mu^{-\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}, \\
(g) &= -2 \mu^{-\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}, \\
(h) &= \frac{1}{2} \mu^{-2\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}^2.
\end{align*}
\]

There are other potentially divergent diagrams, analogous to those depicted in (269), which factorize into convergent diagrams.

The first four terms in (270) are a combination of a diagram divergent at 1-loop order (giving a single pole) times a divergent 1-loop counterterm (which gives another single pole). The fifth term is more peculiar: it is the combination of a convergent diagram (which corresponds to a contact term) times two 1-loop counter-terms (thus giving also a double pole).

Owing to the MOPE, diagrams (a), (e), (f) and (h) give a divergence proportional to the insertion of the local operator $\dagger$. They can be subtracted by adding a counterterm proportional to the divergent part of the integral of the corresponding MOPE coefficients

\[
\frac{1}{2} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{L} \mu^{-\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{L} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1} \\
- \mu^{-\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{L} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{L} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1} + \frac{1}{2} \mu^{-2\varepsilon} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{L} \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}^2.
\]

Since we use the minimal subtraction scheme, we want to subtract only the double and single poles in $\varepsilon$ at $\varepsilon = 0$. To isolate these poles, we have to perform a Laurent expansion of the various terms in (271) and to keep the terms of order $\varepsilon^{-2}$ and $\varepsilon^{-1}$ but to drop the analytic part. Setting the renormalization momentum scale $\mu = L^{-1}$, we obtain the final expression for the renormalization factor $Z$ at 2-loop order

\[
Z = 1 - b (2 - D) \left( \begin{array}{c}
\text{Diagram} \\
\end{array} \right)_{\varepsilon = 1}
\]
Calculations at 2-loop order

\[ + b^2 (2 - D) \left[ \frac{1}{2} \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{a} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon} - \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{a} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}, \varepsilon} \right. \\
\left. - \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}, \varepsilon^{-1}, \varepsilon} + \frac{1}{2} \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}, \varepsilon} \right] \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}}^2 \right. \\
\left. + \mathcal{O}(b^3) \right]. \tag{272}

Here \( \left\langle \right\rangle_{n_1, \ldots, n_p} \) denotes the sum of the terms of order \( \varepsilon^{n_1}, \ldots, \varepsilon^{n_p} \) in the Laurent expansion of \( \left\langle \right\rangle_L \), taken at \( L = 1 \).

Similarly, the diagrams \((b), (c), (e)\) and \((f)\) give a divergence proportional to the bilocal operator \( \ldots \). An analogous analysis leads to the following expression for the coupling-constant renormalization-factor \( Z_b \) at 2 loop order

\[
Z_b = 1 + b \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} \\
+ b^2 \left[ \frac{-2}{3} \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-2}, \varepsilon^{-1}} + 2 \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}, \varepsilon} \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}, \varepsilon} \right] \\
- 2 \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-2}, \varepsilon^{-1}} + 2 \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}, \varepsilon} \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} \right] \\
+ \mathcal{O}(b^3). \tag{273}
\]

### 6.2 Leading divergences and constraint from renormalizability

Renormalizability, once established, completely determines the terms of order \( \frac{b^2}{\varepsilon^2} \) in the renormalization-factors, since the RG-functions have to be finite. In our case the following modified \( Z \)-factors fulfill this requirement up to order \( b^2 \):

\[
Z^{(1)} = 1 - b (2 - D) \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} \\
\times \left[ 1 + \frac{b}{2} \left( \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} - \nu (d + 2) \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} \right) \right] \tag{274}
\]

\[
Z^{(1)}_b = 1 + b \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} \\
\times \left[ 1 + \frac{b}{2} \left( 2 \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} - \nu d \left\langle \begin{array}{c} \text{\phantom{a}} \\ \text{\phantom{a}} \end{array} \right| + \right\rangle_{\varepsilon^{-1}} \right) \right]. \tag{275}
\]

This can be verified by calculating the renormalization-group functions \( \beta(b) \) and \( \nu(b) \) at order 2. Moreover, these functions are identical to their 1-loop counterpart. It is therefore useful, to factorize the \( Z \)-factors loop-wise (with respect to the RG-functions) through the ansatz:

\[
Z = Z^{(1)} \times Z^{(2)} \times \ldots \tag{276}
\]

\[
Z_b = Z^{(1)}_b \times Z^{(2)}_b \times \ldots . \tag{277}
\]

If our considerations are correct, \( Z^{(2)} \) and \( Z^{(2)}_b \) do not contain terms of order \( \frac{b^2}{\varepsilon^2} \), but only terms of order \( \frac{b^2}{\varepsilon} \). Moreover, by construction they just contain the 2-loop contributions to the RG-functions.
Explicitly, from Eqs. (272) and (274), we obtain for \( Z^{(2)} \) the expression:

\[
Z^{(2)} = 1 + b^2(2 - D) \left[ \frac{1}{2} \langle \ \rangle \left\langle \right\rangle_{e-2,e^{-1}} - \left\langle \ \right\rangle_{e-1} \left\langle \ \right\rangle_{e-2,e^{-1}}
- \frac{\nu(d + 2)}{2} \left( \left\langle \ \right\rangle_{e^{-1}} \right)^2 + \frac{1}{2} \left( \left\langle \ \right\rangle_{e^{-1}} \right)^2 \left\langle \ \right\rangle_{e^{-1}} \right],
\]

(278)

where we already used the fact that \( \left\langle \ \right\rangle_{e^0} = 0 \), see Eq. (164). Finally, one has to remember that \( \langle \ldots \rangle_{e^p} \) are the terms of order \( e^p \) of the Laurent series of the integral over distances of the corresponding MOPE coefficient

\[
\langle \ldots \rangle_{L} = \int_{\text{distances} \leq L} \ldots
\]

(279)

Using this fact one obtains the decomposition

\[
Z^{(2)} = 1 + b^2 (2 - D) \left[ F_1 + F_2 + F_3 + O(e^0) \right],
\]

(280)

where in the next subsection, each term will be shown to be expressible as \( 1/e \) times a convergent integral. These terms are:

\[
F_1 = \frac{1}{2} \left\langle \ \right\rangle_{L} - \frac{1}{2} \left\langle \ \right\rangle_{L} \left\langle \ \right\rangle_{L}
\]

(281)

\[
F_2 = \frac{1}{2} \left\langle \ \right\rangle_{e^{-1}} \times
\]

\[
\times \left( \left\langle \ \right\rangle_{L} + \frac{(2 - D)(d + 2)}{2} \left\langle \ \right\rangle_{L} - \left\langle \ \right\rangle_{L} \left\langle \ \right\rangle_{L} \right)
\]

(282)

\[
F_3 = \frac{1}{2} \left\langle \ \right\rangle_{e^{-1}} \left( \left\langle \ \right\rangle_{L} \right) - \left\langle \ \right\rangle_{e^{-1}}
\]

(283)

Even if not written explicitly, we will only calculate the residue of \( F_1, \ldots, F_3 \) at \( L = 1 \). Similarly, dividing (273) by (275) we obtain for \( Z_b^{(2)} \)

\[
Z_b^{(2)} = 1 + b^2 \left[ - \left( \left\langle \ \right\rangle_{e^{-1}} \right)^2 + \frac{(2 - D)d}{4} \left\langle \ \right\rangle_{e^{-1}} \left\langle \ \right\rangle_{e^{-1}}
\]

\[
+ 2 \left\langle \ \right\rangle_{e^{-1}} \left\langle \ \right\rangle_{e^{-1}} \left\langle \ \right\rangle_{e^{-1},e^0} - 2 \left\langle \ \right\rangle_{e^{-2},e^{-1}} \left\langle \ \right\rangle_{e^{-2},e^{-1}}
- 2 \left\langle \ \right\rangle_{e^{-1},e^0} + 2 \left\langle \ \right\rangle_{e^{-1},e^0} \right]
\]

(284)

that we decompose as

\[
Z_b^{(2)} = 1 + b^2 \left[ C_1 + C_2 + C_3 + O(e^0) \right],
\]

(285)
with
\[
C_1 = -\frac{2}{3} \langle \begin{array}{c} \end{array} \rangle_L + \left( \langle \begin{array}{c} \end{array} \rangle_L \right)^2
\]
(286)
\[
C_2 = -2 \langle \begin{array}{c} \end{array} \rangle_L + \langle \begin{array}{c} \end{array} \rangle_L \langle \begin{array}{c} \end{array} \rangle_L
\]
(287)
\[
C_3 = \langle \begin{array}{c} \end{array} \rangle_{\epsilon=1} \left( \langle \begin{array}{c} \end{array} \rangle_L + \frac{(2-D)d}{4} \langle \begin{array}{c} \end{array} \rangle_{\epsilon=1} \right).
\]
(288)

The coefficients \(C_1, C_2\) and \(C_3\) as well as \(F_1, F_2\) and \(F_3\) will be shown to be \(1/\epsilon\) times a convergent integral in the next section.

### 6.3 Absence of double poles in the 2-loop diagrams

The simplest case is the diagram \(C_1\) in Eq. (286). A subdivergence occurs when two dipoles are contracted to a single dipole. When this contraction is performed first, the MOPE coefficient factorizes as
\[
\left( \begin{array}{c} \end{array} \right) \approx \left( \begin{array}{c} \end{array} \right) \left( \begin{array}{c} \end{array} \right).
\]
(289)
There are three different subdivergences and one finally obtains that the double pole associated with this diagram is given by
\[
\langle \begin{array}{c} \end{array} \rangle_{\epsilon=2} = 3 \times \frac{1}{2} \langle \begin{array}{c} \end{array} \rangle_{\epsilon=1} \langle \begin{array}{c} \end{array} \rangle_{\epsilon=1}.
\]
(290)
The factor of \(1/2\) comes from the nested integration [52]: the double pole results from the integration over a “sector” where the distances inside the subdiagram are smaller than all other distances. Note that by subtracting all subdivergences in each individual sector, the only remaining divergence is the integral over the global scale. Using the procedure of section 4.3 it is expressed as \(1/\epsilon\) times a convergent integral.

Similarly, let us consider the diagram \(C_2\) in Eq. (287). A subdivergence occurs when the single dipole to the right of the diagram is contracted to a point. The MOPE coefficient factorizes as
\[
\left( \begin{array}{c} \end{array} \right) \approx \left( \begin{array}{c} \end{array} \right) \left( \begin{array}{c} \end{array} \right).
\]
(291)
Consequently, the double pole for this diagram is
\[
\langle \begin{array}{c} \end{array} \rangle_{\epsilon=2} = \frac{1}{2} \langle \begin{array}{c} \end{array} \rangle_{\epsilon=1} \langle \begin{array}{c} \end{array} \rangle_{\epsilon=1}.
\]
(292)
where the factor of \(1/2\) again comes from the nested integration.

Finally, let us consider the term \(F_1\) in (281). Four sectors contribute to the double pole of its first term, which correspond to the subcontractions depicted here
\[
\left( \begin{array}{c} \end{array} \right) \approx \left( \begin{array}{c} \end{array} \right) \left( \begin{array}{c} \end{array} \right) \left( \begin{array}{c} \end{array} \right),
\]
(293)
Each of the contractions appears with a combinatorial factor of two. The double pole for this diagram is therefore

\[
\langle \begin{array}{c}
\bullet
\end{array} \mid \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-2}} = 2 \times \frac{1}{2} \langle \begin{array}{c}
\bullet
\end{array} \mid \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} \langle \begin{array}{c}
\bullet
\end{array} \langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} + 2 \times \frac{1}{2} \langle \begin{array}{c}
\bullet
\end{array} \mid \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} \langle \begin{array}{c}
\bullet
\end{array} \langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}}. \tag{294}
\]

Thus, also \( F_1 \) is finite.

For the remaining terms \( F_2 \) and \( F_3 \), we have to show, that the singular contributions in the bracket cancel. To this aim, we make use of the equation of motion to compute the effect of the insertion of the operator \( + \) in the counter-terms. In exercise E.5 (page 163), the reader can show that

\[
\langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} = -\nu(d + 2) \langle \begin{array}{c}
\bullet
\end{array} \mid \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} + \langle \begin{array}{c}
\bullet
\end{array} \langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} \langle \begin{array}{c}
\bullet
\end{array} \langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} + O(\varepsilon^0). \tag{295}
\]

\[
\langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} = -\nu d \langle \begin{array}{c}
\bullet
\end{array} \mid \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} + \langle \begin{array}{c}
\bullet
\end{array} \langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} \langle \begin{array}{c}
\bullet
\end{array} \langle \begin{array}{c}
\bullet
\end{array} \rangle_{\varepsilon^{-1}} + O(\varepsilon^0). \tag{296}
\]

Using these identities implies that also \( F_2 \) and \( C_3 \) only contain a single pole.

### 6.4 Evaluation of the 2-loop diagrams

The evaluation of \( F_1, F_2, F_3, C_1, C_2, C_3 \) is only for \( D = 1 \) analytically amenable. For all other dimensions, the convergent integrals have to be calculated numerically. These calculations are plagued by a couple of difficulties: First of all integrable subdivergences have to be eliminated by suitable variable transformations. Two types of subdivergences exist: Divergences for small distances (as predicted by the MOPE) and divergences for small volume of the span of the integration-points, arising from the \( D \)-dimensional measure. They require separate, while dependent variable-transformations. The result of these procedures are convergent integrals over up to 5 variables, which have to be integrated numerically. Since in addition, these integrals are strongly peaked (up to about \( 10^4 \) times the average) only specialized integration routines can handle them. We implemented an adaptive Monte Carlo integration. For details of the numerical integrations, which took about \( 10^3 \)h CPU-time on a work-station, we refer the interested reader to [25].

### 6.5 RG-functions at 2-loop order

For completeness, we still give the explicit form of the renormalization group functions at 2-loop order:

\[
\beta(b) = -\varepsilon b + \left( \frac{1}{2 - D} \frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)} + \frac{(2 - D)d}{4D} \right) b^2 + 2\varepsilon \left[ (C_1 + C_2 + C_3) + \frac{2 - D}{2} d (F_1 + F_2 + F_3) \right] b^3 + O(b^4) \tag{297}
\]

\[
\nu(b) = \frac{2 - D}{2} \left[ 1 + \frac{1}{2D} b + 2\varepsilon (F_1 + F_2 + F_3) b^2 + O(b^3) \right]. \tag{298}
\]

Solving \( \beta(b^*) = 0 \) for \( b^* \) and insertion of \( b^* \) into Eq. (298) then yields \( \nu^*(D, \varepsilon) \). We shall discuss this function in the next section.
7 Extracting the physical informations: Extrapolations

7.1 The problem

In section 3.1, we have seen that the coupling $b$ for self-avoidance has dimension $\varepsilon = 2D - \frac{2-D}{2}d$. We have then performed a perturbative expansion in $b$ to order 1 (section 3.8) and 2 (section 6). As in standard $\phi^4$-theory, the $\beta$-function has a non-trivial fixed point $b^*$ with $\beta(b^*) = 0$ and positive slope, which thus is IR-attractive and governs the behavior of the membrane at large distance. We have seen in Eq. (177), that $b^*$ is of order $\varepsilon$, such that the perturbative expansion of the scaling function $\nu(b)$ can be written as a perturbative expansion in $\varepsilon$ of the form

$$\nu^*(D, \varepsilon) = \frac{2-D}{2} + \nu_1(D)\varepsilon + \nu_2(D)\varepsilon^2 + O(\varepsilon^3).$$ (299)

Let us already mention that similar expansions are valid for other exponents and other models, but let us focus on $\nu^*(D, \varepsilon)$, whose coefficients are plotted in figure 16. Our goal is to obtain informations about self-avoiding polymers and membranes, and to this aim, we can expand about any point on the critical curve $\varepsilon(D, d) = 0$, see figure 12, page 23. For polymers, one usually expands about the point $(D = 1, d = 4)$, i.e. for $D = 1$ fixed. For membranes, naively setting $D = 2$ and any finite $d$ as e.g. $d = 3$, i.e. $\varepsilon = 4$ in Eq. (299), yields the wrong result 0. To obtain the correct result, the general idea is of course to expand about some point $(D_0, d_0)$ on the critical curve $\varepsilon(D_0, d_0) = 0$. If the function $\nu^*(D, \varepsilon)$ were known exactly, the result of this expansion is expected to converge towards $\nu^*(D, \varepsilon)$, but eventually only in some part of the possible range of $D$ in $0 < D < 2$, and possibly only after resummation. However, only one or two terms of the series in $\varepsilon$ are available, such that the result will depend on the expansion point and on the expansion parameters.

The simplest scheme is to extrapolate towards the physical theories for $D = 1, 2$ and $d = 2, 3, \ldots$, using the expansion parameters $D - D_0$ and $d - d_0$. However, this set of expansion parameters is not optimal,
This simply means that the point \( D = 2 \) is a singular point and that it is not possible to perform a direct \( \varepsilon \)-expansion about it.

To define an expansion about any point on the critical curve \((D_0, d_0)\), note that the expansion \((299)\) is exact in \( D \) and of order \( n \) in \( \varepsilon \). For the example of a 2-loop calculation, on which we will focus here, \( n=2 \). Thus \( \nu^*(D, \varepsilon) \) can be expanded up to order 2 both in \( D - D_0 \) and \( \varepsilon \). Then one can change the extrapolation path through any invertible transformation \( \{x, y\} = \{x(D, \varepsilon), y(D, \varepsilon)\} \) and re-express \( D \) and \( \varepsilon \) as functions of \( x \) and \( y \) up to order \( n \) in \( x \) and \( y \) around the point \((x_0, y_0)\) on the critical curve, yielding

\[
\nu^*(D, \varepsilon) = \nu^*(x, y) = \nu^*_{0,0}(x_0, y_0) + \Delta x \nu^*_{1,0}(x_0, y_0) + \Delta y \nu^*_{0,1}(x_0, y_0) + \frac{1}{2}(\Delta x)^2 \nu^*_{2,0}(x_0, y_0) + \Delta x \Delta y \nu^*_{1,1}(x_0, y_0) + \frac{1}{2}(\Delta y)^2 \nu^*_{0,2}(x_0, y_0) + \ldots
\]

The goal is to find an optimal choice of variables \( \{x, y\} \). The guidelines for such a choice are the following: (i) the estimate for \( \nu^* \) should depend as little as possible on the choice of the expansion point on the critical curve, (ii) it should reproduce well the known result for polymers \((D = 1)\), (iii) for membranes \((D = 2)\) the limit \( d \to \infty \) should be non-singular and in agreement with large-\( d \) results. This demand turns out to be quite stringent.

7.2 General remarks about extrapolations and the choice of variables

As already stressed, the \( \varepsilon \)-expansion given by \((299)\) cannot be used directly for membranes, as it can for polymers, since directly setting \( D = 2 \) in the \( \varepsilon \)-expansion for \( \nu^* \) gives a trivial, but absurd, result, since all the terms \( \nu_{n}(2) \) of the \( \varepsilon \)-expansion vanish! Moreover when \( D = 2 \), \( \varepsilon = 4 \) irrespective of the value of \( d \). This simply means that the point \( D = 2 \), \( \varepsilon = 0 \) (which corresponds to \( d_c = \infty \)) is a singular point and that it is not possible to perform a direct \( \varepsilon \)-expansion about it.

Let us now turn to the general analysis.
Finally, one must choose a resummation procedure to extrapolate \( \nu^* \) from the knowledge of the series (300) up to order \( n \). Since only a few terms are available and since not sufficient knowledge is available on the large order behavior of these series or on the analytical structure of the resummed series, sophisticated resummation methods as Borel transforms cannot be used. The series therefore have to be summed and boldly truncated at order 2.

Let us now discuss possible extrapolation variables. Statements about their goodness always apply to the only available 2-loop calculations [25, 26], but are partially transferable to other cases as e.g. in [32, 33].

\( \{D, \varepsilon\} \) This works well for polymers \( (D = 1) \), since both at 1- and 2-loop order we get results which are quite stable with respect to \( D_0 \), but no prediction is possible for membranes.

\( \{D, d\} \) similar to \( \{D, \varepsilon\} \).

\( \{D, D_c(d)\} \) Recall that one of the problems of the expansion in \( d \) is, that for \( D \to 2 \), \( d_c(D) \to \infty \) and therefore \( d - d_c(D) \) becomes large. (In general \( d_c(D) \sim 1/(2 - D) \).) However, by using the above variables, one maps the region of possible values of \( (D, d) \) onto the square \( [0, 2]^2 \), and the critical curve \( D_0, d_c(D_0) \) becomes a straight line. In general, these variables work well, both for polymers and membranes.

\( \{\varepsilon, D_c(d)\} \) Another promising method is the expansion in \( \varepsilon \) and \( D_c(d) \). This expansion is also regular for \( D \to 2 \) and is perhaps more in the spirit of an \( \varepsilon \)-expansion.

Some examples are given for polymers in figure 17 and for membranes on figure 18 (using one more reformulation of the perturbation-expansion as discussed below). In general, one observes a plateau for \( D \approx 1.5 \), and the exponent is evaluated at this plateau, since there, the result depends the least on the expansion point (“minimal sensitivity method”). The breadthiness of the plateau gives a measure of the goodness of the extrapolation variables, and of the result. Another criterion, which evidently can only be used at 2- or higher loop order is to demand, that the results of the 1- and 2-loop extrapolations (or higher if available) agree. In general, this methods yields comparable results to the minimal sensitivity method.

It has also been checked in [25] that these schemes work well for a toy model, i.e. when expanding the result obtained by a variational approach \( \nu_{var} = 2D/d \) in \( x \) and \( y \), by going to higher orders. In that case,
the series becomes convergent \(^5\), but only in a part of the interval \(0 < D < 2\), depending on the dimension \(d\).

Let us conclude by remarking that apart from the above general rules, for any new model and exponent, one has to experiment in order to find the best way to extract the physical information. What may seem to be an disadvantage of the above method, namely its ambiguity, upon careful thinking turns out to be one of its main advantages, since it allows to estimate the confidence of the result, i.e. its systematic error.

### 7.3 Expansion about an approximation

In section 2.3, we discussed approximations for the exponent \(\nu^*\), namely the Flory and the variational approximation. It is an intriguing idea, to use these approximations as a starting point for a systematic expansion. To build such a systematic expansion on the level of diagrams is a rather hopeless task. However, as first observed in \([25, 26]\), this is possible in the perturbative expansion of critical exponents, as we show now. The general idea is that the approximations can be written as a non-trivial relation among the renormalization \(Z\)-factors of the form

\[ Z_b \approx Z^\sigma. \]  
(301)

This relation can be formalized by setting

\[ Z_b = Z_0^\sigma Z_p^\alpha, \]
\[ Z = Z_0 Z_p, \]  
(302)

where we think of \(Z_0\) as a large renormalization factor and \(Z_p\) as a small perturbation. We have also introduced an exponent \(\alpha\), but it will drop out from the final result. In order to eliminate \(Z_0\), one first solves the defining equation (175) for the \(\beta\)-function, expressed in terms of \(Z_0\) and \(Z_p\), for \(Z_0\):

\[ \frac{\partial}{\partial b} \ln Z_0 = \frac{1}{\sigma + d/2} \left[ -\varepsilon b \frac{\alpha + d/2}{\beta(b)} + 1 \right] - \left( \alpha + d/2 \right) b \frac{\partial}{\partial b} \ln Z_p. \]  
(303)

Inserting this into the definition of the anomalous dimension \(\nu(b)\), and using the definition \(\varepsilon = 2D - \frac{2-D}{2}d\), yields

\[ \nu(b)(d + 2\sigma) = 2D + (2 - D)\sigma + \beta(b) \frac{\partial}{\partial b} \ln(Z_b Z^{-\sigma}) + \frac{\beta(b)}{b}. \]  
(304)

The last term vanishes at the fixed point \(b^*\), given by \(\beta(b^*) = 0\). We therefore obtain a non-trivial \(\varepsilon\)-expansion for the critical exponent \(\nu^* = \nu(b^*)\), reading

\[ \nu^*(d + 2\sigma) = 2D + (2 - D)\sigma + \beta(b) \left. \frac{\partial}{\partial b} \ln(Z_b Z^{-\sigma}) \right|_{b=b^*}. \]  
(305)

This expansion is formally obtained, when expanding \(\nu^*(d + 2\sigma)\) instead of \(\nu^*\) in \(\varepsilon\). In the following sections, we will explain and specify to the 2 cases of interests, \(\sigma = 0\) (variational approximation) and \(\sigma = 1\) (Flory approximation).

\(^5\)It is interesting to expand \(\nu_{\text{var}} = \frac{2D}{D}\) in \(D\) and \(\varepsilon\). The result is

\[ \nu_{\text{var}} = \frac{2 - D}{2} \frac{1}{1 - \frac{\varepsilon}{4D}} = \frac{2 - D}{2} \left( 1 + \frac{\varepsilon}{2D} + \frac{\varepsilon^2}{4D^2} + \ldots \right). \]

Thus boldly truncating the series at a given order in \(\varepsilon\) becomes increasingly bad when approaching \(D = 2\).
Figure 19: Example of a contraction towards a dipole.

7.4 Variational method and perturbation expansion

With the help of a Gaussian variational ansatz, which becomes exact when $d \to \infty$, Goulian[64], Le Doussal[65] and Guitter and Palmeri[66] have shown that in the latter limit

$$\nu^* = \nu_{\text{var}} = \frac{2D}{d}.$$  \hfill (306)

Another way to obtain the same result is to suppose, that the dipole does not necessitate any renormalization, thus $Z_b = 1$. Using Eq. (305), and there setting $Z_b = 1$ and $\sigma = 0$ according to Eq. (302), one recovers Eq. (306).

Let us now analyze, whether there is a limit in perturbation theory in which $Z_b = 1$. The counter-term for the renormalization of the wave-function at 1-loop order is given by the residue

$$\langle \begin{array}{c} \bullet \\ \end{array} \mid \bullet \rangle \approx 1 \text{ as } D \to 2,$$  \hfill (307)

whereas the counter-term for the renormalization of the coupling-constant is

$$\langle \begin{array}{c} \bullet \\ \end{array} \mid \bullet \rangle \approx 2^{-2D/(2-D)} \text{ as } D \to 2,$$  \hfill (308)

which is exponentially smaller than Eq. (307) when $D \to 2$. A similar exponential factor appears for the renormalization of the coupling-constant at 2-loop order, but not for the renormalization of the wavefunction. There is no rigorous proof that this persists to all orders in perturbation theory, but a convincing heuristic argument, which we present now.

In figure 7.4 we have depicted an example of a MOPE-coefficient $f(s, t, \ldots)$ from the contraction towards a dipole. The important observation is that $f$ has always the structure

$$f(s, t, \ldots) = f(s^{2-D} + t^{2-D}, \ldots).$$  \hfill (309)

For $\varepsilon \approx 0$, this can be re-expressed as

$$2^{-d/2} f \left( \frac{1}{2} (s^{2-D} + t^{2-D}), \ldots \right).$$  \hfill (310)
(Recall from section 3.2 that typical integrals over the dual of the imbedding space read $\int_p e^{-p^T Q p} \sim \det(Q)^{-\frac{d}{2}}$, leading for a $d \times d$-matrix $Q$ to the exponent of $-d/2$.) For $D \to 2$, the expression $\frac{1}{2} (s^{2-D} + t^{2-D})$ is always of order 1. Supposing that all subdivergences are subtracted, the expression to be integrated is finite and the integral of order 1. The argument is concluded upon remarking that

$$2^{-d/2} = 2^{-2D/(2-D)} \quad (311)$$

If this exponential bound $\ln Z_b \ll \ln Z$ for $D \to 2$ is correct, then $\nu^* - \nu_{\text{var}} \sim 2^{-d/2}$ when $d \to \infty$. For smaller, but still large $d$, it is useful to use Eq. (305) with $\sigma = 0$, i.e. [25]

$$\nu^* d = 2D + \beta(b) \frac{\partial}{\partial b} \ln(Z_b) \bigg|_{b=b^*} \quad (312)$$

for extrapolations.

Let us also mention that in the context of the $\phi^4$-theory, the variational approximation corresponds to the mean-field approximation, for which we know that the specific heat exponent $\alpha$, which for membranes has to be defined as [34]

$$\alpha := 2 - \frac{\nu^* d}{D} \quad (313)$$

vanishes.

### 7.5 Expansion about Flory’s estimate

In the introduction, we have shown that a simple heuristic argument supposing that the elastic energy scales like the contribution from self-avoidance, yields the Flory-estimate

$$\nu_{\text{Flory}} = \frac{2 + D}{2 + d} \quad (314)$$

In perturbation theory, a similar ansatz would be to set $Z = Z_b$. This is the case of $\sigma = 1$ in Eq. (305), which results into [25]

$$\nu^*(d + 2) = D + 2 + \beta(b) \frac{\partial}{\partial b} \ln \left( \frac{Z_b}{Z} \right) \bigg|_{b=b^*} \quad (315)$$

This makes it clear that if the wave-function and coupling-constant renormalizations are the same, or more precisely if $Z_b/Z$ stays finite at the IR fixed point $b^*$, $\beta(b) \frac{\partial}{\partial b} \ln \left( \frac{Z_b}{Z} \right) \bigg|_{b=b^*}$ vanishes and the Flory result becomes exact. Moreover, the $\varepsilon$-expansion of $\nu^*(d + 2)$ is clearly an $\varepsilon$-expansion about $\nu_{\text{Flory}}$.

This expansion seems to be the most satisfying numerically. In particular, the method of minimal sensitivity and that of minimizing the second order term give generally close results. Let us also stress that good expansion-parameters for $\nu^*$ are not necessarily good for $\nu^*(d + 2)$ and vice versa. E.g. the expansion in $D$ and $d$ is bad for $\nu^*$ but works quite well (although not optimal) for $\nu^*(d + 2)$. Examples of a 2-loop extrapolation, using the expansion (315), are presented in figure 18.

### 7.6 Results for self-avoiding membranes

Results of a 2-loop extrapolation for $\nu^*$ are given on figure 20 for membranes ($D = 2$) in $d$ dimensions ($2 \leq d \leq 20$). We see that for $d \to \infty$ the prediction of the Gaussian variational method becomes exact, as argued in section 7.4. For small $d$, the prediction made by Flory’s argument is close to our results. This
is a non-trivial result, since the membrane case corresponds to $\varepsilon = 4$ and in comparison with polymers in $d = 3$, where $\varepsilon = 1/2$, the 2-loop corrections were expected to be large. In fact, they are small when one expands around the critical curve $\varepsilon = 0$ for an adequate range of $D \sim 1.5$ (depending slightly on $d$ and on the choice of variables) and a suitable choice of extrapolation variables. In this case the 2-loop corrections are even smaller than the 1-loop corrections and allow for more reliable extrapolations to $\varepsilon = 4$. This can be understood from the large order behavior, as will be discussed in section 14.

Let us now turn to the physically relevant case of membranes in three dimensions ($D = 2, d = 3$). Our calculations predict an exponent $\nu^* \approx 0.85$ or equivalently a fractal dimension of $d_f \approx 2.4$, which is in agreement with those experiments and simulations which find a fractal phase. As we have discussed in sections 2.5 and 2.6, this is still under debate. Let us therefore give some heuristic argument why eventually no fractal phase exists in $d = 3$ [25, 167], i.e. why self-avoiding membranes might always be flat, as suggested by the numerical simulations discussed in section 2.6. Let us start from phantom membranes (without self-avoidance). The fractal exponent $\nu^*$ is then 0 in the crumpled phase, 1 in the flat phase, and equal to

$$\nu^*_c = 1 - \frac{1}{d} + \mathcal{O}\left(\frac{1}{d^2}\right)$$

at the crumpling transition [70, 51]. This last estimate is the result of a large $d$ expansion. Its applicability to low dimensions is thus not clear a priori, but numerical simulations [17] show that even in 3 dimensions this approximation is reasonable ($\nu^*_c = 2/3$). Let us now ask whether self-avoidance is relevant at the crumpling transition. By naive power-counting we find that this is the case if

$$\mathcal{D} := 2 \cdot 2 - \nu^*_c d > 0$$

i.e. with Eq. (316) for

$$d < 5 .$$

For $\mathcal{D} > 0$ we expect that the fractal exponent $\nu^*_c$ at the crumpling transition is different with or without self-avoidance.

One can now argue that at the crumpling transition and in the presence of self-avoidance, bending rigidity enhances the exponent $\nu^*$, but does not intervene in a proper renormalization of self-avoidance [167]. Following the arguments of section 3.9, this would imply that renormalization of self-avoidance is driven by renormalization of the field, such that $\nu^*$ at the crumpling transition and in the presence of self-avoidance would be given by the variational estimate

$$\nu^*_{c+SA} = \frac{2D}{d} .$$

Values near to this result were obtained in numerical simulations by Grest [116] in $d = 4, 5, 6, 8$ and by Barsky and Plischke [117] in $d = 4, 5$. This suggests that their membranes are at or near to the crumpling transition.

8 Other critical exponents and boundaries

8.1 Correction to scaling exponent $\omega$

The 2-loop calculations presented in section 6 allow in principle to compute other scaling exponents for self-avoiding tethered membranes. The first exponent is the so-called correction to scaling exponent $\omega$
Figure 20: Extrapolation of the 2-loop results in $d$ and $\varepsilon$ for membranes $D = 2$ in $d$ dimensions, using the expansion of $\nu^* (d + 2)$ (squares). The solid line is the prediction made by Flory’s approximation, the dashed line by the variational ansatz.

which governs the corrections to the large $L$ scaling behavior. It is known that this exponent is given by the slope of the $\beta$-function at the IR fixed point $b^*$ [147]

$$\omega = \left. \frac{\partial \beta (b)}{\partial b} \right|_{b = b^*}. \tag{320}$$

Its $\varepsilon$-expansion is given by\(^6\)

$$\omega = \varepsilon + 2 \varepsilon \left( C_1 + C_2 + C_3 \right) + 2 D \varepsilon \left( F_1 + F_2 + F_3 \right) \varepsilon^2 + O(\varepsilon^3) \tag{321}$$

with

$$A = \left\langle \begin{array}{c} \bullet \end{array} \right\rangle \varepsilon = \frac{1}{2 - D} \frac{\Gamma \left( \frac{D}{2 - D} \right)^2}{\Gamma \left( \frac{2 D}{2 - D} \right)}. \tag{322}$$

(For the definition of $C_1, \ldots, C_3$ and $F_1, \ldots, F_3$, see Eqs. (281) to (283) and (286) to (288).) Since for membranes, the term of order $\varepsilon^2$ is much larger than the order $\varepsilon$ term, no numerical value can be extracted [25].

### 8.2 Contact exponents

Another scaling exponent is the so-called bulk contact exponent $\theta_2$ [25]. For a general introduction to contact exponents for polymers and membranes we refer to [168]. The contact exponent $\theta_2$ is related to the

\(^6\)Note that there are some misprints in [25].
probability to find two fixed points $x_1$ and $x_2$ inside the membrane at a relative distance $r = |\vec{r}|$ in external $d$-dimensional space

$$P(r; x_1, x_2) = \left\langle \delta^d(r - [r(x_1) - r(x_2)]) \right\rangle .$$  \hspace{1cm} (323)

For a large membrane, $P$ is expected to take the scaling form

$$P(r; x_1, x_2) = R_{12}^{-d} F(r/R_{12}) ,$$  \hspace{1cm} (324)

where $R_{12}$ is the mean distance between $x_1$ and $x_2$.

$$R_{12}^2 = \frac{1}{2} \left\langle [r(x_1) - r(x_2)]^2 \right\rangle .$$  \hspace{1cm} (325)

The contact exponent $\theta_2$ is given by the small $r$ behavior of the scaling function $F$

$$F \left( \frac{r}{R_{12}} \right) \sim \left( \frac{r}{R_{12}} \right)^{\theta_2} \quad \text{when} \quad r \to 0 .$$  \hspace{1cm} (326)

$\theta_2$ is related to the scaling dimension $\omega_{12}$ of the 2-membrane contact operator

$$\delta_{12}(x_1, x_2) = \tilde{\delta}^d(r_1(x_1) - r_2(x_2))$$  \hspace{1cm} (327)

in the model of two independent self-avoiding membranes, $\mathcal{M}_1$ and $\mathcal{M}_2$. This model is described by the Hamiltonian

$$\mathcal{H} = \frac{Z}{2 - D} \int_{x_1 \in \mathcal{M}_1} \frac{1}{2} (\nabla r_1(x_1))^2 + \frac{Z}{2 - D} \int_{x_2 \in \mathcal{M}_2} \frac{1}{2} (\nabla r_2(x_2))^2$$

$$+ bZ_0 \mu^\varepsilon \left( \int_{x_1 \in \mathcal{M}_1} \int_{y_1 \in \mathcal{M}_1} \tilde{\delta}^d(r_1(x_1) - r_1(y_1)) + \int_{x_2 \in \mathcal{M}_2} \int_{y_2 \in \mathcal{M}_2} \tilde{\delta}^d(r_2(x_2) - r_2(y_2)) \right)$$

$$+ 2tZ_\varepsilon \mu^\varepsilon \int_{x_1 \in \mathcal{M}_1} \int_{y_2 \in \mathcal{M}_2} \tilde{\delta}^d(r_1(x_1) - r_2(y_2)) .$$  \hspace{1cm} (328)

It is renormalized by the same factors for $r$ and $b$ as the single membrane model, but with an additional renormalization for the inter-membrane coupling $t_0 = tZ_t(b, t)Z(b)^{d/2} \mu^\varepsilon$. The new counter-term $Z_t$ contains the same divergent diagrams as those which contribute to $Z_b$, but with different numerical factors. In particular, when $t = b$, $Z_t(b, t = b) = Z_b(b)$, so that the symmetric 2-membrane model reduces to the single membrane model.

As a consequence of this formalism, one can define a new RG function, $\beta_t$, which measures the dimension of $\delta_{12}$ defined in Eq. (327) as

$$\beta_t(b, t) := \mu \frac{\partial}{\partial \mu} \left|_{t=0} \right. \left. \frac{\varepsilon + \beta(b) \left[ \frac{\partial}{\partial b} \ln Z_t(b, t) + \frac{d}{2} \frac{\partial}{\partial b} \ln Z(b) \right]}{1 + t \frac{\partial}{\partial b} \ln Z_t} \right| .$$  \hspace{1cm} (329)

calculate the RG-flow in the $(b, t)$ plane and check that $(b, t) = (b^*, t^*)$ is the IR stable fixed point which governs the scaling behavior of a large membrane. Insertions of $\int_{x_1 \in \mathcal{M}_1} \int_{y_2 \in \mathcal{M}_2} \tilde{\delta}^d(r_1(x_1) - r_2(y_2))$ are generated by varying the inter-membrane coupling $t$ from its fixed-point value $t^*$ and have dimension

$$\omega_{12} := \left. \frac{\partial}{\partial t} \beta_t \right|_{t=t^*} .$$  \hspace{1cm} (330)
In order to obtain $\Theta_2$, recall that (i) Eq. (327) does not contain integrations over $x$ and $y$ at variance with membrane-membrane self-avoidance, (ii) $P(r; x_1, x_2) \sim R_{12}^{-\Theta_2}$, and (iii) $R_{12} \sim |x - y|^{\nu^*}$, to get the final result

$$\Theta_2 = \frac{1}{\nu^*} [\omega_{12} + 2D - \nu^* d] . \tag{331}$$

We can now compare $Z_b$, given in Eq. (273)\(^6\), and $Z_t$:

$$Z_b(b) = 1 + \frac{A}{\varepsilon} b + \left( \frac{A(A + \frac{1}{2})}{4D\varepsilon} + C_1 + C_2 + C_3 \right) b^2 + \ldots$$

$$Z_t(b, t) = 1 + \frac{A}{\varepsilon} t + \left( \frac{A^2}{2\varepsilon^2} + C_1 \right) t^2 + \left( \frac{A}{2\varepsilon^2} - \frac{A}{4D\varepsilon} + C_2 + C_3 \right) bt + \ldots , \tag{332}$$

with $A$ given in Eq. (322). The final result to order $\varepsilon^2$ reads

$$\Theta_2 = \frac{4}{2 - D} \frac{A}{1 + A} \varepsilon + 4 \varepsilon \frac{(3 + A)C_1 + 2(C_2 + C_3) - 4(F_1 + F_2 + F_3)DA}{(2 - D)(1 + A)^3} \varepsilon^2 + O(\varepsilon^3) \tag{333}$$

As in the case of the correction to scaling exponent $\omega$, no reliable estimate for $\Theta_{12}$ can be extracted from the $\varepsilon$-expansion [25].

Let us also mention that if one or both of the points $x_1, x_2$ in Eq. (323) are on the boundary of the membrane, the contact exponent changes. This is first of all due to the reduced available domain of integration for operators approaching $\bar{\delta}^d(r - [r(x_1) - r(x_2)])$, and second to the change of the propagator itself, as discussed in section 8.4 [21, 168, 57].

\subsection{8.3 Number of configurations: the exponent $\gamma$}

In principle, one can also calculate the partition function of the membrane, which is defined as [56, 57]

$$Z(b) = \int \mathcal{D}\{r\} \bar{\delta}^d(r(0)) e^{-\mathcal{H}[r]} . \tag{334}$$

The factor of $\bar{\delta}^d(r(0))$ is included in order to eliminate the 0-mode of the path-integral over $r$, which in dimensional regularization would lead to a factor of 0. Rescaling the membrane by a factor of $\lambda$, the partition function is expected to behave as

$$Z(b; \lambda) = \lambda^{\gamma - 1} C_{\text{Vol}^D} Z(b; 1) . \tag{335}$$

$C$ is a non-universal constant, Vol the volume of the membrane and $\gamma$ a universal exponent. If the dimension $D$ of the membrane is not an integer, $\gamma$ is related to the scaling exponent $\nu^*$ by

$$\gamma - 1 = -\nu^* d , \tag{336}$$

where the r.h.s. is just the full dimension of the operator $\bar{\delta}^d(r(0))$ in the absence of counter-terms proportional to $\bar{\delta}^d(r(0))$. Note that indeed the MOPE of the contraction of any number of dipoles towards $\bar{\delta}^d(r(0))$ does not contain a term proportional to $\bar{\delta}^d(r(0))$ (see exercise E.2 on page 161). The relation (336) will therefore hold, if no other counter-terms as those discussed for an infinite membrane in section 3.8, namely a counter-term for the field and coupling are present. By power-counting, we find that the only additional relevant or marginal operators for finite membranes are boundary operators. The leading boundary-operator
is just its \((D-1)\)-dimensional length, the next to leading operator the size of \((D-2)\)-dimensional “corners” etc. The length of the boundary has dimension \(D - 1\) in internal length-units, and will just be marginal at \(D = 1\). Therefore, for polymers the relation (336) is only valid for closed polymers, whereas open ones involve the independent exponent \(\eta\), known from \(\phi^4\)-theory. (Let us stress that this exponent is unaccessible when studying closed polymers, or equivalently the correlation function \(\langle \phi^2(x)\phi^2(y) \rangle\) in \(\phi^4\)-theory.) However when studying \(D\)-dimensional objects with say \(D\) slightly larger than 1, even though the boundary is a relevant operator, no pole in \(1/\varepsilon\) proportional to this operator is found in an \(\varepsilon\)-expansion, as will be seen in the next section when studying boundaries. The reason is, that the location of the poles associated to singular configurations can always be inferred from power-counting which e.g. in the case of a dipole approaching a boundary will be

\[
\frac{1}{\varepsilon + 1 - D}. \tag{337}
\]

Therefore, in an \(\varepsilon\)-expansion, i.e. when performing the limit \(\varepsilon \to 0\) first, this term will be finite. On the other hand, when first setting \(D \to 1\) before \(\varepsilon \to 0\), there will be a divergence violating Eq. (336), as mentioned above. For membranes, i.e. when setting \(D = 2\) before letting \(\varepsilon \to 0\), divergences are expected proportional to the number of corners. A term proportional to the curvature of the membrane can also appear. However, the scheme explained here is useless in dealing with these divergences, since for \(D = 2\), the regularization parameter \(\varepsilon\) always equals 4.

Some more insight can be gained from a simplified model, in which one uses an infinite membrane (to have the simple propagator of infinite space), but where self-avoidance is only affective in some part of the membrane, from now on simply called membrane \(\mathcal{M}\). The most beautiful way to treat this, was proposed by Duplantier, who introduces the characteristic function of the membrane [21, 168]

\[
\Phi(q) = \int_{\mathcal{M}} d^D x \ e^{ixq}, \tag{338}
\]

with the help of which, the self-energy of the membrane, namely the integral

\[
I := \int_{x \in \mathcal{M}} \int_{y \in \mathcal{M}} \langle \delta^d(r(x) - r(y)) \rangle_0 \tag{339}
\]

can be written as (\(A\) is a geometric well-behaved prefactor)

\[
I = A \int d^D q \Phi(q)\Phi(-q)|q|^{\nu_d - D}. \tag{340}
\]

The analysis in [21, 168] shows, that there are additional contributions for odd integer values of \(D\) for hyper-ellipsoids, and for any integer values of \(D\) for a hyper-torus.

There is another source of terms violating Eq. (336), which is already present for non self-avoiding membranes and to which we turn now. In the latter case, the partition function can be calculated exactly, using methods developed in conformal field theory, which involves the \(\zeta\)-function regularization. The result for \(\gamma_0\), is [21, 168, 57]

\[
\gamma_0 - 1 = \begin{cases} 
-\nu_d & \text{for } D \text{ not integer} \\
-\nu_d + \frac{\chi}{2} & \text{for } D = 1 \text{ with } \chi = 0 \text{ for a closed chain, and } \chi = 1 \text{ for an open one} \\
-\nu_d + \frac{\chi}{2} & \text{for } D = 2 \text{ with } \chi \text{ the Euler characteristics of the manifold}. 
\end{cases} \tag{341}
\]
8.4 Boundaries

Up to now, we only considered infinitely large membranes. An important question, not only theoretically but also experimentally, is how boundaries influence the bulk critical behavior. This is the more important since all available simulations deal with relatively small systems, which may be completely dominated by boundary effects.

There are several things which can be done. Consider first a membrane, obtained by cutting an infinite membrane along the line \( x_\perp = 0 \). We then use a coordinate-system, in which \( x_\perp \) measures the distance from the boundary and \( x_\parallel \) is a \( D-1 \) dimensional vector parallel to the boundary. The correlation-function is as usual obtained by inverting the Laplacian on this half-membrane with Neumann boundary conditions [148]. Note that these conditions have to be fulfilled by \( B(x,y) := \langle r(x)r(y) \rangle_0 \), but are not satisfied by \( C(x,y) = \frac{1}{2} \langle [r(x) - r(y)]^2 \rangle_0 \). Explicitly, this is

\[
\frac{\partial}{\partial x_\perp} B(x,y) \big|_{x_\perp=0} = 0.
\]

The solution to this equation is the usual bulk-correlator plus its image-part:

\[
B(x,y) = B_{\text{bulk}}(x,y) + B_{\text{bulk}}(x,\bar{y}),
\]

where \( \bar{y} \) is the mirror image of \( y \) with respect to the boundary. Denoting \( y = (y_\perp, y_\parallel) \), this is \( \bar{y} = (-y_\perp, y_\parallel) \). For the correlator \( C(x,y) \), this reads explicitly

\[
C(x,y) = \frac{1}{2} \langle [r(x) - r(y)]^2 \rangle_0
= |x - y|^{2-D} + [(x_\parallel - y_\parallel)^2 + (x_\perp + y_\perp)^2]^{\frac{2-D}{2}} - \frac{1}{2} \left( |2x_\perp|^{2-D} + |2y_\perp|^{2-D} \right).
\]

The first term on the r.h.s. is the usual bulk correlator. Due to Hölder’s inequality [169], and for \( x_\parallel - y_\parallel = 0 \) the additional terms satisfy

\[
\left[ (x_\perp + y_\perp)^2 \right]^{\frac{2-D}{2}} - \frac{1}{2} \left( |2x_\perp|^{2-D} + |2y_\perp|^{2-D} \right) \begin{cases} > 0 & D > 1 \\ = 0 & D = 1 \\ < 0 & D < 1 \end{cases}
\]

(345)
This implies that for \( D > 1 \), \( C(x, y) > |x - y|^{2 - D} \). The case \( D < 1 \) is more subtle since when integrating over \( \int d\mathbf{x}_\perp \) and \( \int d^{D-1}x_\parallel \), the latter is only defined by analytic continuation. For a polymer, no additional contribution appears. This is understood by observing in analogy to a random walk that the correlator between two points \( x \) and \( y \) only depends on what happens between these two points.

Let us now proceed to self-avoiding membranes. Our interest is to understand, what happens, when a dipole approaches the boundary. Due to power-counting, the most relevant boundary operator is

\[
\mathcal{H}_{\partial M} = \int d^D x_\parallel \, 1 .
\]  

(346)

As already mentioned in the last subsection, this operator is marginal at \( D = 1 \), and relevant for \( D > 1 \), necessitating a (multiplicative) renormalization only in \( D = 1 \). In the language of \( \phi^4 \)-theory, this renormalization shows up in the scaling exponent \( \eta \).

All other boundary-operators are irrelevant at \( \varepsilon = 0 \).

9 The tricritical point

9.1 Introduction

In many experiments, one encounters tri-critical behavior. In the context of polymers, the tri-critical point exists due to a competition between long-range attractive interactions as e.g. Van der Waals forces and hard-core repulsion. At high temperature repulsion dominates and the polymer is swollen. Lowering the temperature, finally attractive forces will collapse the polymer into a compact state. For a single long polymer, the transition between these two states occurs at the \( \Theta \)-point, which represents a different multi-critical state for the polymer [170] (and [6] for a general review). A similar transition is expected for membranes. As shown in [27], two regimes can be distinguished, depending on the dimension \( D \) of the membrane and the dimension \( d \) of imbedding space. For polymers and for membranes with \( D \) small enough, one expects an effective 3-body repulsive interaction to be relevant to describe the \( \Theta \)-point close to the upper critical dimension \( d_c = 3D/(2 - D) \). For larger \( D \), a modified 2-body interaction, repulsive at short range, but attractive at larger range, is relevant to describe the \( \Theta \)-point close to the upper-critical dimension, now given by \( d'_c = 2(3D - 2)/(2 - D) \). The crossover between these two interactions occurs at \( D = 4/3, d = 6 \). While the modified 2-body interaction can be treated analytically at 1-loop order, this is impossible for the 3-body interaction. Numerical methods to calculate the diagrams involved have been developed in [27]. We will focus here on the crossover between the 2-body and the 3-body interactions which can be studied via a “double \( \varepsilon \)-expansion” about the critical point \( D = 4/3, d = 6 \). As a result it is shown that depending on \( D \) and \( d \), the \( \Theta \)-point is described either by: (a) a Gaussian fixed point, (b) the 3-body repulsive interaction, (c) the modified 2-body interaction. The three corresponding domains in the 2-dimensional \((d, D)\) plane are depicted in figure 22. The fat lines are the separatrices between these domains. The fat dashed line is a linear extrapolation of the 1-loop result. This line separates the domains (b) and (c) indicating that the modified 2-body interaction should be relevant to describe 2-dimensional membranes at the \( \Theta \)-point, independent of the dimension \( d \).

9.2 Double \( \varepsilon \)-expansion

The leading operator studied in the last sections was the 2-body interaction

\[
\int \int_{x \neq y} \cdots = \int \int_{x \neq y} \delta^d(r(x) - r(y)) .
\]  

(347)
By fine-tuning its coupling such that the renormalized 2-body interaction vanishes, one reaches the \( \Theta \)-point separating the swollen from the collapsed phase. In the RG-analysis, the next-to-leading operators then control the physical behavior.

Analyzing the canonical dimensions of all subleading operators, one finds two candidates: The bilocal operator

\[
(-\Delta r)\tilde{\delta}^d(r(x) - r(y)) = \quad ,
\]

and the trilocal operator

\[
\tilde{\delta}^d(r(x) - r(y))\tilde{\delta}^d((x) - r(y)) = \quad .
\]

The Hamiltonian, one has to study is therefore

\[
\mathcal{H}_{u,b} = \frac{Z}{2-D} \int_x + u Z u^{\varepsilon_u} \int_x \int_y \int_z + b Z b^{\varepsilon_b} \int_x \int_y ,
\]

with the same normalizations as in the previous sections (see also appendix A.1). The both couplings have canonical dimension (in momentum units)

\[
\varepsilon_u := [u^{\varepsilon_u}]_\mu = 3D - 2\nu d \quad , \quad \varepsilon_b := [b^{\varepsilon_b}]_\mu = 2D - 2\nu(d+2) \equiv 3D - 2 - \nu d .
\]

They become relevant at the Gaussian fixed point when \( \varepsilon_u \) or \( \varepsilon_b \) respectively become positive. We have drawn the critical curves (\( \varepsilon_b = 0 \) and \( \varepsilon_u = 0 \)) on figure 22. We see that at \( D = 4/3 \) the two operators \( \quad \) and \( \quad \) interchange their role. Below \( 4/3 \) \( \quad \) is more relevant, above \( 4/3 \) \( \quad \). At \( D = 4/3 \) and \( d = 6 \) both operators are marginal: We have the interesting situation of a system with two coupling constants.
In contrast to standard perturbation theory, where in first order of the coupling constant the divergences are single poles, the leading singularity of \( \langle \mathcal{O} \rangle_L \) is a double pole, due to the sequence of divergent contractions

\[
\begin{align*}
&\quad 
\end{align*}
\]

This prevents us from performing the renormalization in the standard way. Let us look at the problem from another point of view. As the modified 2-point interaction renormalizes the elastic energy \( \frac{1}{2} (\nabla r)^2 \), perturbations in this operator have to be controlled by a small coupling \( b \) as is done in the Hamiltonian \( (350) \). On the other hand, the 3-point interaction renormalizes the modified 2-point interaction via the contraction

\[
\begin{align*}
&\quad 
\end{align*}
\]

Therefore there has to be a small parameter controlling the ratio of \( \) and \( \). These demands are satisfied by replacing the 3-point coupling constant \( u \) by \( bg \). The Hamiltonian becomes

\[
H_{g,b} = \frac{Z}{2-D} \int_x + bgZ_bZ_g \mu^{\varepsilon_b+\varepsilon_g} \int \int \int x y z + bZ_b^\mu \int \int .
\]

Perturbation-theory in \( b \) and \( g \) is now performed, by counting orders in \( b \) and \( g \) equivalently. They have canonical dimensions

\[
\varepsilon_b := [b \mu^{\varepsilon_b}]_\mu = 3D - 2 - \nu d , \quad \varepsilon_g := [g \mu^{\varepsilon_g}]_\mu = 2 - \nu d ,
\]

which will lead to poles in \( \varepsilon_b \) and \( \varepsilon_g \) in the perturbation expansion. Note that this situation with two completely independent couplings \( b \) and \( g \) and independent canonical dimensions \( \varepsilon_b \) and \( \varepsilon_g \) is quite unusual in standard perturbation theory. Here it is due to the fact that both the inner dimension \( D \) and the dimension of imbedding space \( d \) can be chosen independently.

The perturbative expansion will be performed in the vicinity of the critical point \( d_c = 6 \) and \( D_c = 4/3 \). The theory will be finite in terms of the renormalized quantities

\[
\begin{align*}
&\quad 
\end{align*}
\]

where the renormalization factors have up to first order in \( b \) and \( g \) the form

\[
Z = 1 + p \frac{g}{\varepsilon_g} + q \frac{b}{\varepsilon_b}
\]

with constants \( p \) and \( q \). We draw the attention to the important point that pole terms in \( \varepsilon_b \) are always proportional to \( b \) as should be evident from dimensional arguments. The same is true for \( \varepsilon_g \) and \( g \). It is also important to note that this would not be the case for the parameterization \( (353) \), nor if one there replaces \( u_0 \) by \( u_0^2 \), what one might be tempted to do.

In order to explicitly perform the calculations, we study as in section 3.1 expectation values of an IR-finite observable \( \mathcal{O} \). The following diagrams contribute to first order in \( g \) and \( b \) at \( D = 4/3 \) (for more
They determine the renormalization factors at 1-loop order:

\[
Z = 1 + \frac{b}{\varepsilon_b} \tag{360}
\]

\[
Z_g = 1 + \frac{3}{4} \frac{g}{\varepsilon_g} + \frac{7}{2} \frac{b}{\varepsilon_b} \tag{361}
\]

\[
Z_b = 1 - \frac{3}{4} \frac{g}{\varepsilon_g} + \frac{b}{\varepsilon_b} \tag{362}
\]

As usually we define the renormalization group \(\beta\)-functions of the two couplings \(b\) and \(g\) as their variation with respect to the renormalization scale \(\mu\) at fixed bare parameters:

\[
\beta_b(b, g) = \mu \frac{\partial}{\partial \mu} b \tag{363}
\]

\[
\beta_g(b, g) = \mu \frac{\partial}{\partial \mu} g. \tag{364}
\]

Inserting the definitions of \(b\) and \(g\), we get two coupled linear equations in \(\beta_b\) and \(\beta_g\), which can be solved after some algebra. They lead to the \(\beta\)-functions at 1-loop order:

\[
\beta_b(b, g) = -\varepsilon_b b - \frac{3}{4} bg + 5b^2 \tag{365}
\]

\[
\beta_g(b, g) = -\varepsilon_g g + \frac{11}{2} bg + \frac{3}{4} g^2. \tag{366}
\]

The scaling function of the field \(\nu(b, g)\) becomes:

\[
\nu(b, g) = \nu - \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z
= \frac{1}{3} + \frac{1}{2} b. \tag{367}
\]

### 9.3 Results and discussion

The system of equations (365), (366) determines four fixed points in the \((g, b)\) plane. The physical couplings must correspond to a repulsive interaction at short distance, hence to the domain \((b \geq 0, \ g \geq 0)\).
One of the fixed points is IR-attractive, one IR-repulsive and the other two have one attractive and one repulsive direction. For special values of the parameters $\varepsilon_b$ and $\varepsilon_g$, fixed points may coincide. Passing through these special values describes the transition from one fixed point to another, resulting in an eventual non-analyticity of the critical exponent $\nu(b, g)$. We first list the different critical points visualized in figure 23.

$P_1$: The Gaussian fixed point $b_c = 0$ and $g_c = 0$: it is stable for $\varepsilon_b < 0$ and $\varepsilon_g < 0$.

$P_2$: The fixed point $b_c = 0$ and $g_c = \frac{4}{3}\varepsilon_g$ describes also a trivial theory, although $g_c$ has a non-trivial value. Indeed, regarding the Hamiltonian (353), we see that both interactions are renormalized to 0. Also the critical exponent $\nu(b, g)$ equals that of the free (Gaussian) theory. The stability condition is
\( \varepsilon_g > 0 \) and \( \varepsilon_b + \varepsilon_g < 0 \).

**P3:** The fixed point \( b_c = \frac{1}{5} \varepsilon_b \) and \( g_c = 0 \): for this non-trivial fixed point only the modified 2-point interaction plays a role. It is stable for \( \varepsilon_b > 0 \) and \( 11\varepsilon_b > 10\varepsilon_g \).

**P4:** The fixed point \( b_c = \frac{2}{27}(\varepsilon_b + \varepsilon_g) \) and \( g_c = \frac{4}{63}(-11\varepsilon_b + 10\varepsilon_g) \) is the most interesting one. Both couplings flow to a finite non-zero value. This point is stable for \( \varepsilon_b + \varepsilon_g > 0 \) and \( 11\varepsilon_b < 10\varepsilon_g \). It corresponds to the fixed point for the case of a 3-point interaction only in the limit of \( D \to 4/3 \) from below. We will explain that in more detail below.

Let us discuss the graphics of figure 23: We can distinguish 8 different regions in the \((d, D)\) plane around the critical point \((d_c = 6, D_c = 4/3)\), named A to H. The separating lines are

1. \( \varepsilon_g = 0 \) separating D,E and A,H
2. \( \varepsilon_b + \varepsilon_g = 0 \) between E,F and A,B
3. \( \varepsilon_b = 0 \) separating F,G and B,C
4. \( 11\varepsilon_b = 10\varepsilon_g \) between C,D and G,H

The flow graphs in figure 23 correspond to these regions A to H, starting with region H in the upper left corner.

Coming back to the general situation depicted in figure 23, the flows are such that:

- In regions C, D and E, the Gaussian fixed point \( P_1 \) or the pseudo-Gaussian fixed point \( P_2 \) are IR-stable. The modified 2-point and 3-point interactions are irrelevant and the large distance properties of the manifold at the \( \Theta \)-point are those of a free Gaussian manifold.

- In regions A, B and H, the fixed point \( P_3 \), described by the modified 2-point interaction only, is IR-stable. The 3-point interaction is irrelevant and the modified 2-point Hamiltonian (353) with \( g \equiv 0 \), also discussed in exercise E.4, is sufficient to describe the large-distance properties of the manifold at the \( \Theta \)-point through an \( \varepsilon_b \)-expansion.

- Finally, in regions F and G the fixed point \( P_4 \), which contains a mixture of 3-point and modified 2-point interactions, is IR-stable. As discussed in [27], this fixed point corresponds to the limit \( D \to 4/3 \) for the 3-point Hamiltonian, i.e. setting \( b \equiv 0 \) in Eq. (350). Therefore the pure 3-point Hamiltonian is sufficient to describe the \( \Theta \)-point in an \( \varepsilon_g \) expansion.

If one extrapolates these 1-loop results, one obtains the picture already summarized in figure 22 for the \( \Theta \)-point as a function of the external dimension of space \( d \) and of the internal dimension of the membrane \( D \): The \((d, D)\) plane is separated into three regions:

- For \( D < 2 \) and \( d \) sufficiently large, both the 3-point interaction and the modified 2-point interaction are irrelevant. The \( \Theta \)-point is described by the Gaussian model.

- For \( d < d_c = 3D/(2 - D) \) and \( D \) sufficiently small, the 3-point interaction is more relevant than the modified 2-point interaction and governs the \( \Theta \)-point.

- For \( d < d'_c = 2(3D - 2)/(2 - D) \) and \( D \) sufficiently large, the modified 2-point interaction is more relevant than the 3-point interaction and governs the \( \Theta \)-point.
At 1-loop order, the separatrix between these two domains is given by line number 4. \((11\varepsilon_b = 10\varepsilon_g, \text{with } \varepsilon_b \text{ and } \varepsilon_g \text{ given by (354))}, \) i.e. by the line

\[ d = 108D - 138 \]  

(368)

Thus, if we trust this picture far from the critical point \((d = 6, \ D = 4/3)\), we expect that for 2-dimensional membranes \((D = 2)\), the modified 2-point interaction will always be the most relevant one to describe the \(\Theta\)-point, even for \(d < 6\). One also checks that the modified 2-point interaction is less relevant than the standard 3-point interaction to describe polymers \((D = 1)\) in two dimensions \((d = 2)\) at the \(\Theta\)-point.

Finally, let us stress that the analysis of the relevance of the two interaction terms leads to results drastically different from naive power counting or approximate schemes. Naive power counting predicts a separating line given by

\[ d = \frac{4}{2-D} \]  

(369)

and that for \(D = 2\) the 3-body interaction is always more relevant than the modified 2-body interaction. Flory-type arguments give a separatrix

\[ d = 3D + 2 \]  

(370)

while a Gaussian variational approximation leads to

\[ d = 6 \]  

(371)

Both approximations predict that for \(D = 2\) the 3-body interaction is relevant for low dimensions \(d \) \((d < 8 \text{ and } d < 6 \text{ respectively})\).

10 Variants

10.1 Unbinding transition

An interesting and much simpler variant of the self-avoiding membrane model (61) is a non self-avoiding (phantom) membrane attracted to a fixed point

\[ \mathcal{H}_{\text{pin}}[r] = \frac{1}{2-D} \int_x \frac{1}{2} (\nabla r(x))^2 - bZ_b\mu^\varepsilon \int_x \delta^d(r(x)) \]  

(372)

where now

\[ \varepsilon = D - \nu d, \quad \nu = \frac{2-D}{2} \]  

(373)

This was originally considered as a toy-model in [171, 53, 54], where it is was used to develop the methods for the proof of perturbative renormalizability of the self-avoiding case. In recent time, it has also found a more axiomatic treatment in [172, 173], where it is proven that there is a true fixed point close to the perturbatively obtained one. The model also appears in the context of wetting, reviewed in [174].

The model (372) only necessitates one renormalization, namely for \(b \) [171, 53, 54]. The elastic energy is not renormalized. Physically this is understood from the observation that Eq. (372) has also to describe the membrane far away from the binding point, and there clearly no renormalization of the elastic energy is required. Consider now renormalization. First note, that the only divergences come from approaching the \(\delta\)-interactions. The leading term is (denoting \(\cdot := \delta^d(r(x))\))

\[ (\bullet \bullet \bullet) = [C(x - y)]^{-d/2} \]  

(374)
leading with \( b_0 = bZ_b \mu^\varepsilon \) to the 1-loop \( \beta \)-function

\[
\beta(b) := \frac{\partial}{\partial \mu} \left. b = -\varepsilon b - \frac{1}{2} \langle \cdots | \delta | \cdots \rangle b^2 + O(b^3) , \quad \langle \cdots | \delta | \cdots \rangle = 1 . \tag{375}
\]

The fixed point lies at \( \varepsilon < 0 \) and is repulsive. The binding of the membrane is described by the derivative \( \omega \) of the \( \beta \)-function at the non-trivial fixed point \( b^* \) with \( \beta(b^*) = 0 \):

\[
\omega := \left. \frac{d}{db} \beta(b) \right|_{b=b^*} = -\varepsilon . \tag{376}
\]

This means that near to the critical point \(|b - b^*| \sim \mu^\omega\), and since the mean distance \( R \) to the binding center scales as \( R \sim \mu^{-\nu/2} \), we have

\[
R \sim |b - b^*|^{-\nu/\omega}. \tag{377}
\]

Let us now turn to the case of polymers, which has extensively been treated in the literature. It is also interesting, since there a striking simplification occurs. For polymers, and polymers only, the result (376) stays true to all orders in perturbation theory. This is proven as follows. Consider the contraction of \( (n+1) \delta \)-interactions. We claim that \( (D = 1) \)

\[
\langle \cdots | \delta | \cdots \rangle = \left[ \langle \cdots | \delta | \cdots \rangle \right]^n , \tag{378}
\]

where the \( n \) arguments on the r.h.s. are the \( n \) nearest neighbor distances of the l.h.s.. This is proven as follows: First, by changing the integration-variables enforcing the \( \delta \)-interactions and ordering the distances, i.e. \( x_i < x_{i+1} \), we can write

\[
\langle \cdots | \delta | \cdots \rangle = \int \int \cdots \int \prod_{i=1}^{n} e^{ik_i \left[ r(x_i) - r(x_{i+1}) \right]} . \tag{379}
\]

The MOPE-coefficient \( \langle \cdots | \delta | \cdots \rangle \) is obtained from Eq. (379) by dropping the integration over \( p \) (which finally gives \( \varepsilon \)), setting \( p = 0 \) and taking the expectation value, leading to

\[
\langle \cdots | \delta | \cdots \rangle = \int \int \cdots \int \langle \prod_{i=1}^{n} e^{ik_i \left[ r(x_i) - r(x_{i+1}) \right]} \rangle_0 . \tag{380}
\]

For polymers, this is easily verified to factorize as

\[
\langle \cdots | \delta | \cdots \rangle = \prod_{i=1}^{n} e^{-k_i^2 |x_i - x_{i+1}|} \tag{381}
\]

\[
= \prod_{i=1}^{n} |x_i - x_{i+1}|^{-d/2} ,
\]

as should be demonstrated.
However when performing perturbation theory in $x$-space, at each order new terms appear, and it is extremely difficult to show that the $\beta$-function can still be recast in the simple form (375). The simplest remedy is to introduce as in section 4.4 a chemical potential $t$ conjugate to the length $L$ of the polymer, and then to integrate over the polymer-length. Since $L = \sum_{i=1}^{n} |x_{i+1} - x_i| + \ell$, where $\ell$ is the length of the free ends, this leads in Eq. (381) to

$$\prod_{i=1}^{n} \int_{k_i} \frac{e^{-(k_i^2 + t)|x_i - x_{i+1}|}}{k_i}$$

(382)

and integration over all positions $x_i$ gives

$$\prod_{i=1}^{n} \int_{k_i} \frac{1}{k_i^2 + \ell}$$

(383)

which factorizes trivially. Finally, one has to sum the perturbation expansion; this is easily done, since the perturbation expansion is a geometric series. Thus (375) is exact, if one there sets

$$\langle \mathbf{\ldots} | \mathbf{\ldots} \rangle = \epsilon \int_{k} \frac{1}{1 + k^2} = \Gamma \left(2 - \frac{d}{2}\right).$$

(384)

(Note that the equivalent problem arises in the presence of non-linear growth [175]. A pedagogical treatment is given in [176].)

Let us now turn back to the general case of membranes. One should note that the model (372) is equivalent to the model of a directed polymer (or membrane) in a $(d + D)$-dimensional space binding to a $D$-dimensional plane (or line)

$$\mathcal{H}_{\text{pin}}[r] = \frac{1}{2 - D} \int_{x} \frac{1}{2} (\nabla r(x))^2 + bZ \mu^\epsilon \int_{x} \tilde{\delta}^d(h(x))$$

$$= \frac{1}{2 - D} \int_{x} \frac{1}{2} (\nabla h(x))^2 + bZ \mu^\epsilon \int_{x} \tilde{\delta}^d(h(x)) + \text{const},$$

(385)

with

$$r(x) = \left(\begin{array}{c} x \\ h(x) \end{array}\right).$$

(386)

$h(x)$ are the orthogonal fluctuations. (For a review see [177].)

If the membrane is rigid, the model is further modified to

$$\mathcal{H}_{\text{pin}}[h] = -\frac{1}{2(4 - D)(2 - D)} \int_{x} \frac{1}{2} (\Delta h(x))^2 + bZ \mu^\epsilon \int_{x} \tilde{\delta}^d(h(x)),$$

(387)

where now

$$\epsilon = D - \nu d, \quad \nu = \frac{4 - D}{2},$$

(388)

which can be renormalized along the same lines as Eq. (372) [53, 54], using now the correlator $C(x - y) = |x - y|^{4-D}$. A mathematically specific intriguing case is the limit of $D \to 2$, since in that limit $\nabla h(x)$ is dimensionless, such that an infinity of couplings appears, of the form $\int_{x} f(\nabla h(x)) \tilde{\delta}^d(h(x))$, with an arbitrary function $f$. This problem was considered in [178] and it was concluded that the fixed point of Eq. (387) is still relevant, but is unlikely to be reached for an arbitrary microscopic models.
10.2 Tubular phase

In section 2.3, we have introduced an effective free energy for isotropically polymerized tethered membranes. An interesting question is, what happens in the case of anisotropy [179, 180]. Anisotropy should be relevant physically for tubules, which are synthesized by polymerizing a fluid membrane with parallelly oriented embedded stiff objects, as e.g. DNA-molecules. This leads in generalization of Eq. (10) to a model with different elastic constants in the parallel ($x_\parallel$) and ($D-1$) orthogonal ($x_\perp$) components

\[
\mathcal{H}[r] = \int_x \frac{\kappa_\perp}{2} \left( \partial_\perp^2 r \right)^2 + \frac{\kappa_\parallel}{2} \left( \partial_\parallel^2 r \right)^2 + \frac{t_\perp}{2} \left( \partial_\perp r \right)^2 + \frac{t_\parallel}{2} \left( \partial_\parallel r \right)^2

+ u_\perp \left( \partial_\perp r \partial_\perp^2 r \right)^2 + (u_\parallel + v_\parallel) \left( \partial_\parallel r \partial_\parallel r \right)^2 + u_\parallel \left( \partial_\parallel r \partial_\parallel r \right)^2

+ v_\perp \left( \partial_\perp r \partial_\perp r \right)^2 + v_\parallel \left( \partial_\parallel r \partial_\parallel r \right)^2

+ \frac{b}{2} \int_x \int_{x'} \delta^d(r(x) - r(x')) ,
\]

(389)

In section 2.3, we have seen that for $t < 0$ rotational symmetry is spontaneously broken and the membrane becomes flat. In the case of two harmonic elastic constants $t_\parallel$ and $t_\perp$, one may well have $t_\parallel < 0$ and $t_\perp > 0$, leading to a tubular phase with length proportional to the microscopic length $L$ and diameter $R_g \sim L^{\nu^*}$, see figure 24. In contrast to isotropic membranes, where the flat phase is stable at all scales, the tubular phase of tethered membranes behaves at large distances as a stiff polymer, which is known to resemble a flexible polymer for even larger scales.

The tubular phase has indeed been found in simulations [181].

Since it is difficult to treat the full model (389), to catch the effects of self-avoidance, it was proposed [182] to study a reduced model by expanding about the mean-field solution

\[
r(x) = \left( \zeta x_\parallel + u(x) \right), \quad \zeta = \frac{1}{2} \frac{|t_\parallel|}{u_\parallel + v_\parallel}
\]

(390)

where $u(x)$ and $h(x)$ are the fluctuations. (Note the difference to the definition (18).) Keeping only the
leading harmonic terms in \( u \) and \( h \), we arrive at

\[
\mathcal{H}'[r] = \frac{1}{2} \int_x \kappa(\partial^2 h)^2 + t_\perp (\partial_\perp h)^2 + 4 |t_\parallel| (\partial_\parallel u)^2 + t_\parallel (\partial_\parallel u)^2 \\
+ \frac{b}{2} \int_x \int_{x'} \delta^{d-1}(h(x) - h(x')) \delta(\zeta(x_\parallel - x'_\parallel) + u(x) - u(x')).
\] (391)

The last term \( \delta(\zeta(x_\parallel - x'_\parallel) + u(x) - u(x')) \) can be simplified, if fluctuations in \( u \) are subdominant with respect to the leading term \( \zeta(x_\parallel - x'_\parallel) \). Let us suppose that this is the case; it was a posteriori verified in [182]. Then \( \delta(\zeta(x_\parallel - x'_\parallel) + u(x) - u(x')) = \frac{1}{D} \delta(x_\parallel - x'_\parallel) \) and the degree of freedom \( u(x) \) decouples, leading to

\[
\mathcal{H}''[h] = \frac{1}{2} \int_x \kappa(\partial^2 h)^2 + t_\perp (\partial_\perp h)^2 + \frac{b}{2\zeta} \int_x \int_{x'} \delta^{d-1}(h(x) - h(x')) \delta(x_\parallel - x'_\parallel).
\] (392)

This model can be studied using the methods of sections 3 and 4. The dimensions in units of \( x_\perp \sim 1/\mu \) are \( [x_\parallel]_{x_\perp} = \frac{1}{2} \) and \( \bar{\xi} := [b/\zeta]_{\mu} = 2D - \frac{3}{2} - \nu(d - 1) \) with \( \nu := [h]_{x_\perp} = \frac{5}{4} - \frac{D}{2} \). Note that since \( \nu(D = 2) > 0 \), direct calculations at \( D = 2 \) are possible in contrast to the isotropic case discussed in the rest of this review. An important simplification arises from the fact that the interaction in Eq. (392) does not depend on the distance in the parallel direction \( x_\parallel - x'_\parallel \), thus can not give a renormalization of \( \kappa_{\parallel} \). The argument is identical to what happens in the dynamics of self-avoiding membranes and is in more details discussed in section 11.3. The calculations at \( D = 2 \) to first order in \( \bar{\xi} \) were performed in [164], and later generalized to arbitrary \( D \) in [165, 166]. As discussed in section 7, the advantage of calculating at arbitrary \( D \) and then extrapolating to \( D = 2 \) is a better numerical precision and control of the error. The results of [165, 166] (\( \nu^*(D = 2, d = 3) \approx 0.62 \)) for \( \nu^* \) are found to be comparable with the Flory-result

\[
\nu^* \approx \nu^\text{Flory} = \frac{1 + D}{1 + d}.
\] (393)

obtained by remarking that the dimensional arguments that led to Eq. (15) apply to the self-avoidance in \( (d - 1) \) dimensions in the \( (D - 1) \) transversal directions; thus leading to the replacements of \( d \to d - 1 \) and \( D \to D - 1 \) in Eq. (15).

However, for larger \( \bar{\xi} \), i.e. when lowering \( d \) further, one can no longer expect that the anharmonic elastic terms in Eq. (389) can be neglected. In [182] it is argued that this happens at \( d < d_1 \approx 6 \). Since we know of no systematic treatment of both self-avoidance and anharmonicity, we refrain from further discussions. Let us only note, that in the case of the non-selfavoiding but anharmonic model, anomalous exponents have been calculated [179, 182], also for the case of unidirectionally polymerized, and otherwise fluid membranes [183].

11 Dynamics

11.1 Langevin-dynamics, effective field theory

In this section, we want to study the dynamics of polymerized tethered membranes, including polymers. To this aim, awe add to the membrane position \( r(x) \) a time argument \( t \), and study the evolution of \( r(x, t) \). The simplest diffusive dynamics which can be constructed, is given by the Langevin-equation

\[
\frac{d}{dt} \tilde{r}_0(x, t) = \lambda_0 \left[ -(2 - D_1) \frac{\delta \mathcal{H}}{\delta \tilde{r}_0(x, t)} + \zeta^i(x, t) \right].
\] (394)
\( \mathcal{H} \) is the static Hamiltonian as given in (61). In order to keep the derivation transparent, we give it in terms of bare quantities, and shall introduce renormalized ones later. The Gaussian noise \( \zeta(x, t) \) has correlations

\[
\left\langle \zeta^i(x, t) \zeta^j(x', t') \right\rangle = \frac{2(2 - D)S_D}{\lambda_0} \delta^D(x - x') \delta^{ij} \delta(t - t').
\]  

(395)

The factors of \((2 - D)\) and \(S_D\) in Eqs. (394) and (395) are introduced in the same spirit as those of the static Hamiltonian in Eq. (61) in order to obtain simple expressions for the response and correlation functions and to compare with the static case. Readers only interested in the general procedure can safely ignore them. Since in our arguments and presentation we closely follow [30], let us also mention that there these factors were hidden in the normalization of the integration measures, but are equivalent to the above. In order to define the time-derivative in Eq. (394), a discretization has to be introduced. We use prepoint (Itô) discretization [152]

\[
\frac{r_0^i(x, t + \tau) - r_0^i(x, t)}{\tau} = \lambda_0 \left[ -(2 - D) \frac{\delta \mathcal{H}}{\delta r_0^i(x, t)} + \zeta^i(x, t) \right].
\]  

(396)

In the terminology of [29], this is model A. In the polymer-community, this is called the Rouse-model [184, 185] and computer scientists will recognize the definition of a molecular dynamics algorithm. The construction ensures that the static correlation-functions obtained from the Boltzmann-weight \( e^{-\mathcal{H}} \) are correctly reproduced, see the discussion in section 12.3 and [3, 152].

This model has been studied using scaling arguments for polymers [186, 187] and membranes [15]. For polymers, a renormalization group analysis has been performed at 1- [188, 189, 190, 191, 192, 193] and 2-loop [194] order. The result to all orders was given in [30], and we follow the procedure outlined there.

The Langevin equation (394) can be transformed into an effective field theory [152, 195] with action

\[
\mathcal{J} [r_0, \tilde{r}_0, \zeta] = \frac{1}{2 - D} \int_{x,t} \left[ \tilde{r}_0(x, t) \left( \dot{r}_0(x, t) + \lambda_0 (2 - D) \frac{\delta \mathcal{H}}{\delta r_0(x, t)} - \zeta(x, t) \right) + \frac{1}{4\lambda_0} \zeta(x, t)^2 \right].
\]  

(397)

Expectation values of an observable \( \mathcal{O} \) are calculated by integrating over all fields \( r_0 \) and \( \tilde{r}_0 \) and the noise \( \zeta \)

\[
\langle \mathcal{O} \rangle_{b_0} = \int \mathcal{D} [r_0] \int \mathcal{D} [\tilde{r}_0] \int \mathcal{D} [\zeta] \mathcal{O} e^{-\mathcal{J}[r_0, \tilde{r}_0, \zeta]},
\]  

(398)

where the normalization is such that \( \langle 1 \rangle_{b_0} = 1 \). The derivation and interpretation of Eq. (397) is simple: The path-integral over \( \tilde{r}_0(x, t) \) enforces the Langevin-equation (394) to be satisfied, and the term proportional to \( \zeta^2 \) reproduces the noise (395). Since \( \mathcal{J} \) is quadratic in \( \zeta \), the corresponding integral can still be performed, leading to

\[
\mathcal{J} [r_0, \tilde{r}_0] = \frac{1}{2 - D} \int_{x,t} \left[ \tilde{r}_0(x, t) \left( \dot{r}_0(x, t) + \lambda_0 (2 - D) \frac{\delta \mathcal{H}}{\delta r_0(x, t)} \right) - \lambda_0 \tilde{r}_0(x, t)^2 \right].
\]  

(399)

Introducing now renormalized fields and couplings \( \tilde{r}_0 = \sqrt{Z} \tilde{r} \), \( \lambda_0 = \lambda Z \lambda \) and \( b \) and \( r \) as in Eq. (174) yields

\[
\mathcal{J} [r, \tilde{r}] = \frac{1}{2 - D} \int_{x,y,t} \left[ \sqrt{Z} \tilde{Z} \tilde{Z} \cdot \cdot \cdot + \lambda \sqrt{Z} Z \lambda \cdot \cdot \cdot - \tilde{Z} Z \lambda \cdot \cdot \cdot \cdot \right]
\]

\[
+ \lambda b \mu \sqrt{Z} Z \lambda Z b \int_{x,y,t} \cdot \cdot \cdot \cdot \cdot \cdot .
\]  

(400)
The symbols for the local operators are in the same spirit as in Eqs. (70) and (71) defined as
\[ \tilde{r}(x, t) \tilde{r}(x, t) = \tilde{r}(x, t)(-\Delta) r(x, t) \],
\[ \tilde{r}(x, t) \tilde{r}(x, t)^2 = \tilde{r}(x, t)^2 \],
(401)
where a wiggly line always indicates a response field. The interaction is:
\[ \tilde{r}(x, t) (ik) e^{ik[\tilde{r}(x,t) - r(y,t)]]} \].
(402)
These notations are collected in appendix A.2.

Note that in Eq. (400), the response-field \( \tilde{r} \) could also be integrated over and thus eliminated. This is sometimes done [3, 152]. However, there are two disadvantages of this procedure: First of all, this would generate a term quadratic in \( \delta H / \delta r(x,t) \), rendering the analysis of divergences rather tedious. Second, the field \( \tilde{r}(x,t) \) has an immediate physical meaning. To see this, add in Eq. (394) a force \( F(x,t) \). This yields an additional term proportional to \( \int \tilde{r}(x,t) F(x,t) \) to the dynamic action (400). The response of the field \( r(x,t) \) to a small applied force \( F(y,t') \) therefore is
\[ \langle r(x,t) \tilde{r}(y,t') \rangle_b \] (403)
It is called response-function.

Perturbation theory is now performed by expanding about the Gaussian theory. We use the free propagator (response-function) \( R \) and correlator \( C \) in position-space
\[ C(x,t) := \frac{1}{d} \left\langle \frac{1}{2} (r(x,t) - r(0,0))^2 \right\rangle_0 \]
\[ = \frac{|x|^{2-D}}{\Gamma \left( \frac{D-2}{2} \right)} \left[ \frac{2}{D-2} \left( \frac{4\lambda |t|}{x^2} \right)^{(2-D)/2} + \int_0^{\frac{x^2}{4\lambda|t|}} ds s^{(D-2)/2} e^{-s} \right] \]
\[ \equiv \frac{|x|^{2-D}}{\Gamma \left( \frac{D}{2} \right)} \left[ \left( \frac{4\lambda |t|}{x^2} \right)^{(2-D)/2} e^{-\frac{x^2}{4\lambda|t|}} + \int_0^{\frac{x^2}{4\lambda|t|}} ds s^{D/2} e^{-s} \right] \] (404)
\[ R(x,t) := \frac{1}{d} \left\langle r(x,t) \tilde{r}(0,0) \right\rangle_0 \]
\[ = \Theta(t) \left( 4\pi \lambda |t| \right)^{-D/2} e^{-x^2/4\lambda|t|} S_D(2-D) \],
(405)
where as usual \( S_D \) is the volume of the unit sphere in \( D \) dimensions, see Eq. (713).

The normalization of the integration measure \( f_x \) is the same as in the static case, compare appendix A.1. This insures that in the static limit \( (x^2 \gg \lambda |t|) \)
\[ C(x,t) \longrightarrow |x|^{2-D} \].
(406)
If \( x^2 \) is much smaller than \( \lambda |t| \), the correlator approaches the finite value
\[ C(x,t) = \frac{(4|t|\lambda)^\nu}{\Gamma(D/2)} + \mathcal{O}(x^2) \].
(407)
It is useful to note that the propagator is simply related to the time-derivative of the correlator by
\[ R(x,t) = \Theta(t) \frac{1}{\lambda} \dot{C}(x,t) \].
(408)
This is the perturbative version of the fluctuation dissipation theorem, further discussed in the context of non-conserved forces in section 12.3.
11.2 Locality of divergences

In analogy to the static case, we can now construct perturbation-theory. The perturbative expansion of an observable $O$ is in analogy with Eq. (72)

$$\langle O \rangle_b = \text{Norm} \sum_n \frac{(\lambda b \mu^\varepsilon Z_h Z_\lambda)^n}{n!} \int \langle O \rangle_0^n ,$$  \hspace{1cm} (409)

where the normalization Norm has to be chosen such that $\langle 1 \rangle_b = 1$ and the integral is taken over all arguments of the interaction. We claim that like in the static case, divergences only occur at short distances and short times. To prove this look at a typical expectation value

$$\langle O \rangle_0^n = \sum_\ell \int_{k_1} \ldots \int_{k_{2n}} f_\ell(x_l - x_m, t_l - t_m, k_l, k_m) e^{-\frac{1}{2} \sum_{i,j} Q_{ij} k_i k_j} ,$$  \hspace{1cm} (410)

where each contribution consists of a function $f_\ell$, which is a product of propagators, correlators and $k$’s and an exponential factor, with

$$Q_{ij} = -C(x_i - x_j, t_i - t_j) .$$  \hspace{1cm} (411)

$f_\ell$ is a regular function of the distances. As we have seen in section 3.2 in the static case, divergences at finite distances can only occur if $Q_{ij}$ is not a positive form. We shall show that $Q_{ij}$ is a positive form for all $k_i$ which satisfy the constraint

$$\sum_i k_i = 0 .$$  \hspace{1cm} (412)

This constraint, implicitly enforced by $f_\ell$, always holds, see Eq. (402). For equal times, positivity of $Q_{ij}$ is just the statement that the Coulomb energy of a globally neutral assembly of charges is positive, as derived in section 3.2. One simply identifies $C$ with the Coulomb-propagator and $k_i$ with the charges. In the dynamic case, write

$$Q_{ij} = (2 - D)S_D \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega}{2\pi} \frac{2\lambda}{\omega^2 + (\lambda p^2)^2} e^{ip(x_i - x_j) + i\omega(t_i - t_j)}$$  \hspace{1cm} (413)

The exponential in (410) now is

$$\sum_{i,j} k_i k_j Q_{ij} = (2 - D)S_D \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega}{2\pi} \frac{2\lambda}{\omega^2 + (\lambda p^2)^2} \sum_{i,j} k_i k_j e^{ip(x_i - x_j) + i\omega(t_i - t_j)}$$

$$= (2 - D)S_D \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega}{2\pi} \frac{2\lambda}{\omega^2 + (\lambda p^2)^2} \left| \sum_i k_i e^{ip x_i + i\omega t_i} \right|^2$$  \hspace{1cm} (414)

Due to equation (412), the integral is ultraviolet convergent and thus positive. It vanishes if and only if the charge-density, regarded as a function of space and time, vanishes. This is possible if and only if endpoints of the dipoles (which form the interaction) are at the same point in space and time. No divergence occurs at finite distances.

11.3 Renormalization

It is easy to see that like in the static case, there exists a multilocal operator product expansion (MOPE), which we have introduced in section 3.5. The general criterion for renormalizability, theorem 59, page 63, then ensures renormalizability.
We show now that the counter-terms which render the static theory (61) finite are also sufficient for the
dynamic case (399). As an illustration we first calculate the 1-loop counter-terms for the renormalization
of the field and of the coupling-constant.

The first singular configuration appears, when both ends of the interaction (402) are contracted towards
a single point. The leading term (MOPE-coefficient) of this expansion is (a more detailed demonstration
of the derivation of the MOPE-coefficients in dynamic theories is given in sections 12.4 ff. in the case of
disorder-dynamics)

\[
\varnothing = 2 \int_k -[\tilde{r}(x, t)(ik)] e^{-k^2|x-y|^2-D} [(ik)(r(x, t) - r(y, t))] + \ldots .
\]  

We now expand \( r(y, t) \) as

\[
r(y, t) = r(x, t) + (y - x)\nabla r(x, t) + \frac{1}{2} \left[(y - x)\nabla\right]^2 r(x, t) + \mathcal{O}(|x - y|^3) .
\]  

The leading term in equation (415) is

\[
\int_k \tilde{r}(x, t)(-\Delta r(x, t)) \frac{(x - y)^2 - k^2}{D} e^{-k^2|x-y|^2}\nu = \tilde{r}(x, t)(-\Delta)r(x, t) \left(\frac{-1}{2D}\right) |x - y|^{D-\nu d} .
\]  

Denoting with \( \varnothing \rightarrow \) the MOPE coefficient of \( \varnothing \) proportional to

\( \rightarrow = \tilde{r}(x, t)(-\Delta)r(x, t) \), this can be written in the form

\[
\varnothing \rightarrow = \varnothing ,
\]  

where

\[
\varnothing = - \frac{1}{2D} |x - y|^{D-\nu d}
\]  

is the static MOPE-coefficient, see Eq. (156). This implies that the 1-loop counter-term for the wave-
function renormalization is the same as in the static case.

Let us now consider the counter-term for the coupling-constant renormalization. Using the techniques
explained in section 3.6 we obtain for the contraction of two interactions towards a single one

\[
\varnothing \rightarrow = \frac{d}{4} [R(x, t) + R(y, t)] \times [C(x, t) + C(y, t)]^{-d/2-1} .
\]  

\( x \) and \( y \) are the distances between the contracted endpoints of the dipoles, and \( t \) is their time-difference.
The trick is now to write this expression with the help of Eq. (408) as

\[
-\frac{1}{2\lambda} \Theta(t) \frac{d}{dt} [C(x, t) + C(y, t)]^{-d/2} .
\]  

To evaluate the diagram, we have to integrate over all times. If we use no cut-off in the time-direction, the
time integral will simply give the value of the function at its lower bound:

\[
\int_0^\infty dt \varnothing = \frac{1}{2\lambda} \varnothing .
\]
The r.h.s. is the MOPE-coefficient of the static theory, see Eq. (139). We easily convince ourselves that this relation implies the same counter-term as in the static case, if we take care of the additional combinatorial factor of 2 for the time-ordering of the interactions.

One knows from general arguments that the divergences associated to short distances in space are removed by the static counter-terms [3]. This implies that new divergences can only appear for short times. Using the general theorem 59 (page 63) about renormalizability, we have to subtract all divergences proportional to marginal and relevant operators. Thus, there may be new divergences proportional to

\[ \tilde{r}(x, t) \tilde{r}(x, t) = \tilde{r}^2 \quad (423) \]

which have to be subtracted in the MOPE. We now consider a general contraction of \( n \) dipoles towards \( \rightarrow \):

\[ \rightarrow \rightarrow \rightarrow \rightarrow = \tilde{r}(x, t) \tilde{r}(x, t) \quad (424) \]

In order to obtain the operator \( \rightarrow \rightarrow \rightarrow \rightarrow \), one has to contract all fields \( r \) and \( \tilde{r} \) except of the field \( \tilde{r}(z, t) \) with the largest time-argument (all other contractions give 0). One also has to leave uncontracted one arbitrarily chosen field \( r \). Due to the structure of the interaction (402), the field \( r \) always appears in the form \( r(x, t - \tau) - r(y, t - \tau) \). So the contraction yields

\[ \tilde{r}(z, t) [r(x, t - \tau) - r(y, t - \tau)] M(\text{distances}) \quad (425) \]

where \( M \) denotes the MOPE-coefficient which depends on the distances in space and time. Now, \( r(x, t - \tau) - r(y, t - \tau) \) has to be expanded about \( (z, t) \). The leading term has at least one spatial gradient. No term of the form (423) can be constructed. Therefore, there is no singular contribution of this type in any order in perturbation theory and no renormalization of \( \rightarrow \rightarrow \rightarrow \rightarrow \) is needed.

The last marginal operator at \( \varepsilon = 0 \) is

\[ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow = \tilde{r}(x, t)^2 \quad (426) \]

Its renormalization can either be obtained from the fluctuation-dissipation theorem (further discussed in section 12.3)

\[ \Theta(t) \frac{\partial}{\partial t} \left\langle \frac{1}{2} (r(x, t) - r(0, 0))^2 \right\rangle_b = \lambda \sqrt{Z} \langle r(x, t) \tilde{r}(0, 0) \rangle_b , \quad (427) \]

which relates the full correlation- and response-functions as in [30]. Let us give a direct derivation here: First note that the interaction can be written as

\[ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow = \int_k \left[ \tilde{r}(x, t) - \tilde{r}(y, t) \right] (ik) e^{ik[r(x, t) - r(y, t)]} . \quad (428) \]

Study now the contraction of \( n \) interactions \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \) towards \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \):

\[ \rightarrow \rightarrow \rightarrow \rightarrow = \tilde{r}(x, t) \tilde{r}(x, t) \quad (429) \]

Using the same arguments as those leading to Eq. (425), the contraction gives

\[ \left[ \tilde{r}(x, t) - \tilde{r}(y, t) \right] \left[ \tilde{r}(x', t') - \tilde{r}(y', t') \right] M(\text{distances}) . \quad (430) \]

The leading term is proportional to \( (\nabla \tilde{r})^2 \), and no term proportional to \( \rightarrow \rightarrow \rightarrow \rightarrow = \tilde{r}^2 \) is generated.

Let us note that this is sufficient to prove renormalizability of polymer dynamics without using the general theorem of renormalizability (theorem 59 on page 63): Two classes of diagrams exist: (i) “static”
diagrams (i.e. those correcting $\leftrightarrow$ and $\Rightarrow\Leftarrow$), which using the fluctuation dissipation theorem are reduced to diagrams of the static theory, and for which in the case of polymers one can use the equivalence to $\phi^4$-theory; (ii) “dynamic” diagrams, proportional to $\rightarrow\leftarrow$ and $\Rightarrow\Leftarrow\Rightarrow\Leftarrow$, which vanish identically.

In Eq. (399), we have introduced renormalization group $Z$-factors for the fields $r$, $\tilde{r}$ and for $\lambda$ and $b$.

The absence of counter-terms proportional to $\rightarrow\leftarrow$ and $\Rightarrow\Leftarrow\Rightarrow\Leftarrow\Rightarrow\Leftarrow$ in Eq. (400) implies that $Z\tilde{Z} = 1$ and $\tilde{Z}Z\lambda = 1$, i.e.

$$\tilde{Z} = \frac{1}{Z} \quad \text{and} \quad Z\lambda = Z.$$  

Eq. (400) takes the simple form

$$J[r, \tilde{r}] = \frac{1}{2-D} \int \int \rho \rho + \lambda Z \Delta \rho - \lambda \Delta \Delta \rho + \lambda b \mu Z \int \int \rho.$$  

Solving the renormalization group equations for $\lambda$ and $\tilde{r}$, one obtains the new dynamic exponent $z^*$ [30].

Let us give a more intuitive derivation here, using the technique of exact exponent identities developed in section 3.9: The absence of a counter-term proportional to $\int_0^t \tilde{r}(x,t) \dot{r}(x,t)$ gives the identity for the full dimensions

$$[\tilde{r}]_t + [r]_t + D [x] = 0.$$  

Analogously, the absence of a counter-term proportional to $\int_0^t \tilde{r}(x,t)^2$ yields

$$2 [\tilde{r}]_t + D [x] + [\lambda t]_t = 0.$$  

Eliminating $[\tilde{r}]_t$ from these equations, gives

$$[\lambda t]_t = 2 [r]_t + D [x].$$

The auto-correlation function therefore scales as

$$\left\langle \frac{1}{2} (r(x,t) - r(x,t'))^2 \right\rangle_b \sim |t - t'|^{2/z^*},$$  

with

$$z^* = 2 + \frac{D}{\nu^*}. $$

For polymers ($D=1$), this relation was given using scaling arguments in [186], for membranes ($D = 2$) in [15]. This result was followed by perturbative calculations for polymers in 1-loop [188, 189, 190, 191, 192, 193] and 2-loop [194] order. The result to all orders is due to [30]. Let us also mention that non-selfavoiding membranes at the crumpling transition or in the flat phase have been treated in [196, 197].

It is interesting to note that for polymers and membranes Eq. (437) can be written in the form

$$z^* = 2 + d_f,$$  

where $d_f$ is the fractal dimension of the membrane or the polymer.

### 11.4 Inclusion of hydrodynamic interaction (Zimm Model)

While the Rouse-model defined in Eqs. (394) and (395) is particularly simple, it cannot be realized in experiments. Experiments on polymers and membranes are always carried out in solution. The question
Figure 25: The different phases for a $D$-dimensional self-avoiding membrane embedded in $d$ dimensions including hydrodynamic interactions. The region with $\delta = 2 - \nu d < 0$ and $\varepsilon < 0$ is the Gaussian phase. Self-avoidance and hydrodynamic interaction are irrelevant. Hydrodynamics is naively relevant for $\delta > 0$, but renormalization reduces the range of relevance to the grey domain. It becomes irrelevant if $d > d_f + 2$, where $d_f = D/\nu^*$ is the fractal dimension of the membrane (Rouse dynamics). We have drawn two different estimates for the cross-over line from the hydrodynamic to the Rouse domain (see main text). Self-avoidance is always relevant when $\varepsilon > 0$.

is, how the dynamic exponent $z^*$, defined in Eq. (436), will change in the presence of additional hydrodynamic degrees of freedom. $z^*$ should experimentally be observable via dynamic light- or neutron-scattering methods. To our knowledge, no such experiment has been performed.

Hydrodynamic interactions for polymers were first introduced by Zimm [198]. He wrote down the following Langevin equation, which will also be used for membranes:

$$\frac{d}{dt} r_0(x, t) = D \cdot \left( - (2 - D) \frac{\delta H}{\delta r_0} + \zeta \right) (x, t).$$

(439)

Here, $\cdot$ denotes the scalar product of the matrix operator $D$ and the vector $\delta H/\delta r_0$, which is defined by

$$f \cdot g := \int_x f^i(x) g^i(x).$$

(440)

The hydrodynamic interaction is

$$D^{ij}(x, y, r, r') = \lambda_0 \delta^{ij} \delta^D (x - y) + \lambda_0 \eta_0 \int_k \left( \frac{\delta^{ij} k^2}{k^2} - \frac{k^i k^j}{k^4} \right) e^{ik[r-r']}.$$  

(441)

We will not repeat the derivation [198] of equation (441) here. Let us however note that one supposes that the hydrodynamic degrees of freedom are fast enough, so that their dynamics can be neglected and that
screening effects are irrelevant. This might be wrong for membranes and in this case our results would only apply to membranes with large holes. (For a discussion of screening-effects for fluid membranes see e.g. [48].) For \( \eta = 0 \), (439) reduces to purely diffusive motion (Rouse model).

The noise correlation is

\[
\left\langle \zeta^i(x, t) (D \cdot \zeta)^j(y, t') \right\rangle = 2(2 - D) S_D \delta^{ij} \delta(t - t') \delta^D(x - y) .
\] (442)

This ensures that the static behavior is correctly reproduced, see the discussion in section 12.3 and [3, 152]. In analogy to Eq. (399), we obtain the dynamic functional in Itô (prepoint) discretization

\[
\mathcal{J}[\tilde{r}_0, \tilde{r}_0] = \frac{1}{2 - D} \int_t \left[ \tilde{r}_0 \frac{d}{dt} \tilde{r}_0 - \tilde{r}_0 \cdot D \cdot \tilde{r}_0 \right] + \int_t \tilde{r}_0 \cdot D \cdot \frac{\delta H}{\delta r_0} .
\] (443)

This model has to be renormalized. Again divergences only occur at small distances. They can be analyzed via a multilocal operator product expansion (MOPE). Renormalizability is ensured if counter-terms for all possible marginal and relevant operators are included into the action. It has been shown in [31] that the model is renormalized by introducing one additional renormalization for \( \eta \), leading to a dynamic renormalization different from the Rouse model. The two regularization parameters \( \varepsilon \) and \( \delta \) are the canonical dimensions of the coupling-constants

\[
\varepsilon := [b_0]_\mu = 2D - \nu d \\
\delta := [\eta_0]_\mu = 2 - \nu d .
\]

As in section 9, this leads to a double-\( \varepsilon \) expansion, which has to be performed about the point \(( \delta = 0, \varepsilon = 0 )\), i.e. \(( D = 1, d = 4 )\).

The dynamic exponent \( z^* \) is derived along the same lines as for the Rouse-model. In contrast to the latter, now a counter-term proportional to \( \tilde{r}^2 \) is needed, such that Eq. (434) no longer holds. At 1-loop order, this is due to the contraction of the hydrodynamic interaction. Denoting

\[
\tilde{r}^i(x, t) \tilde{r}^j(y, t) \int_k \left( \frac{\delta^{ij}}{k^2} - \frac{k^i k^j}{k^4} \right) e^{i[k \cdot [r(x, t) - r(y, t)]]} ,
\] (444)

the diagram

\[
\left\langle \tilde{r}^i(x, t) \tilde{r}^j(y, t) \tilde{r}^k(x, t) \right\rangle = \frac{d - 1}{2d(d - 2)}
\] (445)

is thus positive and ensures the stability of a fixed point \( \eta^* > 0 \) at least for small \( \delta \).

Since the hydrodynamic interaction is long-range, as discussed in section 3.9, it is not renormalized. Accordingly, the exponent identity in Eq. (434) is replaced by

\[
2 [\tilde{r}]_t + (2 - d) [r]_t + 2D [x] + [\lambda t]_t = 0 .
\] (446)

Together with Eq. (433) this yields for the dynamical exponent \( z^* \)

\[
z^* = d .
\] (447)

The stability condition for the fixed point \( \eta = 0 \) is therefore

\[
\delta < (d - 2)(\nu^* - \nu) .
\] (448)
At 1-loop order, the separating line is

$$\delta = (d-2)\frac{c}{8}. \quad (449)$$

Numerical evaluation yields the thin line separating the regions with \(z^* = d\) and \(z^* = 2 + D/\nu^*\) in figure 25. There is however a priori no reason to trust this estimate for membranes, i.e. \(c = 4\). We know however that in any dimension the Flory-estimate \(\nu_{\text{Flory}} = (2 + D)/(2 + d)\) is quite a good approximation for \(\nu^*\) in the fractal phase, for polymers as well as for membranes [26, 25]. Inserting this relation we obtain for the separatrix

$$d = 2(D+1). \quad (450)$$

In figure 25, this is the fat line between the regions with \(z^* = d\) and \(z^* = 2 + D/\nu^*\). Let us stress that we only use the Flory-approximation to estimate \(\nu^*\), but not any of the systematically wrong assumptions which have to be used to derive it.

Another possibility to get (450) is to require that the value of \(z^*\) is continuous on the phase separation line. The equivalence of the results obtained by the two methods is a consequence of the general structure of the renormalization group.

We can also give a rigorous bound for the phase separation line. As \(\nu^* \leq 1\), hydrodynamics is always relevant for

$$d < D + 2. \quad (451)$$

12 Disorder and non-conserved forces

In this section, we want to study the dynamics of polymers and \(D\)-dimensional elastic manifolds \((0 < D < 2)\) diffusing and convected in a static random flow. The velocity pattern of the flow \(\vec{v}(r)\) is constant in time and leads to convection of the polymer in addition to diffusion. We are interested in the general case of a non-potential flow: the extreme example is the hydrodynamic divergenceless flow, with \(\nabla \cdot \vec{v}(r) = 0\), but
mixtures of potential and divergenceless flows are also considered. See figure 27 for a visualization. This is a generalization of the Rouse- and Zimm-dynamics discussed in the last section.

This study is interesting for several reasons: Technically, disorder can be treated with the help of the same tools as self-avoidance: The non-local term is nothing but the disorder-correlator, decorated with two response-fields. Short-range, i.e. $\delta$-correlated disorder thus leads to $\delta$-interactions between different points on the membrane, very much in the spirit of the $\delta$-interaction enforcing self-avoidance. Physically, static disorder is relevant in most of the accessible cases. Potential disorder normally leads to trapping, ultra-slow dynamics, anomalously small response to external perturbations and pinning which cannot be studied using RG-techniques [199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211]. Non-potential disorder has much less been studied. Whereas physical realizations of frozen non-potential disorder are rare, similar problems appear for driven dynamics in the presence of quenched disorder [212, 213, 214, 215] or domain growth in the presence of shear [216]. The problem itself is intriguing: Does the non-potential part of disorder still lead to glassy dynamics? We will show below that this is not the case, but that physical observables are described by a new perturbative fixed point, which is universal with a large domain of attraction. Moreover, the fluctuation-dissipation theorem is broken, which can be interpreted as a rise in temperature under renormalization. The origin of this phenomenon is easy to understand: Non-potential disorder can only be kept up by constantly pumping energy into the system, thus raising temperature. This leads to a dynamic exponent $\zeta$ not substantially larger than 2. This prevents possible computer simulations from the usual problems of ultra-slow dynamics.

This section heavily relies on [33], for which the reader is referred to for further details and discussion. Our main aim here is to explain the underlying principles and to show how the techniques developed in this review serve to analyze disorder situations. This is also the reason, why we will restrain our attention to one of the situations studied in [32, 33], namely an isotropic membrane in short range correlated disorder, visualized in figure 26.

### 12.1 The model

We again consider a $D$-dimensional manifold imbedded into $d$ dimensions, including particles ($D = 0$), polymers ($D = 1$) and membranes ($D = 2$). In generalization of Eq. (394), we study the Langevin...
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\[ \dot{r}_0^i(x, t) = \lambda_0 \left( \Delta r_0^i(x, t) + \zeta^i(x, t) + (2 - D) F^i[r_0(x, t)] \right), \quad (452) \]

where the term proportional to \( \Delta r_0^i(x, t) \) is the derivative of the internal (entropic) elasticity. The Gaussian thermal noise \( \zeta^i(x, t) \) is the same as in Eq. (395).

\( F^i[r] \) is a Gaussian quenched random force field with correlations:

\[ F^i[r] F^j[r'] = \Delta_{ij}^0(r - r'). \quad (453) \]

Disorder averages (over \( F \)) are denoted by overbars, whereas angular brackets denote thermal averages (over the noise \( \zeta \)).

We consider a statistical rotationally invariant force field with both a potential (L) ("longitudinal") and a divergence-free (T) ("transversal") part whose correlations depend only on the distance \( r - r' \). Both parts contribute separately to the correlator:

\[ \Delta_{ij}^0(r) = -\frac{\partial}{\partial r^i} \frac{\partial}{\partial r^j} \Delta_{ij}^L(r) - \left( \delta_{ij} \sum_l \frac{\partial}{\partial r^l} \frac{\partial}{\partial r^l} - \frac{\partial}{\partial r^i} \frac{\partial}{\partial r^j} \right) \Delta_{ij}^T(r). \quad (454) \]

In Fourier space

\[ \Delta_{ij}^0(k) = \int \Delta_{ij}^0(r) e^{ikr}, \quad (455) \]

where the normalization of the \( k \)-integral is as usual (see appendix A.1). Several cases for the correlation of the random force are of interest. We will mainly focus on forces with short range correlations, for which the force correlator scales like a \( \delta \)-distribution \( \Delta_{ij}^0(r - r') \sim \delta^d(r - r') \), with however a non-trivial index structure. In Fourier space, the correlator then reads at small \( k \)

\[ \Delta_{ij}^0(k) = g^L_{ij} P_{ij}^L(k) + g^T_{ij} P_{ij}^T(k), \quad (456) \]

where \( P_{ij}^L(k) = k^i k^j / k^2 \) is the longitudinal and \( P_{ij}^T(k) = \delta^j \) - \( P_{ij}^T(k) \) the transversal projector, already encountered in Eq. (441); see also appendix A.3. As we shall show below, this is the generic situation, relevant for all short-ranged correlations, even those, for which one starts with a force correlator which is formally shorter ranged than the \( \delta \)-distribution (e.g. decaying faster than \( 1 / r^d \)).

Long-range correlations can also be studied [33], but for simplicity of presentation are omitted here.

Note also that if the forces can be derived from a potential \( V(r) \) in the Hamiltonian \( \mathcal{H}[r] = \frac{1}{2-D} \int_x + V[r(x)] \), and denoting by \( \tilde{V}(k) \) the Fourier transform of \( V(r) \), then

\[ \Delta_{ij}^0(k) = k^i k^j \tilde{V}(k), \quad (457) \]

or

\[ \Delta_{ij}^0(r) = V(r). \quad (458) \]

(This suggested the terminology “potential disorder”.)

Before developing a field theoretic description for the above model, let us state the (systematically wrong, but numerically often satisfying) predictions made by a Flory-type ansatz; supposing that Eq. (452) imposes the scaling for the radius of gyration

\[ \frac{R_G}{t} \sim \frac{R_G}{x^2} \sim \frac{1}{\sqrt{R_G^d}}. \quad (459) \]

leads to

\[ R_G \sim x^{\frac{4}{d+2}} \sim t^{\frac{2}{d+2}}. \quad (460) \]

However these estimates are certainly much too crude to catch the physics of the problem, especially the difference between potential and transversal disorder.
12.2 Field theoretic treatment of the renormalization group equations

We start from the equation of motion (452) and convert it into an effective field theory analogous to what we did in Eq. (399). The effective dynamical action in prepoint (Itô) discretization reads

\[
J [\bar{r}_0, \bar{r}_0] = \frac{1}{2 - D} \int \frac{d^d x, t}{\bar{r}_0^i(x, t)} \left[ \bar{r}_0^i(x, t) + \lambda_0 \left( -\Delta \bar{r}_0^i(x, t) + \zeta^i(x, t) + (2 - D) F^i[\bar{r}_0^i(x, t)] \right) \right].
\]  

(461)

The noise \( \zeta(x, t) \) as well as the disorder force \( F[r] \) are Gaussian, see Eqs. (395) and (453), and can thus be integrated over. This gives the dynamic functional

\[
J [\bar{r}_0, \bar{r}_0] = \frac{1}{2 - D} \int \frac{d^d x, t}{\bar{r}_0(x, t)} \left( \bar{r}_0(x, t) - \lambda_0 \Delta \bar{r}_0(x, t) \right) - \lambda_0 \bar{r}_0(x, t)^2 - \frac{1}{2} \lambda_0^2 \int_{x, y, t, t'} \bar{r}_0^i(x, t) \Delta_{\bar{r}_0}^{ij} \left[ \bar{r}_0^j(x, t) - r_0(y, t') \right] \bar{r}_0^i(y, t').
\]  

(462)

Here and in the following, contraction over indices is implied, where ever confusion is impossible. Note also that the factor of \( 1/2 \) disappears, if we use the time-ordered disorder interaction.

The disorder correlator is

\[
\Delta_{\bar{r}_0}^{ij} [r_0(x, t) - r_0(y, t)] = \int_k e^{ik[r_0(x, t) - r_0(y, t)]} \left( g_0^L P_L^{ij} (k) + g_0^T P_T^{ij} (k) \right).
\]  

(463)

In order to simplify notations, we introduce the following graphical symbols

\[
\begin{align*}
\boxdot & = \bar{r}(x, t)(-\Delta x) \bar{r}(x, t) \\
\boxdot \boxplus & = \bar{r}(x, t) \dot{\bar{r}}(x, t) \\
\boxplus \boxtimes & = \bar{r}(x, t)^2 \\
\boxtimes_{i \rightarrow j} & = \int_k \bar{r}(x, t) e^{ik[r(x, t) - r(y, t')]} \bar{r}(y, t') \\
\boxtimes_{\text{L}} & = \int_k P_L^{ij} (k) \bar{r}(x, t) e^{ik[r(x, t) - r(y, t')]} \bar{r}(y, t') \\
\boxtimes_{\text{T}} & = \int_k P_T^{ij} (k) \bar{r}(x, t) e^{ik[r(x, t) - r(y, t')]} \bar{r}(y, t').
\end{align*}
\]  

(464)

The action takes the symbolic form

\[
J [\bar{r}_0, \bar{r}_0] = \frac{1}{2 - D} \int_{x, t} \left( \boxdot_{\text{L}} + \lambda_0 \boxdot_{\text{T}} - \lambda_0 \boxtimes_{\text{L}} \right)
\]  

\[
- \frac{\lambda_0^2}{2} \int_{x, t} \int_{y, t'} \left( \boxtimes_{\text{L}} g_0^L + \boxtimes_{\text{T}} g_0^T \right),
\]  

(465)

where the index \( \text{o} \) denotes bare quantities. The dimension of the coupling is

\[
\varepsilon := \left[ g_0^L \right]_{\mu} = \left[ g_0^T \right]_{\mu} = 2 + D - \frac{2 - D}{2} d.
\]  

(466)
Disorder is relevant for $\varepsilon > 0$, i.e.
\[
d < d_c(D) = \frac{4 + 2D}{2 - D}.
\]
As we have done for self-avoiding membranes, we want to renormalize the model within an $\varepsilon$-expansion. To this aim, renormalized fields and $\lambda$ are introduced by setting
\[
\begin{align*}
r_0 &= \sqrt{Z} r \\
\tilde{r}_0 &= \sqrt{Z} \tilde{r} \\
\lambda_0 &= Z_\lambda \lambda.
\end{align*}
\]
(468)
It is more complicated to introduce renormalized couplings $g^L$ and $g^T$. Set
\[
\begin{align*}
i_{\gamma} \cdots i_{\gamma} \left( P_{ij}^L g^L + P_{ij}^T g^T \right) \lambda^2 \mu^\varepsilon &= i_{\gamma} \cdots i_{\gamma} \left( P_{ij}^L g_0^L + P_{ij}^T g_0^T \right) \lambda_0^2.
\end{align*}
\]
(469)
This equation is to be understood such that quantities on the l.h.s. are renormalized, and those on the r.h.s. are bare.

The non-interacting ($g^L = g^T = 0$) theory is the same as in Eq. (399), such that also the free response $R(x, t)$ and correlation functions $C(x, t)$ are the same as in Eqs. (405) and (404).

### 12.3 Fluctuation-dissipation theorem and Fokker-Planck equation

Before embarking onto the analysis of divergences, we have to clarify an important point, related to the fluctuation-dissipation-theorem (FDT). The latter states that as long as all forces in Eq. (452) can be derived from a potential, then the full correlation and response-function are related by [152]
\[
\Theta(t) \frac{\partial}{\partial t} \left\{ \frac{1}{2} [r(x, t) - r(0, 0)]^2 \right\} = \lambda \sqrt{Z \tilde{Z} \lambda} \left\langle r(x, t) \tilde{r}(0, 0) \right\rangle.
\]
(470)
This relation is violated in the presence of non-potential forces, for our model in the case of $g^T \neq 0$.

Also note that in the case of purely potential disorder, the equation of motion (452) can be recast in the form (394). The latter implies that trajectories of the Langevin-equation sweep out configuration space (as long as the dynamics is ergodic) and that the probability to find the membrane in a given configuration $r(x)$ is given by $e^{-\mathcal{H}[r(x)]}$. Equal-time expectation values (the “statics”) are simply obtained by studying the partition-function with the weight $e^{-\mathcal{H}[r(x)]}$, as was done in sections 3 to 9. This is proven by going from the Langevin-equation (452) to the (functional) Fokker-Planck equation (suppressing all indices $0$ for bare quantities)
\[
\begin{align*}
\frac{1}{\lambda} \frac{d}{dt} \mathcal{P}[r(x), t] &= -\sum_{i=1}^d \frac{\delta}{\delta r^i(x)} \left\{ (\Delta r^i(x) + (2 - D)F^i[r(x)]) \mathcal{P}[r(x), t] \right\} \\
&\quad + (2 - D) S_D \sum_{i=1}^d \left( \frac{\delta}{\delta r^i(x)} \right)^2 \mathcal{P}[r(x), t].
\end{align*}
\]
(471)
Supposing that $F^i[r(x)]$ can be derived from a potential, $F^i[r(x)] = -\frac{\delta}{\delta r^i(x)} V[r(x)]$, then Eq. (471) can be rewritten as
\[
\begin{align*}
\frac{1}{\lambda} \frac{d}{dt} \mathcal{P}[r(x), t] &= (2 - D) S_D \sum_{i=1}^d \frac{\delta}{\delta r^i(x)} \left\{ \mathcal{P}[r(x), t] \frac{\delta}{\delta r^i(x)} \mathcal{H}[r(x)] + \frac{\delta}{\delta r^i(x)} \mathcal{P}[r(x), t] \right\},
\end{align*}
\]
(472)
where $\mathcal{H}[r(x)] = \int x \frac{1}{2}\partial^2 + V[r(x)]$. Assuming that equilibrium is reached ($\frac{d}{dt} \mathcal{P}[r(x), t] = 0$) the solution of Eq. (472) reads

$$\mathcal{P}[r(x), t] \sim e^{-\mathcal{H}[r(x)]},$$

what should be demonstrated. The most important consequence of the above demonstration is that in the case of purely potential disorder, no non-potential (transversal) disorder can be generated.

### 12.4 Divergences associated with local operators

We now analyze the model, using the techniques of the multilocal operator product expansion (MOPE) as explained in sections 3.5 and 3.6.

In order to simplify notations, we shall in the following suppress the factor of $\lambda$, i.e. set

$$\lambda t \rightarrow t.$$  \hfill (474)

This is not problematic, as $\lambda$ always appears with time. At the end of the calculations one has to replace $t$ by $\lambda t$ which is necessary in order to get the renormalization factors correct.

The first class of divergences stems from configurations where the two end-points of the interaction are approached. The interaction is:

$$\begin{align*}
\mathcal{L} \rightarrow t + T \mathcal{T} g^T
\end{align*}$$

In order to extract the divergences for small $x - y$ and $t - t'$, the first possibility is not to contract any response-field. We then start by normal-ordering the r.h.s. of (475). Within dimensional regularization, $:e^{ikr(x,t)}: = e^{ikr(x,t)}$ and we can use the identity (analogous to Eq. (129))

$$
:e^{ikr(x,t)} : e^{-ikr(y,t')} = :e^{ikr(x,t)} e^{-ikr(y,t')} : e^{-k^2 C(x-y,t-t')}.
$$

Expanding the normal-ordered vertex-operators on the r.h.s. for small $x - y$ and $t - t'$, the leading contribution is

$$1 e^{-k^2 C(x-y,t-t')} ,$$

yielding the first term in the short-distance expansion of (475) (for the normalization of the $k$-integral cf. (716) ff.):

$$
\int_k e^{-k^2 C(x-y,t-t')} :e^{ikr(x,t)} e^{-ikr(y,t')} : (P_L^{ij}(k)g^L + P_T^{ij}(k)g^T)
\begin{align*}
&= \hat{r} \left( \frac{x + y}{2}, \frac{t + t'}{2} \right)^2 \left( g^T \left( 1 - \frac{1}{d} \right) + g^L \frac{1}{d} \right) C(x - y, t - t')^{-d/2} + \text{subleading terms}
\end{align*}
$$

The second contribution to the normal-ordered product of (475) is obtained upon contracting one response field. Due to causality, this must be the field with the smaller time-argument. For simplicity, let us take $t > 0$ and put $y = t' = 0$. By the same procedure as above, we obtain the contribution:

$$
\int_k :e^{ikr(x,t)} e^{-ikr(0,0)} : R(x,t)(ik)^j e^{-k^2 C(x,t)} (P_L^{ij}(k)g^L + P_T^{ij}(k)g^T) .
$$

(479)
The next step is to expand \( :e^{ik[r(x,t)−r(0,0)]} : \) about \((x, t)\). (It is important to expand about \((x, t)\) as otherwise \(\tilde{r}(x, t)\) had to be expanded, too.) This expansion is
\[
:e^{ik[r(x,t)−r(0,0)]} : = 1 + (ik)^l \left( t \partial_t r(x,t) + (x \nabla) r(x,t) − \frac{1}{2} (x \nabla)^2 r(x,t) \right) + \text{subleading terms} .
\]

Upon inserting (480) into (479) and integration over \(k\), only terms even in \(k\) survive. We can also neglect the term linear in \(x\), which is odd under space reflection. The remaining terms are
\[
\int_k :\tilde{r}^i(x,t) \left( t \partial_t r(x,t) − \frac{1}{2} (x \nabla)^2 r(x,t) \right) : (ik)^j (ik)^l R(x,t)e^{-k^2C(x,t)} \times
\]
\[
\times \left( P^{ij}_T(k) g^L + P^{ij}_L(k) g^T \right)
\]
\[
= -\frac{1}{2} R(x,t)C(x,t)^{-d/2-1} \left( t \nabla + \frac{x^2}{2D} \right) g^L .
\]

For the contribution proportional to \(\rightarrow\), we have retained from the tensor operator \(\tilde{r}(x,t)(x \nabla)^2 r(x,t)\) only the diagonal contribution \(\frac{1}{D} \tilde{r}(x,t)x^2(\Delta) r(x,t)\), which is sufficient at 1-loop order. For the subtleties associated with the insertion of this operator at the 2-loop level cf. section 6. Using the perturbative FDT, (408), this can still be simplified to
\[
\frac{\partial}{\partial t} C(x,t)^{-d/2 \left( t \nabla + \frac{x^2}{2dD} \right) g^L .
\]

Eqs. (478) and (482) contain all possible divergent terms in the short-distance expansion of (475) and all terms which have to be taken into account in 1-loop order. Notably, due to causality, no term independent of \(\tilde{r}\) appears.

### 12.5 Renormalization of disorder (divergences associated with bilocal operators)

In analogy to Eq. (139), there are also UV-divergent configurations associated to bilocal operators, which renormalize the disorder, and which are depicted on figure 28. Up to permutations of the two interaction vertices, there are two possibilities to order their end-points in time, namely \(D_1\) and \(D_2\) on figure 28. We first calculate \(D_1\), starting from
\[
\tilde{r}^i(y,t) \int_k \left( g^T P^{ij}_T(k) + g^L P^{ij}_L(k) \right) e^{ikr(y,t)−r(x,0)} \tilde{r}^j(x,0) \times
\]
\[
\times \tilde{r}^l(y′,t−\sigma) \int_p \left( g^T P^{lm}_T(p) + g^L P^{lm}_L(p) \right) e^{ipr(y′,t−\sigma)−r(x′,−\tau)} \tilde{r}^m(x′,−\tau) .
\]

For small \(x−x′, y−y′, \tau\) and \(\sigma\), with \(\tau, \sigma > 0\), there is one contribution for the renormalization of the interaction. First, due to causality, \(\tilde{r}^l(y′,t−\sigma)\) and \(\tilde{r}^m(x′,−\tau)\) have to be contracted with a correlator field in order to obtain two response fields at the end. Then the short-distance expansion for nearby vertex-operators reads:
\[
e^{ikr(y,t)} e^{ipr(y′,t−\sigma)} \approx e^{ikr(y,t)} e^{ipr(y′,t−\sigma)} = e^{ikr(y,t)} e^{ipr(y′,t−\sigma)} e^{kpc(y−y′,\sigma)}
\]
\[
\approx e^{i(k+p)r(y,t)} e^{kpc(y−y′,\sigma)} e^{i(k+p)r(y′,t)} e^{kpc(y−y′,\sigma)} ,
\]

---

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where the first and last equality are due to analytic continuation (see section 3.6). Analogously, we find for the other pair of points
\[ e^{-ikr(x,0)} e^{-ipr(x', -\tau)} \approx e^{-i(k+p)r(x,0)} e^{kpC(x-x', \tau)}. \] (485)

This yields up to subleading terms:
\[ \int \int \bar{r}^i(y, t) \bar{r}^j(x, 0) e^{i(k+p)[r(y,t)-r(x,0)]} \left( g^T P_{ij}^T(k) + g^L P_{ij}^L(k) \right) \left( g^T P_{lm}^T(p) + g^L P_{lm}^L(p) \right) \]
\[ (ik)^l (-ik)^m R(y - y', \sigma) R(x - x', \tau) e^{kp(C(y-y',\sigma)+C(x-x',\tau))}. \] (486)

In the next step, first \( k \) and second \( p \) are shifted:
\[ k \longrightarrow k - p, \quad p \longrightarrow p + \frac{k}{2}. \] (487)

The result is
\[ \int \int \bar{r}^i(y, t) \bar{r}^j(x, 0) e^{ik[r(y,t)-r(x,0)]} \left[ \int p \left( g^T P_{ij}^T(p - k/2) + g^L P_{ij}^L(p - k/2) \right) \right] \]
\[ \left( g^T P_{lm}^T(p + k/2) + g^L P_{lm}^L(p + k/2) \right) (k/2 - p)^l (k/2 - p)^m \]
\[ R(y - y', \sigma) R(x - x', \tau) e^{(k^2/4 - p^2)(C(y-y',\sigma)+C(x-x',\tau))}. \] (488)

To compute the correction proportional to the disorder, the expression in the rectangular brackets is expanded for small \( k \). As the integral has a well-defined limit for \( k \rightarrow 0 \), convergent for \( d > 2 \), no term of the form \( k^i k_j / k^2 \) can be generated. The leading term of the above expansion is then (the \( p \)-integral being defined in (716) ff.)
\[ \int \int \bar{r}^i(y, t) \bar{r}^j(x, 0) e^{i[k(r(y,t)-r(x,0))]} \left[ \int p \left( g^T \left( \delta^{ij} - p^{-2} p^i p^j \right) + g^L p^{-2} p^i p^j \right) g^L p^2 e^{-p^2(C(y-y',\sigma)+C(x-x',\tau))} \right. \]
\[ \times R(y - y', \sigma) R(x - x', \tau) \]
\[ = \int \int \bar{r}^i(y, t) \bar{r}^j(x, 0) \left( g^T \left( 1 - \frac{1}{d} \right) + g^L \frac{1}{d} \right) g^L \times \]
Figure 29: The function $\bar{I}(D)$.

\[
\times \frac{d}{2} (C(y - y', \sigma) + C(x - x', \tau))^{-d/2 - 1} \dot{C}(y - y', \sigma) \dot{C}(x - x', \tau)
\]

\[= \left( g^T \left( 1 - \frac{1}{d} \right) + g^L \frac{1}{d} \right) g^L \times \]

\[
\times \frac{2}{d - 2} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \sigma} (C(y - y', \sigma) + C(x - x', \tau))^{-d/2 + 1} .
\]  

(489)

Note also that we have used the (perturbative) FDT, (408), for the first transformation.

The second possible way to do the contraction, see $D_2$ on figure 28, is performed similarly. The leading term is

\[
\times \frac{1}{d} (g^L)^2 \frac{2}{d - 2} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \sigma} (C(y - y', \sigma) + C(x - x', \tau))^{-d/2 + 1} .
\]

(490)

Note that there are two other possible contractions, which can be obtained from $D_1$ and $D_2$ by replacing $\tau$ and $\sigma$ with $-\tau$ and $-\sigma$ respectively. Together they add up to

\[
g^L g^T \frac{4}{d - 2} \left( 1 - \frac{1}{d} \right) \frac{\partial}{\partial \tau} \frac{\partial}{\partial \sigma} (C(y - y', \sigma) + C(x - x', \tau))^{-d/2 + 1} .
\]

(491)

This result is remarkable in several respects: First, the contribution to the renormalization of disorder from the disorder–disorder contraction is isotropic. We will see that this stabilizes the isotropic fixed point. Second, there is no divergent contribution in the purely transversal or purely longitudinal case at 1-loop order. (Note however that there are finite contributions in the transversal case, see appendix C of [33]).

### 12.6 The residues

The dimensional regularization parameter is

\[
\varepsilon = 2 + D - \nu d .
\]

(492)

We now follow the general procedure outlined in section 3.8. In the context of dynamical critical phenomena, we both have to put a cut-off $L$ on the space integration and a cut-off $\lambda L^2$ (where we did set $\lambda = 1$).
onto the time integration. Note also that we have chosen normalizations and notations as in the rest of this article, which differ from those in [33].

The term proportional to \( \langle \cdots \rangle \) is

\[
\begin{align*}
\langle \cdots \rangle_{L} &= \int_{0}^{L} \frac{dx}{x} x^{D} \int_{0}^{L^{2}} dt \left( \cdots \right) \\
&= \int_{0}^{L} \frac{dx}{x} x^{D} \int_{0}^{L^{2}} dt \ C(x, t)^{-d/2} \\
&= L^{D} \varepsilon \int_{0}^{\infty} dt \ C(1, t)^{-d/2} + \text{finite}, \tag{493}
\end{align*}
\]

where in addition \( d \) has been set to \( d_{c} \), defined by \( \varepsilon(D, d_{c}) = 0 \). The residue is

\[
\begin{align*}
\langle \cdots \rangle_{\varepsilon} &= \int_{0}^{\infty} dt \ C(1, t)^{-d_{c}/2} \\
&= :I(D). \tag{494}
\end{align*}
\]

We did not find a closed form to calculate this. Using the approximation for the correlator

\[
C(1, t) \approx \frac{2^{2-D}}{\Gamma\left(\frac{D}{2}\right)} \left( |t| + \frac{\Gamma\left(\frac{D}{2}\right)}{4} \right)^{2-D}, \tag{495}
\]

which is exact\(^7\) for \( t = 0 \) and \( t \to \infty \), we obtain for the integral \( I(D) \)

\[
I(D) = \frac{1}{2D} \Gamma\left(\frac{D}{2}\right)^{2-D} \bar{I}(D). \tag{496}
\]

Were the approximation in (495) exact, \( \bar{I}(D) \) would equal 1. This is not the case, and the value for \( \bar{I}(D) \) obtained from numerical integration is plotted on figure 12.6.

Proceeding equivalently, the diagram correcting the friction coefficient is calculated as

\[
\begin{align*}
\langle \cdots \rangle_{\varepsilon} &= -\frac{1}{d} \int_{0}^{\infty} dt \ C(1, t)^{-d/2} \\
&= -\frac{1}{d} I(D). \tag{497}
\end{align*}
\]

We obtain the interesting relation

\[
\langle \cdots \rangle_{\varepsilon} = -d \langle \cdots \rangle_{\varepsilon}, \tag{498}
\]

\(^7\)Note that this approximation is better than the one used in [33], implying that \( \bar{I}(D) \) is closer to 1.
which can be used to simplify the RG-calculations. It is a reflection of the FDT, valid in perturbation theory, and in the longitudinal case in the full theory.

The elasticity of the membrane is renormalized through

\[
\langle \Omega \mid \tau \rangle_L := \int_0^L \frac{dx}{x} x^D \int_0^L \frac{dx}{x} x^D \int_0^L \frac{dx}{x} x^D \int dt x^2 \frac{\partial}{\partial t} C(x, t)^{-d/2}
\]

\[
= -\frac{1}{2dD} \int_0^L \frac{dx}{x} x^D C(x, 0)^{-d/2} + \text{finite}
\]

\[
= -\frac{1}{2dD} \frac{L^\epsilon}{\epsilon} + \text{finite}.
\]

The residue is therefore given by

\[
\langle \Omega \mid \tau \rangle_L = -\frac{1}{2dD}.
\]

Finally, the diagram correcting the disorder is evaluated as follows:

\[
\langle \Omega \mid \tau \rangle_L :=
\]

\[
= \int_0^L \frac{dx}{x} x^D \int_0^L \frac{dy}{y} y^D \int_0^L \frac{dx}{x} x^D \int_0^L \frac{dy}{y} y^D \frac{4}{d-2} \left(1 - \frac{1}{d}\right) \frac{\partial}{\partial \tau} \frac{\partial}{\partial \sigma} (C(x, \tau) + C(y, \sigma))^{-d/2+1}
\]

\[
= \frac{4}{d-2} \left(1 - \frac{1}{d}\right) \int_0^L \frac{dx}{x} x^D \int_0^L \frac{dy}{y} y^D \frac{4}{d-2} \left(1 - \frac{1}{d}\right) \frac{\partial}{\partial \tau} \frac{\partial}{\partial \sigma} (C(x, \tau) + C(y, \sigma))^{-d/2+1} + \text{finite terms}
\]

\[
= \frac{4}{d-2} \left(1 - \frac{1}{d}\right) \int_0^\infty \frac{dx}{x} x^D (C(x, 0) + C(1, 0))^{-d/2+1} \frac{L^\epsilon}{\epsilon} + \text{finite terms}
\]

\[
= \frac{4}{d-2} \left(1 - \frac{1}{d}\right) \frac{\Gamma^2 \left(\frac{D}{2-D}\right)}{\Gamma \left(\frac{2D}{2-D}\right)} \frac{L^\epsilon}{\epsilon} + \text{finite terms}.
\]

The residue is given by the expression

\[
\langle \Omega \mid \tau \rangle_L = \frac{4}{d-2} \left(1 - \frac{1}{d}\right) \frac{1}{2-D} \frac{\Gamma^2 \left(\frac{D}{2-D}\right)}{\Gamma \left(\frac{2D}{2-D}\right)}.
\]

The factor involving the \(\Gamma\)-functions is familiar from the renormalization of the interaction for self-avoiding membranes, see Eq. (171).

### 12.7 Results and discussion

In this section we analyze the general renormalization group flow given by the two \(\beta\)-functions for longitudinal and transversal disorder. We identify the fixed points and compute the critical exponents at these
fixed points. To present conveniently the analysis below we introduce the following notation for the three independent coefficients, computed in section 12.6 as

\[ A := (2 - D) \langle \ldots \rangle \]
\[ B := \langle \ldots \rangle \]
\[ C := -d(2 - D) \langle \ldots \rangle \]

The explicit expressions for these coefficients are given above in Eqs. (494), (502) and (500). They are non-negative.

We are now in a position to define the $\beta$-functions, quantifying the flow of the renormalized theory upon a variation of the renormalization scale, through

\[ \beta^L(g_L, g_T) := \mu \frac{\partial}{\partial \mu} \bigg|_{\mu_0} g_L \]
\[ \beta^T(g_L, g_T) := \mu \frac{\partial}{\partial \mu} \bigg|_{\mu_0} g_T \].

In terms of the three coefficients $A$, $B$ and $C$, they read:

\[ \beta^L(g_L, g_T) = -\varepsilon g^L + \frac{1}{2} \left( \left( 1 - \frac{1}{d} \right) (d + 2) A - B \right) g_L g_T - \frac{d - 2}{2d} C (g^L)^2 \]  
\[ \beta^T(g_L, g_T) = -\varepsilon g^T - \frac{1}{2} \left( B + \frac{d - 2}{d} C \right) g_T g_T + \left( 1 - \frac{1}{d} \right) \frac{d + 2}{2} A (g^T)^2 . \]

Let us now discuss possible observables. As in Eqs. (176) and (436), we study the roughness exponent $\nu^*$ and dynamic exponent $z^*$, which are defined as

\[ \langle [r(x, t) - r(x', t)]^2 \rangle \sim |x - x'|^{2\nu^*} , \quad \langle [r(x, t) - r(x', t')]^2 \rangle \sim |t - t'|^{2/z^*} \].

We assume scaling behavior

\[ \langle [r(x, t) - r(0, 0)]^2 \rangle \sim |x|^{2\nu^*} \tilde{B}(t/x^{3^*}) \]

with $3^* = \nu^*/z^*$. Let us note that in the literature (and also in the original article [33]) different conventions are chosen; they are obtained by replacing $\nu^* \rightarrow \zeta^*$, $z^* \rightarrow 1/\nu^*$ and $3^* \rightarrow z^*$.

The drift velocity under a small additional applied force $f$ in (452) is

\[ v \sim \langle r(x, t) \rangle / t \]

and scales as

\[ v \sim f^{\phi^*} \]

at small $f$, with

\[ \phi^* = \frac{3^* - \nu^*}{2 - \nu^* + \beta^*} > 1 \]

indicating trapping of the membrane by the flow. $\beta^*$ is the anomalous dimension of the elasticity of the membrane. For a derivation, see [33].
Figure 30: RG flow diagram for SR disorder. The physics is controlled by the fixed point \( I^* \) at \( g^T = g^L \).

The expressions for the exponents then read to lowest order:

\[
\nu(g^L, g^T) = \frac{2 - D}{2} + \frac{g^T}{2} \left( 1 - \frac{1}{d} \right) A - \frac{g^L}{2d} C \tag{513}
\]

\[
\delta(g^L, g^T) = 2 + \frac{1}{d} (A - C) g^L \tag{514}
\]

\[
\beta(g^L, g^T) = -g^L \frac{1}{d} C . \tag{515}
\]

Note that in the end the exponents will depend on the amplitudes only through their ratios \( B/A \) and \( C/A \) which are universal.

For an isotropic manifold, we find from the RG equations that the RG flow is as depicted on figure 30, with the following fixed points:

(1) **Gaussian fixed point**

The Gaussian fixed point at \( g^L = g^T = 0 \) is completely unstable for \( \varepsilon = 2 + D - \frac{d}{2}(2 - D) > 0 \).

(2) **Potential disorder**

The line \( g^T = 0 \) is preserved under renormalization, and we find a flow towards strong coupling. This problem describes the dynamics of an isotropic manifold in a long-range correlated random potential (short range correlated force). The statics of this problem has been much studied and is indeed expected to be described by strong disorder. (For examples see [210, 217, 218, 219, 220, 221].)

(3) **Isotropic disorder fixed point**

Note by taking the difference of Eqs. (506) and (507) that the line \( g^T = g^L \) is preserved by the flow; we thus find an isotropic fixed point at

\[
g^L = g^T = g^* := \frac{2\varepsilon d}{(d - 1)(d + 2)A - dB - (d - 2)C} + \mathcal{O}(\varepsilon^2) . \tag{516}
\]

We have checked numerically, that the denominator of (516) is always positive, which is necessary for this fixed point to be stable and to be in the physical domain. We have also checked numerically that this
Figure 31: Results for isotropic polymers and membranes at the isotropic fixed point, SR-disorder. Results for $\nu$ are obtained from the extrapolation of $\nu d$, and are the most reliable ones. One observes a nice plateau in extrapolations for $\frac{1}{z}$, but no corrections to the Flory approximation can be deduced. The exponents $\phi$ and $\beta$ do not allow for direct extrapolations, but are always corrected upwards. Results for $\beta$ are significant for polymers. For membranes only a bound seems to emerge from extrapolations.

The critical exponents at this fixed point, $\nu^*$ and $\phi^*$, defined in Eqs. (508) and (509), are with the same diagrams

$$\nu^* = \nu(g^*, g^*) = \frac{2 - D}{2} + \frac{(d - 1) A - C - \varepsilon}{(d - 1)(d + 2)A - dB - (d - 2)C} + \mathcal{O}(\varepsilon^2).$$

(517)

The first coefficient in the numerator, $A$, is positive as before, since it arises from the upward corrections to the temperature. However, the elasticity is also renormalized upwards (the polymer tends to shrink to take advantage of favorable regions). This produces the second coefficient $\varepsilon - C$, which is negative. The competition between the two opposite effects finally gives a positive sum and the membrane is stretched. Also note that the $\varepsilon$-correction vanishes like $1/d$ for $D \rightarrow 2$.

A similar formula is valid for the exponent $\phi^*$:

$$\phi^* = \phi(g^*, g^*) = 2 + \frac{2(A - C)\varepsilon}{(d - 1)(d + 2)A - dB - (d - 2)C} + \mathcal{O}(\varepsilon^2).$$

(518)

The $\varepsilon$-correction is always positive, but vanishes like $1/d^2$ for $D \rightarrow 2$.

The elasticity is also renormalized upwards and gives rise to a non-trivial exponent $\beta^*$

$$\beta^* = \beta(g^*, g^*) = \frac{-2C\varepsilon}{(d - 1)(d + 2)A - dB - (d - 2)C} + \mathcal{O}(\varepsilon^2).$$

(519)

The other exponents can be obtained as $1/z^* = \nu^*/\phi^*$ and $\phi^* = (\phi^* - \nu^*)/(2 - \nu^* + \beta^*)$. One notes that in the limit $D \rightarrow 0$ one recovers the results for the particle [222, 223, 224] for $\frac{1}{z}$ and $\phi^*$.

For a polymer, $D = 1$, the disorder becomes relevant below $d = 6$ and setting $D = 1$, we find that the above results yield:

$$\nu^* = 0.5 + 0.130792\varepsilon, \quad \phi^* = 2 + 0.03996\varepsilon, \quad \beta^* = -0.015446\varepsilon$$

(520)
with $\varepsilon = \frac{1}{2}(6-d)$, as well as $\frac{1}{\varepsilon} = 1/4 + 0.060401 \varepsilon$ and $\phi^* = 1 + 0.0369373 \varepsilon$. The most naive extrapolation to $d = 3$ would be (setting $\varepsilon = 1.5$) that $\nu^* = 0.70$, $\delta^* = 2.06$ and $\beta^* = -0.023$ and in $d = 2$ that $\nu^* = 0.76$, $\delta^* = 2.08$ and $\beta^* = -0.031$.

One can try to obtain more reliable estimates for these critical exponents from expressions (517), (518) and (519) for polymers ($D = 1$) and membranes ($D = 2$) in three or two dimensions by optimizing on the expansion point. This is a tedious task since $\varepsilon$ is rather big. The numerical values obtained by the methods detailed in section 7 are not very precise, as we could not find a combination of the exponents and $D$ or $d$, which in suitable extrapolation variables builds up a nice plateau. Different extrapolation schemes yielded strongly varying results. Some indicative values obtained by this methods are summarized in figure 31.

Since $\nu^*$ seems to increase rapidly as $d$ decreases, an interesting question is whether there is a dimension $d_l$ below which the polymer will be fully stretched ($\nu^* = 1$). The result of figure 31 seems to indicate that $d_l$ could be around $d_l = 2$. Our calculations are not precise enough to decide on whether the polymer is already over-stretched or not in $d = 2$ but that would be an interesting point for numerical simulations.

(4) Transversal disorder fixed point

The transversal fixed point ($g^L = 0$) is at

$$g^L = 0, \quad g^T = \frac{2\varepsilon}{\left(1 - \frac{1}{d}\right)(d+2)A} + \mathcal{O}(\varepsilon^2),$$

where the diagram is given in Eq. (503). It is unstable towards perturbations of $g^L$. For the critical exponents $\nu^*$ and $\delta^*$, we recover the Flory-result at 1-loop order:

$$\nu^* = 2 - \frac{D}{2} + \frac{\varepsilon}{d+2} + \mathcal{O}(\varepsilon^2),$$

$$\delta^* = 2 + \mathcal{O}(\varepsilon^2).$$

As was discussed for the directed case, this result was also found to be true for the particle ($D = 0$) to all orders, but for $D > 0$ finite non-transversal terms are generated in perturbation theory which will drive the system towards the isotropic fixed point discussed above (see appendix C of [33]).

12.8 Long-range correlated disorder and crossover from short-range to long-range correlated disorder

It is equivalently interesting to study long-range disorder. We have discussed in section 3.9 that LR-disorder can be treated on the same footing as short-range disorder, with one important difference: There are no disorder corrections, i.e. $\langle \varepsilon \delta \phi \rangle = 0$. We have discussed that under these circumstances there are additional exact relations among the scaling exponents, which sometimes even allow to determine them. Since under renormalization long-range correlated disorder always generates short range correlated one, short-range disorder may be relevant in situations where power-counting indicates that long-range disorder dominates. As discussed in [33], the crossover is very complicated and contains some unexpected features. The most striking one is that under certain circumstances, the canonical dimension of an irrelevant subdominant operator serves as expansion parameter instead of the canonical dimension of the leading operator, as is usually the case. Due to a lack of place, we refer the interested reader to [33].
13 $N$-colored membranes

Field theories have strong connections to geometrical problems involving fluctuating lines. For example, summing over all world-lines representing the motion of particles in space-time, is the Feynman path integral approach to calculating transition probabilities, which can also be obtained from a quantum field theory. Another example is the high-temperature expansion of the Ising model, where the energy-energy correlation function is a sum over all self-avoiding closed loops which pass through two given points. Generalizing from the Ising model to $N$ component spins, the partition function of a corresponding $O(N)$ “loop model” is obtained by summing over all configurations of a gas of closed loops, where each loop comes in $N$ colors, or has a fugacity of $N$. In the limit $N \to 0$, only a single loop contributes, giving the partition function of a closed self-avoiding polymer.[7]

There are several approaches to generalizing fluctuating lines to entities of other internal dimensions $D$; it is important to note that such extensions are not unique. The most prominent generalizations are string theories and lattice gauge theories, both describing $D = 2$ world sheets[225]. The low temperature expansion of the Ising model in $d$ dimensions also results in a sum over surfaces that are $d - 1$ dimensional. Each of these extensions has its own strengths, and offers new insights into field theory. Here we introduce a generalization based on tethered membranes as defined in section 3.1, which have fixed internal connectivity, and thus are the simplest generalization of linear polymers. The resulting manifold theory depends on two parameters $N$ and $D$, with limiting behaviors related to well known models as depicted in Fig. 32. The model is defined by its perturbation series, and as in string theory not obviously derivable from a local Hamiltonian.

The work discussed here has first been published in [34, 35]. We give a shortened presentation of the main ideas, focussing on the construction of the generalized manifold model. This is important from a conceptual point of view, especially since an alternative subtraction scheme is used, in which no divergences proportional to $(\nabla r)^2$ appear at the 1-loop level. We therefore felt the necessity to present in detail the derivation as well as the discussion of the relation between the two models. On the other hand, the original article shall not be rewritten, and therefore most of the applications are only sketched; for more details, the interested reader is referred to the original publications.

This section is organized as follows: We first review the high-temperature expansion, to motivate the formal constructions done later. In a second step, we derive results of the $O(N)$-model in the polymer-language. This is well known, but is done in a way, which can be generalized to membranes. Finally, we discuss some applications: More precise estimations of exponents in the $O(N)$-model, the limit of

\[
\begin{array}{ccc}
O(N)$-field theory & \xrightarrow{D \to 1} & O(N, D)$-manifold model \\
\downarrow N \to 0 & & \downarrow N \to 0 \\
self-avoiding polymers & \xleftarrow{D \to 1} & self-avoiding \\
& & D$\text{-dimensional tethered membranes}
\end{array}
\]

Figure 32: Schematic description of the new model, and its limits.
$N \to \infty$, cubic anisotropy, the random temperature Ising model, and finally a conjecture for the nature of droplets, which govern the Ising model at criticality.

### 13.1 The $O(N)$-model in the high-temperature expansion

In this section, we briefly review the high-temperature expansion of the $O(N)$ model. (For more extensive reviews, see Refs. [13] and [226].) The Hamiltonian is

$$\mathcal{H} = -JN \sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j ,$$

where the sum runs over all nearest neighbors of a $d$-dimensional cubic lattice. To obtain the partition function, we have to integrate over all $\vec{S}_i$ subject to the constraint that $|\vec{S}_i| = 1$, resulting in ($K = \beta J$)

$$Z = \int_{\{\vec{S}_i\}} e^{-\beta \mathcal{H}} = \int_{\{\vec{S}_i\}} \prod_{(i,j)} e^{NK\vec{S}_i \cdot \vec{S}_j} .$$

(525)

The high-temperature expansion is obtained by expanding the exponential factors in Eq. (525) as

$$e^{NK\vec{S}_i \cdot \vec{S}_j} = 1 + NK\vec{S}_i \cdot \vec{S}_j + \cdots .$$

(526)

Typically, only the first two terms in the Taylor-expansion are retained. This is justified as we are only interested in universal quantities, for which the weight is already not unique and may be modified $[\exp(K\vec{S}_i \cdot \vec{S}_j) \to 1 + NK\vec{S}_i \cdot \vec{S}_j]$ in order to cancel subsequent terms in the Taylor expansion.

We can represent the various terms in the perturbation expansion in the following manner (see Refs. [13, 227]): For each term $NK\vec{S}_i \cdot \vec{S}_j$, we draw a line connecting sites $i$ and $j$. At any given site $i$, up to $2d$ such lines may join. The integral over the spin $\vec{S}_i$ is non-zero, if and only if an even number of bonds end at site $i$. For calculational convenience, we normalize the integrals by the corresponding solid angle such that

$$\int d\vec{S}_i = 1 .$$

(527)

Let us now study the first few terms in the perturbation expansion (see figure 33). The diagram (a) is

$$\begin{align*}
(a) &\quad = (KN)^4 \int d\vec{S}_1 \ldots d\vec{S}_4 \, S_1^\alpha S_2^\alpha S_3^\beta S_4^\beta S_2^\gamma S_3^\gamma S_4^\delta S_1^\delta .
\end{align*}$$

(528)
To do the integrations, note that
\[ \int d\vec{S}_i \vec{S}_i^2 = \int d\vec{S}_i 1 = 1, \] (529)
and therefore
\[ \int d\vec{S}_i S_i^\alpha S_i^\beta = \frac{1}{N} \delta^{\alpha\beta}. \] (530)
Performing all but the last integration in Eq. (528), we obtain
\[ (a) = K^4 N \int d\vec{S}_1 \vec{S}_1^2 = K^4 N. \] (531)
For any non-intersecting loop, this result is easily generalized to
\[ K^{\text{number of links}} N, \] (532)
i.e. every closed loop contributes a factor of $N$. Let us now analyze what happens when loops intersect and
to this aim calculate configuration (b). Doing all but the integration over $\vec{S}_1$, we obtain
\[ (b) = K^8 N^2 \int d\vec{S}_1 (\vec{S}_1^2)^2 = K^8 N^2 = (a)^2. \] (533)
Two graphs which have one common site thus give the same contribution in the high-temperature expansion
as if they were disjoint. This is not the case if they have one bond in common, see (c). The integral contains
an odd power of the field $\vec{S}_1$, and therefore
\[ (c) = 0. \] (534)
This high-temperature series can thus be reinterpreted as the sum over all self-avoiding (non-intersecting)
loops. Bonds are \textit{totally} self-avoiding, see e.g. configuration (c), while vertices are also \textit{partially}
self-avoiding as can be seen from the following argument. There are 3 possible ways to build up configuration
(b): One may take 2 small loops, but there are also 2 possibilities to use one loop only (Note that these
configurations come with a different power of $N$). The latter have to be excluded from the partition function.
(There are additional constraints associated with multiple intersections.) On the other hand, as we are
only interested in universal quantities, taking precise account of these configurations should be irrelevant
as long as bond-self-avoidance is present. In the direct polymer approach of Edwards and Des Cloizeaux
[4, 5, 6] discussed below, this corresponds to taking a smaller initial (bare) coupling constant.
A single loop can now be viewed as a random walk, i.e. as the trace of a particle moving under Brownian
motion. The corresponding Hamiltonian is
\[ H_0 = \int_0^L dx \frac{1}{4} (\nabla r(x))^2 + Lt, \] (535)
where $r(x) \in \mathbb{R}^d$ is the trajectory of the particle at time $x$ (equivalently, $x$ is the polymer arc-length). The
total length of the loop is $L = \int dx$. In addition, one has to demand that the particle returns to its starting-
point, i.e. that the polymer is closed. To make it self-avoiding, Edwards and Des Cloizeaux [4, 5, 6] added
an explicit repulsive interaction upon contact, leading to
\[ H = \int_0^L dx \frac{1}{4} (\nabla r(x))^2 + \frac{b\mu e}{4} \int_0^L dx \int_0^L dy \delta^d (r(x) - r(y)) + Lt. \] (536)
The factors of $1/4$, as well as the normalization hidden in $\delta$, are the same as for membranes in Eq. (61), see
also appendix A.1. $\mu$ sets the renormalization scale. In the high-temperature expansion, there appear loops
of all sizes. We thus have to sum over all different lengths of the polymer, weighted by a chemical potential $t$ conjugate to the length, mimicking the constant $K$ in Eq. (525). To avoid possible confusion, let us stress that although closely related, $\ln K$ and $t$ are not identical. While $K$ is defined as the fugacity for the length of the lattice walk, the chemical potential $t$ is conjugate to the coarse-grained length. In principle, the same lattice walk can be represented by curves $r(x)$ of different length $L$. However, as far as universal quantities are concerned, this is unimportant. Both parameters have to be tuned to reach the critical point, and only their deviations from the critical value, but not the critical value itself, have some physical correspondence.

13.2 Renormalization group for polymers

We now discuss the perturbation expansion of the Hamiltonian in Eq. (536). Let us start with the correlation functions of the free (non self-avoiding) polymer. One has to be careful in distinguishing between open and closed polymers which will be denoted by subscripts $o$ and $c$ respectively. For open (or closed, but infinitely long) polymers, the correlation function

$$C_o(x) = \frac{1}{d} \left\langle \frac{1}{2} (r(x) - r(0))^2 \right\rangle_o,$$

is the solution of the Laplace equation

$$\frac{1}{2} \Delta C_o(x) = \delta(x),$$

which is easily found to be

$$C_o(x) = \frac{|x|}{L},$$

and is the same as Eq. (721) for $D = 1$. For closed polymers, Eq. (539) has to be modified. The reason is that the information has two equivalent ways to travel around a polymer loop of size $L$, leading to

$$C_c(x) = \frac{|x|(|L - |x|)|}{L}, \quad \forall |x| < L.$$

We next calculate the weight of a polymer of length $L$. For open polymers this is simply

$$e^{-Lt},$$

where $t$ is the chemical potential. For closed polymers, an additional factor of

$$\left\langle \tilde{\delta}^d (r(L) - r(0)) \right\rangle_o$$

has to be added, which measures the probability to find a closed polymer among all open polymers. The expectation value therefore is taken with respect to the weight for an open polymer, and calculated as follows:

$$\left\langle \tilde{\delta}^d (r(L) - r(0)) \right\rangle_o = \int_k \left\langle e^{ik(r(L) - r(0))} \right\rangle_o$$

$$= \int_k e^{-k^2C_o(L)}$$

$$= \int_k e^{-k^2L}$$

$$= L^{-d/2}.$$

The normalizations of $\tilde{\delta}^d$ and $\int_k$ are chosen as usual, see appendix A.1.
To get the quantities obtained in the high-temperature expansion of the loop model introduced above, we still have to integrate over all possible lengths of the polymer. We define the free density of a single polymer as (see also footnote 8, page 128)

$$Z_1^{(0)} = \frac{1}{2} \int \frac{dL}{L} LL^{-d/2} e^{-L t} = \frac{1}{2} \Gamma \left(1 - \frac{d}{2}\right) e^{d/2 - 1}.$$ (544)

We have chosen to integrate over a logarithmic scale ($\int dL/L$) in order to make the integration measure dimensionless. The factor $L$ counts the number of points which may be taken as origin, and the factor of $\frac{1}{2}$ has been introduced to reproduce the results of the free (Gaussian) field theory, see below.

Additional insight is obtained from a different way to calculate $Z_1^{(0)}$. If we do not perform the last integral in Eq. (543), Eq. (544) becomes

$$Z_1^{(0)} = \frac{1}{2} \int dL e^{-L t} \int_k e^{-k^2 L} \frac{1}{k^2 + t}.$$ (545)

This term of the polymer-perturbation theory is equivalent to a term in the perturbation theory of the field-theoretical description of the $O(N)$-model. If not explicitly noted, the diagram is regularized by the chemical potential $t$. In the usual treatment of the $O(N)$ field theory, the hard constraint of $|\vec{S}| = 1$ is replaced in favor of a soft constraint, implemented by the Hamiltonian

$$\mathcal{H}_\phi = \int d^4 r \left[ \frac{1}{2} \left( \nabla \vec{\phi}(r) \right)^2 + \frac{t}{2} \vec{\phi}^2(r) + \frac{b \mu^2}{16} \left( \vec{\phi}^2(r) \right)^2 \right].$$ (546)

In this description, one has to take the limit $N \to 0$ in order to allow for only 1 connected piece. (Remember that every closed loop counts a factor of $N$.) This equivalence, first pointed out by De Gennes [7], is not accidental and was demonstrated in section 4.4. It reflects the fact that both the field-theoretic formulation of the $O(N)$-model, as well as its lattice equivalent, belong to the same universality class.

We can now also comment on the factor of $\frac{1}{2}$ introduced in Eq. (544). According to Eq. (253), $\mathcal{O} = \left\langle \frac{1}{2} \left( \vec{\phi}(r) \right)^2 \right\rangle_0$ is the Laplace-transform of $\int dx \left\langle \tilde{\delta}^d(r(x)) \right\rangle_0 = L \times L^{D/2}$, such that $\int dL/L$ has to be accompanied by a factor of $\frac{1}{2}$. Also note that a factor of $(4\pi)^{d/2}$ has implicitly been incorporated into the measure, since in Eq. (253) the $\delta$-interaction appears as $\delta^d(r(x))$ and not as $\tilde{\delta}^d(r(x))$ (for the definition of $\tilde{\delta}^d(r(x))$ see appendix A.1).

We now perform the perturbation expansion of the polymer Hamiltonian in Eq. (536). The first term is the expectation value of one $\tilde{\delta}$-interaction with respect to the free theory of a closed polymer, integrated over all positions of the interaction on the polymer of length $L$, and then over all polymer-lengths. This is explicitly

$$\frac{1}{2} \int_0^\infty dL L^{-d/2} e^{-L t} \int_0^L dx \int_0^L dy \int_k e^{ik(r(x)-r(y))} c$$

$$= \frac{1}{2} \int_0^\infty dL e^{-L t} \int_0^L dx \int_0^L dy \left[ \frac{|x-y|(L-|x-y|)}{L} \right]^{-d/2} L^{-d/2}$$

$$= \int_0^\infty dL e^{-L t} \int_0^L dx \int_0^x dz \left[ (L-z) \right]^{-d/2}$$

$$= \int_0^\infty dz \int_0^\infty dx \int_0^\infty dy' e^{-t(x'+y')-z} \int_p e^{-p^2 z} \int_k e^{-k^2(x'+y')}$$
\[
\begin{aligned}
&= \int k \, \int p \frac{1}{(k^2 + t)^2} \frac{1}{p^2 + t} \\
&\equiv 2 \bigcirc \bigcirc \equiv 4 \bigcirc \times \bigcirc .
\end{aligned}
\]  
\text{(547)}

(Remember the factors of 1/2 for each connected component, introduced in Eq. (544).) The relation to \(\phi^4\)-theory is again apparent: The integrals in Eq. (547) are ultra-violet divergent. The leading divergence is subtracted via a finite part prescription, the sub-leading term is treated via dimensional regularization as a pole in
\[
\varepsilon = 2 - d/2 .
\]  
\text{(548)}

(Note the factor of 2 difference from the more usual definition of \(\epsilon = 4 - d\).)

Let us now introduce a renormalized Hamiltonian. Since in contrast to the model introduced in section 3.1, an additional chemical potential appears, three renormalizations may be required: A renormalization of the field \(r\), of the coupling constant \(b\), and of the chemical potential \(t\). Denoting the bare quantities with a subscript \(0\), we set
\[
\begin{aligned}
& r_0 = \sqrt{Z} r , \\
& t_0 = Z_t t , \\
& b_0 = \mu \varepsilon Z^{d/2} Z b .
\end{aligned}
\]  
\text{(549)}

This yields the renormalized Hamiltonian
\[
\mathcal{H} = Z \int \frac{d}{4} \left( \nabla r(x) \right)^2 + \frac{b \mu \varepsilon Z_b}{4} \int \frac{d}{4} \int \frac{dy}{4} (r(x) - r(y)) + Z_t t \int d x,
\]  
\text{(550)}

where \(\mu\) sets the renormalization scale. It is possible to subtract at the scale of the renormalized chemical potential \(t\), but this turns out to be rather confusing when deriving the renormalization group equations. We can now eliminate the divergence in Eq. (547) by setting (the index \(\varepsilon\) means as in Eq. (165) just the pole-term in 1/\(\varepsilon\) of the diagram.)
\[
Z_t = 1 - \frac{b}{\varepsilon}. 
\]  
\text{(551)}

This is seen by expanding \(e^{-\mathcal{H}}\) with \(\mathcal{H}\) given in Eq. (550). From Eqs. (544) and (545), we read off the numerical value of \(\bigcirc\), yielding
\[
Z_t = 1 + \frac{b}{2\varepsilon}. 
\]  
\text{(552)}

The next step is to study the renormalization of the interaction, to which the following two diagrams contribute
\[
\begin{aligned}
& = \int \int \left( e^{i(k_0 + k) \left[ r(x_0 + x) - r(x_0 + y) \right]} e^{i \left( \frac{\Omega_x}{2} - k \right) \left[ r(x_0 + x) - r(y_0 + x) \right]} e^{i \left( \frac{\Omega_y}{2} - k \right) \left[ r(x_0 + y) - r(y_0 + y) \right]} \right) ,
\end{aligned}
\]  
\text{(554)}
short distance singularities appear in the integration over \( x \) and \( y \). The leading term in the short distance expansion is in analogy to the MOPE (see the derivation following Eq. (139))

\[
\int_{k_0,x_0,y_0} e^{ik_0(r(x_0)−r(y_0))} e^{−t(Ω_x+Ω_y)} \int_{k,x,y} e^{−k^2(C_c(x)+C_c(y))} e^{−t(x+y)}. \tag{555}
\]

For small arguments, the correlation function can be approximated by its infinite volume limit, leading up to subleading terms to

\[
\int_{k} e^{−(k^2+t)(x+y)} = \int_{k} \frac{1}{(k^2+t)^2}. \tag{556}
\]

The final result is

\[
\int_{k} e^{−(k^2+t)(x+y)} = \int_{k} \frac{1}{(k^2+t)^2} = \Gamma \left(2 - \frac{d}{2}\right) t^{2-d/2}. \tag{557}
\]

The second diagram in Eq. (553) has already appeared in Eq. (547), and we can symbolically write

\[
\int_{k} e^{−(k^2+t)(x+y)} = \int_{k} \frac{1}{(k^2+t)^2} = \Gamma \left(2 - \frac{d}{2}\right) t^{2-d/2}. \tag{558}
\]

This diagram appears with a combinatorial factor of 2 for its left-right asymmetry, and another factor of 2 for the up-down symmetry of the leftmost interaction.

Adding these contributions yields the following renormalization factor at 1-loop order (note that the combinatorial factors of 4 cancel with that of the \( b/4 \) in the Hamiltonian Eq. (550)),

\[
Z_b = 1 + b \frac{\varepsilon}{2} \left(1 + \int_{k} e^{−(k^2+t)(x+y)} \right) = 1 + 2b. \tag{559}
\]

No field renormalization is necessary (\( Z = 1 \)). We will discuss the apparent difference to the renormalization factors in Eqs. (172) and (173) later.

The next step is to calculate the renormalization group functions, which measure the dependence of the renormalized quantities upon a change of the renormalization scale \( \mu \), while keeping the bare values fixed. The derivation of these functions is given in appendix A.4, and results in a so-called \( \beta \)-function

\[
\beta(b) = \frac{\partial}{\partial b} \ln \left( \frac{Z_b}{Z}\right) = \frac{-\varepsilon b}{1 + b \frac{\varepsilon}{2} \ln Z_b + \frac{b^2}{2} \frac{\varepsilon}{2} \ln Z}, \tag{560}
\]

and a scaling function for the field \( r \)

\[
\nu(b) = \frac{1}{2} - \frac{1}{2} \beta(b) \frac{\partial}{\partial b} \ln \left( Z \right) . \tag{561}
\]

We are now in a position to calculate the exponent \( \nu^* \) in 1-loop order. The \( \beta \)-function is at this order

\[
\beta(b) = -\varepsilon b + b^2 \left(1 + \int_{k} e^{−(k^2+t)(x+y)} \right) + O(b^3) + O(b^2 \varepsilon) = -\varepsilon b + 2b^2 + O(b^3) + O(b^2 \varepsilon), \tag{562}
\]
and the scaling function $\nu(b^*)$ becomes
\begin{align*}
\nu(b^*) &= \frac{1}{2} - \frac{1}{2} \epsilon b^* + O(\epsilon^2) \\
&= \frac{1}{2} - \frac{\epsilon}{2} b^* + O(\epsilon^2) \\
&= \frac{1}{2} + \frac{\epsilon}{8} + O(\epsilon^2).
\end{align*}
(563)

This renormalization scheme is also used in $\phi^4$-theory. At 1-loop order, no renormalization of the wavefunction is necessary. Only the reduced “temperature” $t$ is renormalized. There is another scheme, equally useful, to perform the renormalization of polymers, which is also used in the broader context of polymerized membranes. This scheme also works for infinite membranes. Naturally, for infinite membranes, no renormalization of $t$ can occur as it is identically 0. It is also known for the renormalization of standard field-theories that one has the choice to work either in a massive ($t \neq 0$) or a massless ($t = 0$) scheme.

For the polymer model, let us find a renormalization scheme where $t$ is not renormalized, and therefore the limit $t \to 0$ can be taken without problem. The key observation is that only the combinations $Z Z_t$ and $Z_b Z^{d/2}$ enter the renormalization group calculations, and these combinations are left invariant by changing the $Z$-factors to
\begin{align*}
Z'_t &= 1, \\
Z'_t &= Z Z_t, \\
Z'_b &= Z_b Z_t^{\varepsilon - 2}.
\end{align*}
(564)

For a derivation of this property as a consequence of the rescaling-invariance of the underlying Hamiltonian, see appendix A.5. In terms of the modified $Z$-factors, we obtain
\begin{align*}
\beta(b) &= -\frac{\epsilon b}{1 + b \frac{\partial}{\partial b} \ln Z'_b + \frac{d}{2} b \frac{\partial}{\partial b} \ln Z'}, \\
\nu(b) &= \frac{1}{2} - \frac{1}{2} \beta'(b) \frac{\partial}{\partial b} \ln Z'.
\end{align*}
(565)

This is the scheme used in the rest of this review, and it is the only suitable one for higher loop calculations. On the other hand, it may lead to some confusion as it necessitates a renormalization of the field, even in the case of polymers. This may not have been expected from the 1 to 1 correspondence on the level of diagrams for the $N \to 0$ limit of $\phi^4$-theory, and polymers. As shown above, the two schemes are completely equivalent and one may use the one better suited to the problem at hand.

Let us stress another important difference between the two approaches. When using the MOPE, the divergence of a dipole on approaching its ends was obtained in Eq. (135). Specializing to polymers gives the result
\begin{align*}
|x - y|^{-d/2} 1 - \frac{1}{2} |x - y|^{1-d/2} + \ldots.
\end{align*}
(566)

The divergence proportional to the operator $1$ can be subtracted by analytical continuation. In the absence of any boundary and for infinite membranes, this term has no effect on the renormalization functions. The second term is more serious and has to be subtracted. This is done by renormalization of the field, thus introducing the renormalization factor
\begin{align*}
Z = 1 + b \frac{1}{2 \epsilon}.
\end{align*}
(567)
Upon expanding the Hamiltonian, this yields a counter-term proportional to $+$, which cancels the divergence.

Let us now study the renormalization of the coupling constant in this scheme. Using the MOPE, we can write down the following two UV-divergent configurations

$$\begin{align*}
\text{and } & \quad \text{ and } \quad \text{,}
\end{align*}$$

(568)

from which we shall extract terms proportional to the interaction $\rightarrow$, which we denote as

$$\begin{align*}
\langle & \quad \text{ and } \quad \text{,}
\end{align*}$$

(569)

where $L = t^{-1}$ plays the role of the IR-cutoff. The first is written in the notation of polymer theory as

$$\begin{align*}
\langle & \quad \text{.}
\end{align*}$$

(570)

Indeed this diagram was subtracted when we renormalized the interaction in Eq. (559), where we also subtracted the term,

$$\begin{align*}
\langle & \quad \text{.}
\end{align*}$$

(571)

The MOPE Eq. (152) now tells us that

$$\begin{align*}
\text{= .}
\end{align*}$$

(572)

This result implies that having introduced a counter-term for $\text{,}$ i.e. a renormalization of the field, no counter-term for the diagram in Eq. (571) is needed.

We can check for consistency by comparing the $\beta$-functions from the two schemes at 1-loop order. In the massive scheme, we had

$$\begin{align*}
\beta(b) = -\varepsilon b + b^2 \left( e + \varepsilon \right) + O(b^3) + O(b^2 \varepsilon) .
\end{align*}$$

(573)

In the massless scheme, we obtain

$$\begin{align*}
\beta(b) = -\varepsilon b + b^2 \left( \langle \text{.} \rangle_{\varepsilon} - \frac{d}{2} \langle \text{.} \rangle_{\varepsilon} \right) + O(b^3) + O(b^2 \varepsilon) .
\end{align*}$$

(574)

It is now easy to see that expressions (573) and (574) are equivalent up to order $O(b^3)$ and $O(b^2 \varepsilon)$, since

$$\begin{align*}
\langle & \quad \text{ = .}
\end{align*}$$

(575)

and

$$\begin{align*}
-\frac{d}{2} \langle & \quad = \frac{d}{4} = 1 + O(\varepsilon) .
\end{align*}$$

(576)

Another observation is that in the massless scheme, vertex operators like $e^{ik[r(x) - r(y)]}$ are finite, whereas in the massive scheme they require an additional renormalization. Let us also note that these relations can be understood within the concept or redundant operators, discussed in section 4.1.
13.3 Generalization to $N$ colors

Having performed a careful analysis of the different renormalization schemes, we are now in a position to generalize to the case $N > 0$, i.e. to an arbitrary number of self-avoiding polymer loops. To this aim, we introduce polymers of $N$ different colors, and for the time-being, work in the massive scheme. In addition to , which renormalizes the chemical potential $t$, there is now a second contribution, namely

$$ \text{(577)} $$

This diagram is easily factorized as

$$ \text{(578)} $$

and is therefore equivalent to the digram already encountered in Eq. (547) and absorbed in $Z_t$ (for $N = 0$ in Eq. (551)).

Let us now determine the combinatorial factor: A configuration

$$ \text{(579)} $$

can be made out of 1 polymer or out of two polymers. The latter comes with an additional factor of $N$, accounting for the $N$ different colors introduced above as well as with a relative factor of $1/2$ for the additional connected component as introduced in Eq. (544). $Z_t$ is thus modified to

$$ Z_t = 1 - \frac{b}{\varepsilon} \varepsilon \left( 1 + \frac{N}{2} \right) + \mathcal{O}(b^2) $$

$$ = 1 + \frac{b}{2\varepsilon} \left( 1 + \frac{N}{2} \right) + \mathcal{O}(b^2). \quad \text{(580)} $$

This is indeed the same combinatorial factor as derived from $N$-component $\phi^4$-theory. For the renormalization of the coupling-constant, in addition to

$$ \text{(581)} $$

there is the possibility that an additional loop mediates the interaction between two given polymers, described by a configuration

$$ \text{(582)} $$

The configurations in Eq. (581) are realized in 2 different ways each in the high-temperature expansion, while for Eq. (582) there is only one realization which comes with a factor of $N$ for the $N$ different colors. $Z_b$ is therefore modified to

$$ Z_b = 1 + \frac{b}{\varepsilon} \left( \varepsilon \varepsilon + \varepsilon \varepsilon + \frac{N}{2} \varepsilon \varepsilon \right) $$

$$ = 1 + \frac{b(8 + N)}{4\varepsilon}. \quad \text{(583)} $$

---

8Note that in the original article [34], no factor of $\frac{1}{2}$ was introduced in the definition of Eq. (544); in compensation, it was argued that $\varepsilon\varepsilon$ could be made out of one polymer in two different ways rather than in one way. The same additional factor of 2 was associated to the configurations in Eq. (581), leading to the same final result for $\nu^*$ as given here. For membranes, the normalizations are accordingly modified, see footnote 9, page 129.
Evaluating the critical exponent $\nu^*$ as before now yields

$$\nu^* = \frac{1}{2} + \frac{\varepsilon}{28 + N}.$$  \hfill (584)

It is again possible to switch to the massless scheme. At this stage this is not very enlightening, as for polymers all diagrams are essentially equivalent. We will therefore discuss this scheme in the context of membranes, which are introduced in the next section.

### 13.4 Generalization to membranes

We shall now apply the above construction to polymerized tethered membranes, as introduced in section 3. Including a chemical potential $t_0 = t Z_t$, the Hamiltonian reads

$$H = \frac{Z}{2 - D} \int_x \frac{1}{2} (\nabla r(x))^2 + b \mu \varepsilon Z_b \int_x \int_y \delta^d(r(x) - r(y)) + t Z_t \Omega,$$  \hfill (585)

with the same conventions as before (see appendix A.1), and

$$\Omega := \int d^d x = S_D \int_x.$$  \hfill (586)

In Eq. (75), we had obtained the free correlation function for infinite membranes as

$$C_0(x) = |x|^{2-D}.$$  \hfill (587)

For the correlator to satisfy the above condition, the usual Laplace equation, $\Delta C_0(x) \sim \delta^D(x)$, has to be modified to

$$\frac{1}{(2 - D) S_D} \Delta C_0(x) = \delta^D(x) - \frac{1}{\Omega},$$  \hfill (588)

where $\Omega$ is the volume of the compact manifold. The numerical prefactors come from our choice of normalizations in Eq. (585), see appendix A.1. In the infinite-volume limit, the correction term disappears, and the usual equation is regained. It is then easy to deduce that

$$C_0(x) = \frac{1}{d} \left\langle \frac{1}{2} (r(x) - r(y))^2 \right\rangle_0 = |x - y|^{2-D} - \nu S_D \frac{D}{D \Omega} |x - y|^2 + \text{subleading terms}.$$  \hfill (589)

The coefficient of the correction term clearly agrees for $D = 1$ with the exact result for closed polymers in Eq. (540).

The considerations of section 13.2 can now be generalized to the case of membranes. The free density of a single polymer, i.e. the sum over all sizes of a non-interacting polymer, Eq. (544), is generalized to the membrane-density

$$Z_1^{(0)} := c(D) \frac{S_D^2}{D} \frac{1}{8} \int \frac{d \Omega}{\Omega} \Omega^{-\nu d/D} e^{-t \Omega} = c(D) \frac{S_D^2}{D} \frac{1}{8} \Gamma \left( \frac{\varepsilon}{D} - 1 \right) t^{\varepsilon/D - 1}.$$  \hfill (590)

\[\footnote{As in definition (544) (see footnote 8, page 128), an additional factor of 1/2 has to be introduced in Eq. (590) at variance with the original article [34]. Here, we furthermore introduce a factor of $S_D^2/4$, with which we reproduce the final result of [34]. The reason is a conceptual improvement: Whereas in [34], the combinatorial factor for an additional connected piece was taken in analogy to the polymer-case, here we succeed in its explicit calculation, see Eqs. (611) and (615).} \]
We have chosen to integrate over a logarithmic scale, \( \frac{d\omega}{\omega} = \frac{1}{D} \frac{d\Omega}{\Omega} \). To emphasize the arbitrariness of this choice, we have included an additional factor of \( c(D) \), which is further discussed in the next section, and a factor of \( S^2_D/8 \). The latter factor is chosen in order to render the final result as simple as possible. These factors are important, as they also appear in the ratio of divergences due to self-interactions of one membrane, and those of interactions with other membranes. The factor \( \Omega \) in the integrand of the above equation originates from the possible choices of a point \( x_0 \) on the membrane, while the factor

\[
\Omega^{-\nu d/D} \sim \langle \tilde{\delta}^d(r(x_0)) \rangle_0 ,
\]

is the probability that at this point the membrane is attached to a given point in space. As usual, we have introduced a chemical potential proportional to the size of the membrane.

Let us now generalize Eq. (547) for the effect of one \( \tilde{\delta}^d \)-insertion from the expansion of the interaction. For the time being, we fix the size of the membrane to \( \Omega \), and evaluate

\[
\int_x \int_y \langle \tilde{\delta}^d(r(x) - r(y)) \rangle_0 .
\]

This integral is (see Eq. (543))

\[
\frac{\Omega}{S_D} \int_x C_0(x)^{-d/2} ,
\]

and we have to remove all UV-divergent contributions. To do so, we expand \( C_0(x)^{-d/2} \) for small \( x \). Up to UV-convergent terms, this is (using Eq. (589))

\[
\frac{\Omega}{S_D} \int \frac{dx}{x} x^D \left( x^{-\nu d} + \frac{d \nu S_D}{2 D \Omega} x^{D-\nu d} + \ldots \right) .
\]

The first term is strongly UV-divergent and has to be subtracted by a finite part prescription, while the second is (up to terms of order \( \epsilon^0 \)) equal to

\[
\frac{1}{\epsilon} \Omega^\epsilon/D.
\]

Note that we have cut off the integral at the upper bound \( x_{\text{max}} = \Omega^{1/D} \). This procedure may appear rather crude, but the residue of the pole in \( 1/\epsilon \) is not affected, see section 4.3.

Upon integrating over all scales, the membrane density (to first order) reads

\[
Z_1^{(1)} = 1 - \frac{b \epsilon}{S_D^2} + O(t^2)
\]

\[
= \frac{c(D) S^2_D}{D} \frac{1}{8} \int \frac{d\Omega}{\Omega} \Omega^{-\nu d/D} e^{-\alpha} \left[ 1 + \frac{b}{\epsilon} (\Omega t)^{\epsilon/D} + \ldots \right] ,
\]

which upon integration over \( \Omega \) results in

\[
Z_1^{(1)} = Z_1^{(0)} \left[ 1 + \frac{b}{2\epsilon} + \ldots \right] .
\]

Note the difference in factor of 2 between Eqs. (595) and (597), which is due to the nested integrations as discussed in section 6.3. This factor of 2 can be interpreted as being geometric. The counter-term is only needed in the half-sector \( x < \Omega^{1/D} \) and not in the half-sector \( x > \Omega^{1/D} \). Introducing now a counter-term for \( t \) yields

\[
Z_t = 1 + \frac{b}{2\epsilon} .
\]
The bare and renormalized quantities are then related by generalizing Eq. (549) to

\[ r_0 = \sqrt{Z} r , \quad t_0 = Z t , \quad b_0 = \mu ^{\varepsilon} Z^{d/2} Z b , \]

leading to the renormalization group functions (compare with Eqs. (560) and (561))

\[ \beta (b) = \mu \frac{\partial}{\partial \mu} b = \frac{-\varepsilon b}{1 + b \frac{\partial}{\partial b} \ln Z + \frac{d}{2} b \frac{\partial}{\partial b} \ln Z} , \]

\[ \nu (b) = \frac{2 - D}{2} - \frac{1}{2} \beta (b) \frac{\partial}{\partial b} \ln \left( Z Z^{(2-D)/D} \right) . \]

(The derivation is given in appendix A.4.)

The combinations \(ZZ^{(2-D)/D}\) and \(Z_\varepsilon^{d/2}\), which enter the renormalization group calculations, are left invariant by changing the \(Z\)-factors to

\[ Z'_t = Z_t / Z_\alpha , \quad Z' = Z Z^{(2-D)/D} / Z_t , \quad Z'_b = Z_b Z^{\varepsilon/D-2} . \]

For a derivation of this property as a consequence of the rescaling-invariance of the underlying Hamiltonian, see Appendix A.5.

In order to eliminate the renormalization of \(t\), we chose

\[ Z_\alpha = Z_t , \]

resulting in

\[ Z'_t = 1 , \quad Z' = Z Z^{(2-D)/D} , \quad Z'_b = Z_b Z^{\varepsilon/D-2} , \]

and the renormalization group functions

\[ \beta (b) = \frac{-\varepsilon b}{1 + b \frac{\partial}{\partial b} \ln Z + \frac{d}{2} b \frac{\partial}{\partial b} \ln Z'} , \]

\[ \nu (b) = \frac{2 - D}{2} - \frac{1}{2} \beta (b) \frac{\partial}{\partial b} \ln Z' . \]

With this change of variables, Eq. (598) is replaced by \(Z'_t = 1\), and

\[ Z' = 1 + \frac{2 - D}{2D} b . \]

The above result is precisely that obtained by using the multilocal operator product expansion technique for infinite membranes, see Eqs. (165) and (172). The interpretation of this formula is simple, as

\[ \left\langle \left\langle \begin{array}{c} \cdots \\ \downarrow \end{array} \right\rangle \right\rangle = \frac{-1}{2D} . \]

\[ N\text{-colored membranes} \]
is just the residue of the diverging contribution form the MOPE, of one $\delta^d$-insertion. For $N = 0$, the renormalization of the coupling constant in the massless scheme is analogously (see Eq. (173))

$$Z_b' = 1 + \frac{b}{\varepsilon} \langle \langle \cdots \rangle \rangle_\varepsilon,$$

with

$$\langle \langle \cdots \rangle \rangle_\varepsilon = \frac{1}{2 - D} \frac{\Gamma(D/2)^2}{\Gamma(D/2 - B)}.$$

Alternatively, in the massive scheme ($Z = 1$),

$$Z_b = \left( 1 + \frac{b}{\varepsilon} \langle \langle \cdots \rangle \rangle_\varepsilon \right) \times Z_t^2 = 1 + \frac{b}{\varepsilon} \left( \langle \langle \cdots \rangle \rangle_\varepsilon + 1 \right).$$

Let us now study the generalization to $N$ components in the massive scheme. The membrane-density to first order in $b$ is

$$Z_1^{(1)} = \bigcirc - \frac{b\mu^\varepsilon}{S_D} \bigcirc - 2N \frac{b\mu^\varepsilon}{S_D} \bigcirc + O(b^2).$$

The factor of 2 in front of $\bigcirc$ is due to the fact that the ends of the $\delta$-interaction may be interchanged between the two membranes, whereas the factor of $N$ counts the number of different colors of the additional membrane. We now calculate the diagram $\bigcirc$ for membranes, as

$$\bigcirc = \frac{c(D) S_D^2}{D} \Gamma \left( \frac{\varepsilon}{D} \right) \int \frac{d\Omega}{\Omega} \Omega^2 \Omega^{-\nu_d} e^{-\Omega}.$$

We already have given the derivation of a similar integral in Eq. (590). The only difference is that now a second factor of $\Omega$ appears to take into account the additional point which moves on the membrane. Integration over $\Omega$ yields

$$\bigcirc = \frac{c(D) S_D^2}{D} \Gamma \left( \frac{\varepsilon}{D} \right) t^{-\varepsilon/D} = \frac{c(D) S_D^2}{D} \frac{1}{\varepsilon} t^{\varepsilon/D} + O(\varepsilon^0).$$

Eq. (598) is therefore modified in the same manner as Eq. (580) to

$$Z_t = 1 + \frac{b}{2\varepsilon} \left( 1 + \frac{c(D)N}{2} \right).$$

There are several possibilities to derive the modification to the renormalization factor of the interaction. For a direct derivation generalizing Eq. (583) we note that the effective interaction is modified by a new term proportional to $N$,

$$\begin{array}{c}
\bigcirc \rightarrow \bigcirc - 2N \frac{b\mu^\varepsilon}{S_D} \bigcirc.
\end{array}$$

The renormalization factor $Z_b$ therefore becomes

$$Z_b = 1 + \frac{b}{\varepsilon} \left( \langle \langle \cdots \rangle \rangle_\varepsilon + 1 + \frac{c(D)N}{4} \right).$$
It is now easy to derive the renormalization group functions

$$\beta(b) = -\varepsilon b + b^2 \left( \langle \overrightarrow{\vec{S}}^2 \rangle_{\varepsilon} + 1 + \frac{c(D)N}{4} \right) + \mathcal{O}(b^3) + \mathcal{O}(b^2 \varepsilon) ,$$  \hspace{1cm} (617)

and

$$\nu(b) = \frac{2 - D}{2} \left( 1 + \frac{b}{2D} \left( 1 + \frac{c(D)N}{2} \right) \right) + \mathcal{O}(b^2) .$$  \hspace{1cm} (618)

At the non-trivial (IR-stable) fixed point, this yields the critical exponent to order $\varepsilon$

$$\nu^* = \frac{2 - D}{2} \left( 1 + \frac{\varepsilon}{2D} \frac{1 + \frac{c(D)N}{2}}{\frac{1}{2 - D} \Gamma\left( \frac{D}{2 - D} \right) \left( \frac{D}{2 - D} \right) + 1 + \frac{c(D)N}{4}} \right) ,$$  \hspace{1cm} (619)

representing our central result for the generalized $O(N)$-model.

### 13.5 The arbitrary factor $c(D)$

In calculating the free partition function in Eq. (590), we introduced an arbitrary factor of $c(D)$. In principle, any function of $D$ which satisfies

$$c(1) = 1 ,$$  \hspace{1cm} (620)

reproduces the correct result for linear objects. The additional freedom (or ambiguity) is apparently a reflection of the non-uniqueness of the generalization to manifolds. Even after restricting to the class of hyperspheres, there is a remaining ambiguity in the choice of the measure for the size of these manifolds. This arbitrariness carries over to our generalization of the $O(N)$-model to $N$-colored membranes. (Note also that $c(D)$ is independent of the introduction of factors like the $\frac{1}{2-D}$ in Eq. (585).)

Two choices have been studied in [34], namely $c(D) = 1$ and $c(D) = D$. The second one seems to be the best suited for numerical extrapolations, and we shall in the remainder focus on it.

### 13.6 The limit $N \to \infty$ and other approximations

As in the $O(N)$ model, it is possible to derive the dominant behavior for large $N$ exactly. In the standard $\phi^4$-theory, one starting point is the observation that

$$\langle (\overrightarrow{\vec{S}}^2)^2(r) \rangle = \langle \overrightarrow{\vec{S}}^2(r) \rangle^2 ,$$  \hspace{1cm} (621)

since in the limit $N \to \infty$, spin-components of different colors decouple [3, 228, 147].

Here, we pursue a slightly different approach, based on the diagrammatic expansion. Note that for $N \to \infty$, only simply connected configurations survive. (The vertices are made out of membranes, the links out of $\tilde{\delta}^4$-interactions.) For example, the diagram $\bigcirc \bigtriangledown \bigtriangledown$ which is doubly connected, and the diagram $\bigcirc \bigtriangleup \bigtriangleup \bigcirc$ which includes a self-interaction, each have one factor of $N$ less than the simply connected graph.
The leading diagrams for the membrane density at the origin are then given by (we suppress a couple of geometric factors which are unimportant for the derivation)

\[ f := \bigcirc + \bigcirc \ldots + \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc + \ldots . \]  

(622)

The above sum can be converted into a self-consistent equation for \( f \) by noting the following: Successive diagrams can be obtained from the first (bare) diagram by adding to each point of a manifold a structure that is equivalent to \( f \) itself. This is equivalent to working with a single non-interacting manifold for which the chemical potential \( t_0 \) is replaced by an effective value of \( t_0 + b_0 f \). Calculation of \( f \) for this manifold proceeds exactly as in Eq. (590), and results in the integral

\[ f = \int \frac{d\Omega}{\Omega} \Omega^{1-\nu/d} e^{-\Omega(t_0+b_0 f)} . \]  

(623)

The above integral is strongly UV-divergent, and leads to a form

\[ f = B(t_0 + b_0 f)^{\frac{2-D}{2d-\nu}} + A , \]  

(624)

where \( B \) is a constant. The strong UV-divergence, controlled with an explicit UV-cutoff, is absorbed in the constant \( A \). It is dropped in a dimensional regularization scheme, as in Eq. (590).

The radius of gyration \( R \) is now related to \( f \) as follows: From Eq. (623) we note that \( t_0 + b_0 f \) is the physical chemical potential conjugate to \( \Omega \), thus leading to a typical volume of \( \Omega \sim 1/(t_0 + b_0 f) \). Since there are no self-interactions in the effective manifold introduced above, its radius can be related to the volume by \( R \sim \Omega^{\nu/D} \). Thus, up to a numerical factor which is absorbed into the definition of \( R \), we obtain

\[ R^{-\frac{2D}{2d-\nu}} = t_0 + b_0 f . \]  

(625)

Eliminating \( f \) in Eq. (624) with the help of Eq. (625) yields

\[ R^{-\frac{2D}{2d-\nu}} = (t_0 + b_0 A) + b_0 B R^{\frac{2D}{2d-\nu} - d} . \]  

(626)

Identifying the difference in temperature to the critical theory as

\[ \bar{t} = t_0 + b_0 A , \]  

(627)

the critical theory is approached upon taking \( \bar{t} \to 0 \) and \( R \to \infty \). This occurs if and only if \( d \) is larger than the lower critical dimension

\[ d > d_l = \frac{2D}{2-D} . \]  

(628)

If \( d \) is in addition smaller than the upper critical dimension, i.e.

\[ d < d_u = \frac{4D}{2-D} , \]  

(629)

the left hand side of Eq. (626) vanishes faster than the \( R \)-dependent term on the right hand side, and we obtain the scaling relation

\[ R \sim \bar{t}^{-1/(d-\frac{2D}{2d-\nu})} . \]  

(630)
In the large $N$ limit, the exponent $\nu^*$ is therefore given by

$$\nu_{N \to \infty}^* = \frac{D}{d - \frac{2D}{2-D}}. \quad (631)$$

We can verify that the standard result[3] is correctly reproduced for $D = 1$ as

$$\nu_{N \to \infty}^*(D = 1) = \frac{1}{d - 2}. \quad (632)$$

Note that for $d > d_u$, the leading behavior from Eq. (626) is

$$R \sim \bar{t}^{-\frac{2 + D}{2D}}. \quad (633)$$

implying the free theory result

$$\nu = \frac{2 - D}{2}. \quad (634)$$

### 13.7 Some more applications

In this section, we briefly discuss some possible applications.

As explained in section 7, to extract the physically relevant $O(N)$ exponent for $D = 1$, one has the freedom to expand Eq. (619) about any point $(D_0, d_0)$ on the critical curve $\epsilon(D_0, d_0) = 0$, see figure 12. As discussed in section 7 and depicted in Fig. 34, the resulting extrapolation for $\nu^*$ varies with the extrapolation point. (We have found the extrapolation for $\nu^* d$ (see section 7) to be the best converging one, and chosen it for the example.) Guided by previous results for polymers and membranes[25], the criterion for selecting a particular value from such curves is that of minimal sensitivity to the expansion point, and we thus evaluate $\nu^*$ at the extrema. The broadness of the extremum then provides a measure
Figure 35: Test of Eq. (635) for Ising models in \( d = 2.25, 2.5, 2.75 \) and 3. The upper curves are from the standard extrapolation \((D = 1)\), while the lower curves are from the dual description \((D = d - 1)\). The exponent \( \nu^* \) is estimated from the maximum of each curve, which are obtained by extrapolating \( \nu^* D \) with \( c(D) = D \), linearizing in \( N \), and dividing the result by \( dD \).

for the quality of the result, and the expansion scheme. Although we examined several such curves, only a selection is reproduced in Fig. 34. Our results are clearly better than the standard 1-loop expansion of \( \nu^* = 1/2 + (N + 2)/[4(N + 8)] \).

In analogy to tethered membranes [38], we expect the above expansion scheme to be better controlled than the traditional \( \varepsilon \)-expansion. (The \( \varepsilon \)-expansion should become quasi-convergent for \( D \to 2 \).) However, since the exponents of the \( O(N) \) model are already known to high accuracy, the generalization to \( O(N, D) \) is valuable if it offers insights beyond the standard field theory. Furthermore, the scheme will have limited appeal, if it cannot be extended to other types of field theories. In the rest of this section we shall demonstrate that: (a) The model provides insights about the boundaries of droplets at criticality in Ising models. (b) A generalized manifold model is constructed with cubic anisotropy, which exhibits a reverse Coleman–Weinberg mechanism not present in standard field theory. Furthermore, it provides us with a 1-loop fixed point for the random bond Ising model.

For the Ising model \((N = 1)\), a different geometrical description is obtained from a low temperature expansion: Excitations to the uniform ground state are droplets of spins of opposite sign. The energy cost of each droplet is proportional to its boundary, i.e. again weighted by a Boltzmann factor of \( e^{-\Omega} \). Thus, a low temperature series for the \( d \)-dimensional Ising partition function is obtained by summing over closed surfaces of dimension \( D = d - 1 \). For \( d = 2 \), the high and low temperature series are similar, due to self-duality. For \( d = 3 \), the low temperature description is a sum over surfaces. What types of surfaces dominate the above sum? Since there is no constraint on the internal metric, it may be appropriate to examine fluid membranes. However, there is no practical scheme for treating interacting fluid membranes, and the excluded volume interactions are certainly essential in this case. Configurations of a single surface for \( N = 0 \), self-avoiding or not, are dominated by tubular shapes (spikes) which have very large entropy[229]. Such “branched polymer” configurations are very different from tethered surfaces. However, for \( N \neq 0 \), it may be entropically advantageous to break up a singular spike into a string of many bubbles. If so, describing the collection of bubbles by fluctuating hyper-spherical (tethered) manifolds may not be too off the mark. To test this conjecture, we compare the predictions of the dual high and low-temperature descriptions.

Singularities of the partition function are characterized by the specific-heat critical exponent \( \alpha(D, d, N) \), or (using hyper-scaling) through \( \ln Z_{\text{singular}} \sim |t - t_c|^{\nu^* d/D} \). The equality of the singularities on approaching the critical point from low or high temperature sides, leads to a putative identity

\[
\nu^*(1, d, 1) = \frac{\nu^*(d - 1, d, 1)}{d - 1}.
\]
Numerical tests of the conjecture in Eq. (635) are presented in Fig. 35. The extrapolated exponents (the maxima of the curves) from the dual expansions are in excellent agreement. Nevertheless, higher-loop calculations would be useful to check this surprising hypothesis.

The simplest extension of the $O(N)$ model breaks the rotational symmetry by inclusion of cubic anisotropy[147]. In the field theory language, cubic anisotropy is represented by a term $u \sum_i \phi_i^4$, in addition to the usual interaction of $b \sum_{ij} \phi_i^2 \phi_j^2$. In the geometric prescription of high temperature expansions, the anisotropic coupling $u$ acts only between membranes of the same color, while the interaction $b$ acts
irrespective of color. Stability of the system of colored membranes places constraints on possible values of $b$ and $u$. To avoid collapse of the system, energetic considerations imply that if $u < 0$, the condition $u + b > 0$ must hold, while if $u > 0$, we must have $u + Nb > 0$ [147, 34]. These stability arguments may be modified upon the inclusion of fluctuations: In the well-known Coleman-Weinberg mechanism [147], the RG flows take an apparently stable combination of $b$ and $u$ into an unstable regime, indicating that fluctuations destabilize the system. In the flow diagrams described below, we also find the reverse behavior in which an apparently unstable combination of $b$ and $u$ flows to a stable fixed point. We interpret this as indicating that fluctuations stabilize the model, a reverse Coleman-Weinberg effect, which to our knowledge is new. We have shaded in grey, the unphysical regions in the flow diagrams of Fig. 36. As in their $O(N)$ counterpart, the RG equations admit 4 fixed points: The Gaussian fixed point with $b^*_G = u^*_G = 0$; the Heisenberg fixed point located at $b^*_H \neq 0, u^*_H = 0$; the Ising fixed point with $b^*_I = 0, u^*_I \neq 0$; and the cubic fixed point at $b^*_c \neq 0, u^*_c \neq 0$. Furthermore, as depicted in Fig. 36, there are six different possible flow patterns. In the $O(N)$ model, the flows in (i) and (ii) occur for $N < 4$ and $N > 4$, respectively. The other patterns do not appear in the standard field theory, as is apparent from their domain of applicability in the $(N, D)$-plane on Fig. 36. Note that there are two stable fixed points in three out of these four cases.

The $N \to 0$ limit of the above models is interesting, not only because of its relevance to self-avoiding polymers and membranes, but also for its relation to the Ising model with bond disorder. The latter connection can be shown by starting with the field theory description of the random bond Ising model, replicating it $N$ times, and averaging over disorder [230]. The replicated system is controlled by a Hamiltonian with positive cubic anisotropy $u$, but negative $b = -\sigma$ ($\sigma$ is related to the variance of bond disorder). From the “Harris criterion” [231], new critical behavior is expected for the random bond Ising system. But in the usual field theory treatments [230], there is no fixed point at the 1-loop order. In our generalized model, this

Figure 37: Extrapolations of $\nu^*$ from the expansion of $\nu^* d$ with $c(D) = D$, for the $O(N)$ model in $d = 3$. Exponents at the Heisenberg fixed point for $N = 0$ are compared to those of the Ising and cubic fixed points. The crossing of the latter curves yields an estimate of $\nu^* = 0.6315$ for the $3d$ Ising model.
Large orders

is just the borderline between cases (i) and (iii). However, we now have the option of searching for a stable fixed point by expanding about any $D \neq 1$. Indeed, for $N = 0$ and $1 < D < 1.29$, the cubic fixed point lies in the upper left sector ($u > 0$ and $b < 0$) and is completely stable, as in flow pattern (iii).

The extrapolation for $\nu^*$ at the cubic fixed point is plotted in Fig. 37, where it is compared to the results for the Heisenberg and Ising fixed points. The divergence of $\nu^*$ on approaching $D = 1$ from above, is due to the cubic fixed point going to infinity as mentioned earlier. Upon increasing $D$, the Ising and cubic fixed points approach, and merge for $D = 1.29$. For larger values of $D$, the cubic fixed point is to the right of the Ising one ($b^*_{c} > 0$), and only the latter is stable. Given this structure, there is no plateau for a numerical estimate of the random bond exponent $\nu^*_{DO}$, and we can only posit the inequality $\nu^*_{DO} > \nu^*_{Ising}$, which is also a consequence of an exact argument [232]. While this is derived at 1-loop order, it should also hold at higher orders since it merely depends on the general structure of the RG flows. One may compare this to four loop calculations of the random bond Ising model [233], which are consistent with $\alpha = 0$, i.e. at the border-line of the Harris criterion[231], with $\nu^* = 2/3$.

14 Large orders

If perturbative calculations are simple at first order, as discussed in section 6, they present considerable difficulties at second order, and require a lot of analytical and numerical work.

An important issue is to understand if these calculations make sense beyond perturbation theory, or if non-perturbative effects destroy the consistency of the approach. A first step is to understand the large order behavior of perturbation theory. In this section we shall formulate the problem of the large order behavior for the Edwards model, in a way which is directly applicable both to polymers and to membranes.

Using the formulation of the self-avoiding membrane model (SAM model) as a model of a “phantom” membrane (without self-avoidance) in a random imaginary external potential $V$, we show that the large orders are controlled by a real classical configuration for this potential $V$, which is the analog for SAM of the instanton for $\phi^4$-theory [36, 37, 234]. This “SAM instanton” potential $V$ is the extremum of a non-local functional $S[V]$, which cannot be calculated exactly. We obtain the general form for the asymptotics of the term of order $n$, which is

$$n^{d/2} (-C)^n (n!)^{1-\varepsilon/D},$$

where as usual $D$ is the internal dimension of the membrane, $d$ the dimension of bulk space, $\varepsilon = 2D - d(2-D)/2$ the engineering dimension for self-avoidance and $C$ a positive constant depending on $D$ and $\varepsilon$ (or $d$). This behavior is universal: the constant $C$ obtained from the instanton does not depend on the internal shape or topology of the membrane.

This section closely follows [38], where the problem was first addressed.

14.1 Large orders and instantons for the SAM model

We consider a $D$-dimensional manifold $\mathcal{M}$ with size $L$ and volume $V = L^D$ (typically the $D$-dimensional torus $T_D = [0, L]^D$) in $d$-dimensional Euclidean bulk space. The partition function is

$$\mathcal{Z}(b; L) = \int \mathcal{D}[r] e^{-\mathcal{H}[r; b, L]}$$

with the Hamiltonian

$$\mathcal{H}[r; b, L] = \int_L d^Dx \left( \frac{1}{2} (\nabla r(x))^2 + \frac{b}{2} \int_L d^Dx \int_L d^Dy \delta^d(r(x) - r(y)) \right).$$

(638)
\[ b > 0 \] is the repulsive 2-point interaction coupling which describes self-avoidance, and \( r \) throughout this section the \( \text{bare} \) field. Also note the difference in normalizations to Eq. (61); rescaling the field \( r(x) \to r(x)\sqrt{(2 - D)S_D} \) in Eq. (61), \( b \) and \( b_0 \) are related by

\[
b = b_0 \frac{2}{S_D^2} \left( \frac{4\pi}{(2 - D)S_D} \right)^{d/2}. \tag{639}\]

The functional integration measure \( D[r] = \prod_x d^d r(x)/Z_0 \) is normalized such that the partition function of the free Gaussian manifold \( Z(b = 0; L) = 1 \).

By dimensional analysis, the partition function (637) only depends on the dimensionless coupling constant

\[ g = b L^\varepsilon \tag{640} \]

via

\[
Z(b; L) = Z(g; L = 1) = Z(g) \tag{641}
\]

and is defined as a series

\[ Z(g) = \sum_{n=0}^{\infty} z_n g^n. \tag{642} \]

Of course, \( Z(g) \) also depends on the shape of the manifold \( \mathcal{M} \).

Let us assume that \( Z(g) \) is analytic around the origin for \( -\pi < \arg(g) < \pi \), and has a discontinuity along the negative real axis. This assumption is natural, since for \( g < 0 \), the membrane is collapsed and perturbation expansion is performed around an unstable classical state. Then we can write \( z_n \) as a dispersion integral

\[
z_n = \int \frac{dg}{2i\pi} g^{-n+1} Z(g) = \int_{-\infty}^{0} \frac{dg}{\pi} g^{-n+1} \text{Im}(Z(g + i0^+)). \tag{643} \]

To obtain the behavior for large \( n \), it turns out that it is sufficient to evaluate the integral in (643) in a saddle point approximation. Indeed, we shall show that, at least for \( 0 < \varepsilon < D \), the integral is at large \( n \) dominated by the discontinuity of \( Z(g) \) at small negative \( g \). Moreover, \( Z(g) \) is dominated by a saddle point, when re-expressed as a functional integral over properly defined auxiliary fields.

The Hamiltonian (638) is non-local and involves a distribution of the field \( r \). It is convenient to rewrite the path-integral as

\[
Z(b; L) = \int D[r] \int D[V] \, e^{-\mathcal{H}'[r,V;b,L]} \tag{644} \]

with the new effective Hamiltonian

\[ \mathcal{H}'[r,V;b,L] = \int L d^D x \left( \frac{1}{2}(\nabla r(x))^2 + V(r(x)) \right) - \frac{1}{2b} \int d^d r \, V(r)^2. \tag{645} \]

This representation is nothing but the generalization of the well known formulation of the Edwards model as a model of free random walks in an (imaginary) annealed random potential. As above, \( Z \) is a function of the dimensionless coupling \( g \) and we replace \( b \to g \) and \( L \to 1 \) as in Eqs. (640) and (641).

As argued before, we aim at calculating the partition function for small negative \( g \). For that purpose, it is convenient to rescale the coordinates and the potential \( V(r) \)

\[
x \to (-g)^{\frac{1}{D-\varepsilon}} x , \quad r \to (-g)^{\frac{2}{(D-\varepsilon)}} r , \quad V \to (-g)^{\frac{D}{D-\varepsilon}} V , \tag{646} \]

so that we now consider a membrane with size \( \bar{L} = (-g)^{\frac{1}{D-\varepsilon}} \) and volume

\[ \bar{V} = L^D = (-g)^{\frac{D}{D-\varepsilon}}. \tag{647} \]
This yields the rescaled Hamiltonian
\[ H'_{\text{resc}}[r, V; \bar{L}] = \int L d^p x \left( \frac{1}{2} (\nabla r(x))^2 + V(r(x)) \right) - \frac{\bar{V}}{2} \int d^d r \, V(r)^2. \] (648)

The integral over \( r \) for fixed potential \( V \) defines the free energy density \( \mathcal{E}[V] \) of a “phantom” (i.e. non self-avoiding) membrane in the external potential \( V \)
\[ e^{-\bar{V} \mathcal{E}[V]} = \int \mathcal{D}[r] e^{-\int L d^p x \frac{1}{2} (\nabla r(x))^2 + V(r(x))}. \] (649)

The partition function finally becomes
\[ Z(g) = \int \mathcal{D}[V] e^{-\bar{V} [\mathcal{E}[V] + \frac{1}{2} \int d^d r \, V(r)^2]}. \] (650)

The crucial point of this formulation is that according to Eq. (647), as long as
\[ 0 < \varepsilon < D, \] (651)
the limit \( g \to 0^- \) corresponds to the thermodynamic limit when the volume \( \bar{V} \to \infty \). In this limit the free energy density \( \mathcal{E}[V] \) has a finite limit so that the volume appears only as a global prefactor in the exponential of (650). Hence in the large \( \bar{V} \) limit the integral (650) is dominated by a saddle point \( V_{\text{inst}} \), which is an extremum of the effective energy \( S[V] \) for an infinite and flat membrane. The latter is defined as
\[ S[V] = \mathcal{E}[V] + \frac{1}{2} \int d^d r \, V(r)^2, \] (652)
where \( \mathcal{E}[V] \) is defined in (649) as the free energy density of an infinite flat membrane in the potential \( V \). This saddle point \( V_{\text{inst}}(r) \) is the non-trivial instanton, since the action \( S \) of the trivial extremum \( V(r) = 0 \) is real and does not contribute to the discontinuity of \( Z(g) \). Moreover, as the instanton is obtained through the thermodynamic limit \( \bar{L} \to \infty \), it is independent of the shape of the initial membrane. This implies that the large order behavior of perturbation theory is universal, and does not depend on the internal geometry of the membrane. Let us now derive the saddle-point equations:

The variation of the free energy density is in general
\[ \frac{\delta \mathcal{E}[V]}{\delta V(r)} = \langle \delta[r] \rangle\big|_V, \] (653)
where \( \delta[r] \) is the normalized density of the membrane
\[ \delta[r] = \rho[r] = \frac{1}{V} \int d^p x \, \delta^d(r - r(x)) \] (654)
(which has a finite limit when the volume becomes infinite), and \( \langle \rangle\big|_V \) denotes the expectation value for the phantom membrane in the potential \( V \), as defined in Eq. (649). Hence extremizing \( S[V] \) leads to the variational equation for the instanton potential \( V_{\text{inst}} \)
\[ 0 = \langle \delta[r] \rangle\big|_{V_{\text{inst}}} + V_{\text{inst}}(r). \] (655)

Let us postpone the solution of Eq. (655) and first ask what the consequences of the existence of an instanton for the large order behavior are. Denoting by \( S_{\text{inst}} \) the action for the instanton \( S[V_{\text{inst}}] \), we deduce from Eqs. (647) and (649) that for small negative \( g \), the discontinuity of \( Z(g) \) behaves as
\[ \text{Im}(Z(g)) \sim \exp \left[ -(-g)^{\frac{D}{D - 1}} S_{\text{inst}} \right] \] (656)
and the integral representation for \( z_n \) (643) can be evaluated by the saddle point method at large \( n \). This saddle point is at
\[
g_c = - \left[ \frac{S_{\text{inst}}}{n(1 - \varepsilon/D)} \right]^{1-\varepsilon/D}
\]
and replacing the integral in (643) by its value at \( g_c \) gives the large \( n \) behavior at leading order
\[
z_n \approx (-C)^n (n!)^{1-\varepsilon/D}, \quad C = \left[ \frac{1 - \varepsilon/D}{S_{\text{inst}}} \right]^{1-\varepsilon/D}.
\]

Let us briefly discuss this result. For \( 0 \leq \varepsilon < D \), perturbation theory is divergent with alternating signs. For \( \varepsilon = 0 \), one recovers the typical factorial behavior \((-C)^n n!\) of field theories, provided that \( S_{\text{inst}} \) remains UV finite. As we shall see in the next subsection, our result (658) coincides for \( D = 1 \) with the large order behavior deduced from the \( \phi^4 \) model with \( n = 0 \) components. The reasoning seems to break down at \( \varepsilon = D \), but we shall see that in fact the factor of \( C \), when considered as a function of \( D \) and \( \varepsilon \), is regular at \( \varepsilon = D \) and can be continued to the region \( \varepsilon \geq D \). Thus, the asymptotics (658), although derived for \( 0 < \varepsilon < D \), is valid in the whole physical domain \( 0 < \varepsilon < 2D \). A more rigorous argument is as follows: Eqs. (656) and (657) are still valid for \( \varepsilon > D \); the instanton then governs the behavior of the discontinuity of \( Z(g) \) at large \( g \). This means that the saddle point of Eq. (658) for large \( n \) now is at large negative \( g \).

To go beyond this estimates, one must (i) compute the instanton and its action, and (ii) integrate the fluctuations around the instanton in (650). If one assumes that this calculation goes along the same lines as in standard field theory, one must first isolate the zero modes, i.e. the collective coordinates of the instanton. As we shall see later, the instanton \( V_{\text{inst}} \) is rotationally invariant and is characterized by its position in \( d \)-dimensional space only. Thus it has \( d \) zero modes, each of them gives a factor of \( V^{1/2} \) (by a standard collective coordinates argument), and the remaining fluctuations \( \delta_{\perp}V \) (orthogonal to the translational variations \( \delta_{\parallel}V \sim \frac{\partial V_{\text{inst}}}{\partial r} \)) give a finite determinant \( \mathcal{A} \). Therefore we expect the semiclassical estimate for the discontinuity to be
\[
\text{Im}(Z(g)) \approx \mathcal{A}^{-1/2} \bar{V}^{d/2} e^{-\bar{V} S_{\text{inst}}}
\]
and that the large \( n \) behavior is more precisely
\[
z_n = \mathcal{A}' n^{d/2} (-C)^n (n!)^{1-\varepsilon/D} \left[ 1 + \ldots \right].
\]

Finally we shall see that the action of the instanton remains finite in the limit \( \varepsilon \to 0 \). As in standard \( \phi^4 \) theory, one expects UV divergences to appear only for fluctuations around the instanton, and that these divergences are canceled by the same renormalizations as in perturbation theory. This would imply that our large order estimate (658) is also valid for the renormalized theory at \( \varepsilon = 0 \), in particular for the renormalization group functions which enter into the \( \varepsilon \)-expansion of the scaling exponents. Renormalization however has to be taken into account when evaluating the constant \( \mathcal{A}' \) in (660).

### 14.2 The polymer case and physical interpretation of the instanton

Before discussing membranes, let us study in detail the special case \( D = 1 \), where the model reduces to the Edwards model for polymers. Using the well known mapping between the problem of a Brownian walk in a potential \( V(r) \) and quantum mechanics of a single particle in the same potential, the free energy density \( \mathcal{E}[V] \) of a linear chain fluctuating in a potential \( V(r) \) is in the thermodynamic limit given by the lowest eigenvalue \( E_0 \) of the operator
\[
H = -\frac{\Delta}{2} + V(r),
\]

Let us briefly discuss this result. For \( 0 \leq \varepsilon < D \), perturbation theory is divergent with alternating signs. For \( \varepsilon = 0 \), one recovers the typical factorial behavior \((-C)^n n!\) of field theories, provided that \( S_{\text{inst}} \) remains UV finite. As we shall see in the next subsection, our result (658) coincides for \( D = 1 \) with the large order behavior deduced from the \( \phi^4 \) model with \( n = 0 \) components. The reasoning seems to break down at \( \varepsilon = D \), but we shall see that in fact the factor of \( C \), when considered as a function of \( D \) and \( \varepsilon \), is regular at \( \varepsilon = D \) and can be continued to the region \( \varepsilon \geq D \). Thus, the asymptotics (658), although derived for \( 0 < \varepsilon < D \), is valid in the whole physical domain \( 0 < \varepsilon < 2D \). A more rigorous argument is as follows: Eqs. (656) and (657) are still valid for \( \varepsilon > D \); the instanton then governs the behavior of the discontinuity of \( Z(g) \) at large \( g \). This means that the saddle point of Eq. (658) for large \( n \) now is at large negative \( g \).

To go beyond this estimates, one must (i) compute the instanton and its action, and (ii) integrate the fluctuations around the instanton in (650). If one assumes that this calculation goes along the same lines as in standard field theory, one must first isolate the zero modes, i.e. the collective coordinates of the instanton. As we shall see later, the instanton \( V_{\text{inst}} \) is rotationally invariant and is characterized by its position in \( d \)-dimensional space only. Thus it has \( d \) zero modes, each of them gives a factor of \( V^{1/2} \) (by a standard collective coordinates argument), and the remaining fluctuations \( \delta_{\perp}V \) (orthogonal to the translational variations \( \delta_{\parallel}V \sim \frac{\partial V_{\text{inst}}}{\partial r} \)) give a finite determinant \( \mathcal{A} \). Therefore we expect the semiclassical estimate for the discontinuity to be
\[
\text{Im}(Z(g)) \approx \mathcal{A}^{-1/2} \bar{V}^{d/2} e^{-\bar{V} S_{\text{inst}}}
\]
and that the large \( n \) behavior is more precisely
\[
z_n = \mathcal{A}' n^{d/2} (-C)^n (n!)^{1-\varepsilon/D} \left[ 1 + \ldots \right].
\]

Finally we shall see that the action of the instanton remains finite in the limit \( \varepsilon \to 0 \). As in standard \( \phi^4 \) theory, one expects UV divergences to appear only for fluctuations around the instanton, and that these divergences are canceled by the same renormalizations as in perturbation theory. This would imply that our large order estimate (658) is also valid for the renormalized theory at \( \varepsilon = 0 \), in particular for the renormalization group functions which enter into the \( \varepsilon \)-expansion of the scaling exponents. Renormalization however has to be taken into account when evaluating the constant \( \mathcal{A}' \) in (660).
where $\Delta$ is the Laplacian in $d$ dimensions. Thus we have

$$E[V] = E_0.$$  

(662)

Denoting by $\Psi_0(r)$ the ground state wave-function, and using Eq. (655) and the standard result from first order perturbation theory

$$\langle \delta[r] \rangle_V = \frac{\delta E_0[V]}{\delta V(r)} = \langle \Psi_0 | \frac{\delta H}{\delta V(r)} | \Psi_0 \rangle = |\Psi_0(r)|^2,$$  

(663)

we obtain the instanton potential

$$V_{\text{inst}}(r) = - (\Psi_0(r))^2.$$  

(664)

The eigenvalue equation $H \Psi_0 = E_0 \Psi_0$ becomes non-linear:

$$\frac{1}{2} \Delta \Psi_0 + E_0 \Psi_0 + \Psi_0^3 = 0.$$  

(665)

Since $\Psi_0$ obeys the normalization condition

$$\|\Psi_0\|^2 = \int d^d r |\Psi_0(r)|^2 = 1,$$  

(666)

the wave function $\Psi_0$ and the ground state energy $E_0$ are fully determined by Eqs. (665) and (666). Eq. (665) has nontrivial normalizable solutions for $2 < d < 4$ and $E_0 < 0$. In addition, the ground state $\Psi_0$ is rotationally symmetric, i.e. does not vanish at finite $r$. The action for the instanton (652) finally reads

$$S_{\text{inst}} = E_0 + \frac{1}{2} \int d^d r \Psi_0^4.$$  

(667)

To make contact with the instanton analysis in the Landau Ginzburg Wilson (LGW) $\phi^4$-theory with $n = 0$ components, note that Eqs. (665)-(666) hold \textit{if and only if} $\Psi_0$ and $E_0$ are extrema of the action

$$S'[\Psi, E] = E + \int d^d r \left[ \frac{1}{2} (\nabla \Psi)^2 - E \Psi^2 - \frac{1}{2} \Psi^4 \right].$$  

(668)

This is the standard Landau Ginzburg Wilson action with negative coupling associated to $\Psi^4$ and mass $m^2 = -2E$. Moreover, \textit{at the extrema}, the two actions are equal:

$$S_{\text{inst}}[\Psi_0, E_0] = S'[\Psi_0, E_0].$$  

(669)

The relation becomes clearer by the change of variables

$$\Psi(r) = \left( \frac{-2E}{4-d} \right)^{\frac{1}{2}} \phi \left( \left( \frac{-2E}{4-d} \right)^{\frac{1}{2}} r \right).$$  

(670)

The action $S'$ then reads

$$S'[\Psi, E] = E + \left( \frac{-2E}{4-d} \right)^{2-\frac{d}{2}} S_{\text{LGW}}[\phi]$$  

(671)

with

$$S_{\text{LGW}}[\phi] = \int d^d r \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{4-d}{2} \phi^2 - \frac{1}{2} \phi^4 \right].$$  

(672)
We can extremize (671) with respect to $E$ and $\phi$ independently, and denoting by $\phi_0$ and $E_0$ these extremizing solutions, we get

$$E_0 = \left(\frac{d}{2} - 2\right) S_{\text{LGW}}[\phi_0]^{\frac{1}{\sqrt{d^2 - 1}}}.$$

(673)

The change of variables in Eq. (670) was constructed such that the instanton action takes the simple form

$$S_{\text{inst}} = S'[^{d}_{\text{LGW}}[\phi_0]^{\frac{1}{\sqrt{d^2 - 1}}}.$$

(674)

Since for polymers ($D = 1$) $d/2 - 1 = 1 - \varepsilon/D$, we can use Eq. (658) to write the large order constant $\mathcal{C}$ of the Edwards model as

$$\frac{1}{\mathcal{C}} = S_{\text{LGW}}[\phi_0].$$

(675)

This result could have been derived directly from the standard field theoretical formulation of the Edwards model as a $n = 0$ component $(\phi^2)^2$ model.

The equation for the instanton derived from the action (672) admits a regular solution $\phi_0(|\vec{r}|)$ for any $0 \leq d \leq 4$, so that nothing special occurs at the point $d = 2$ (i.e. $\varepsilon = D = 1$) as one might have expected from Eq. (658). Let us note that since the “mass” in Eq. (672) is equal to $4 - d$, it is positive for $d < 4$ but vanishes at the critical dimension $d = 4$, so that the instanton solution $\phi_0$ still exists for $d = 4$. In Fig. 38 we plot $C^{-1}(d)$ for $0 \leq d \leq 4$, as obtained from numerical integration. Note that for $d > 4$, no solution for the instanton with finite action exists.

It is interesting to give a physical interpretation of the instanton for the Edwards model, since this interpretation is the same for membranes with $D \neq 1$. Let us first recall the standard interpretation of the instanton for the LGW model with action (672), i.e. negative $\phi^4$-coupling. The classical false vacuum $\Psi(r) = 0$ is separated from the true vacua $\Psi(r) = \mp \infty$ by a finite barrier. The instanton solution $\Psi_0$ describes a metastable droplet of true vacuum (with $\Psi_0(r) \neq 0$ inside the droplet) in the false vacuum, which is on the verge to nucleate. Indeed, if the droplet is slightly larger, the positive surface energy dominates and the droplet shrinks and finally vanishes, while if it is slightly smaller, the negative volume energy dominates and the droplet expands.
Consider the energy density $S[V]$ given by Eq. (652). It corresponds to the total free energy of a polymer globule trapped in the potential well $V(r) < 0$, where this effective potential results from the attractive 2-point interaction between elements of the polymer (since we are at negative coupling, $b < 0$). To see how $S$ varies with the average radius of gyration of the polymer, it is convenient to consider the following scale transformation on $V$

$$V(r) \rightarrow V_\lambda(r) = \lambda^{2D/\eta} V(\lambda r) .$$  

(676)

Simple dimensional analysis shows that under (676)

$$\mathcal{E}[V] \rightarrow \lambda^{2D/\eta} \mathcal{E}[V] , \quad \int V^2 \rightarrow \lambda^{2\varepsilon/\eta} \int V^2$$

(677)

(here $D = 1$ and $\varepsilon = 2 - d/2$). As long as $\varepsilon < D$, and for large $\lambda$, i.e. when shrinking the polymer globule, it is the first term $\mathcal{E} < 0$ which dominates and the total free energy $S$ becomes large and negative; while for small $\lambda$, i.e. when expanding the globule, it is the second term on the r.h.s. of (652) which is larger than 0, and which dominates. Thus in this mean field picture, i.e. neglecting thermal fluctuations around the instanton, large globules tend to expand, while small globules tend to collapse. This has a simple physical interpretation: the polymer trapped in its own potential is subject to two opposite forces, (i) attractive forces between its elements which would like to make the polymer collapse, (ii) entropic repulsion which exerts a pressure on the well and would like to expand the polymer (until it becomes a free random walk). What our calculation implies is the simple fact that for large radius (i.e. small $\lambda$) entropic repulsion dominates, while at small radius (large $\lambda$) attraction dominates and the polymer collapses. Thus the instanton solution describes a polymer with attractive interactions on the verge to collapse into its dense (and most stable) phase; this is similar to the instanton in the LGW theory which describes a bubble of true vacuum on the verge to nucleate and to destroy the false vacuum.

### 14.3 Gaussian variational calculation

For $D \neq 1$ (and in general for $0 < D < 2$ non-integer) we know of no exact method to calculate the instanton. A simple and natural approximation is the variational method, i.e. the Hartree approximation.

To evaluate the free energy density $\mathcal{E}[V]$ of the free, i.e. non-interacting membrane in a potential $V$, and described by the Hamiltonian

$$\mathcal{H}_V = \int d^D x \left( \frac{1}{2} (\nabla r)^2 + V(r) \right) ,$$

(678)

we introduce the trial Gaussian Hamiltonian

$$\mathcal{H}_{\text{var}} = \int d^D x \int d^D y \frac{1}{2} r(x) K(x - y) r(y)$$

$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \tilde{r}(k) \tilde{K}(k) \tilde{r}(-k) ,$$

(679)

where $\tilde{\cdot}$ denotes the Fourier transform. The free energy for the trial Hamiltonian is

$$\mathcal{E}_{\text{var}} = -\frac{1}{V} \ln \left[ \int \mathcal{D}[r] e^{-\mathcal{H}_{\text{var}}} \right]$$

$$= \frac{d}{2} \int \frac{d^D k}{(2\pi)^D} \ln \left[ \tilde{K}(k)/k^2 \right] ,$$

(680)
and the factor of $1/k^2$ comes from the normalization of the measure $\mathcal{D}[r]$ taken such that $\mathcal{E}[V = 0] = 0$. $\mathcal{V}$ is the total volume of the membrane.

The Hartree Fock approximation amounts to replacing $\mathcal{E}[V]$ by the best variational estimate $\mathcal{E}_{\text{var}}[V]$

$$\langle \rangle \leq \mathcal{E}_{\text{var}}[V] = \mathcal{E}_{\text{var}} + \frac{1}{\mathcal{V}} \langle \mathcal{H}_\mathcal{V} - \mathcal{H}_{\text{var}} \rangle_{\text{var}} .$$  (681)

$\langle \rangle_{\text{var}}$ denotes the average with respect to the trial Hamiltonian $\mathcal{H}_{\text{var}}$ and one must look for the trial Hamiltonian $\mathcal{H}_{\text{var}}$ (i.e. the kernel $K$) which minimizes $\mathcal{E}_{\text{var}}[V]$. Denote by $\tilde{V}(p)$ the Fourier transform of the potential $V(r)$. Since the variational Hamiltonian is Gaussian, it is easy to compute the second term on the r.h.s. of (681), $\mathcal{V}^{-1} \langle \mathcal{H}_\mathcal{V} - \mathcal{H}_{\text{var}} \rangle_{\text{var}}$ in the infinite volume limit:

$$\langle V(r(0)) \rangle_{\text{var}} + \frac{1}{2} \left[ \langle \nabla r(0) \rangle_{\text{var}}^2 - \int d^Dx \ K(x) \langle r(x)r(0) \rangle_{\text{var}} \right]$$

$$= \int \frac{d^dp}{(2\pi)^d} \tilde{V}(p) \langle e^{ipr(0)} \rangle_{\text{var}} + \int \frac{d^Dk}{(2\pi)^D} \frac{k^2 - \bar{K}(k)}{2} \langle \bar{r}(k)^2 \rangle_{\text{var}}$$

$$= \int \frac{d^dp}{(2\pi)^d} \tilde{V}(p) \exp \left[ -\frac{p^2}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\bar{K}(k)} \right] + \frac{d}{2} \int \frac{d^Dk}{(2\pi)^D} \left( \frac{k^2}{\bar{K}(k)} - 1 \right) .$$  (682)

Combining Eqs. (652), (680) and (682), we finally obtain the variational estimate for the total energy of the instanton

$$\mathcal{S}_{\text{var}}[V] = \mathcal{E}_{\text{var}}[V] + \frac{1}{2} \int d^dr \ V(r)^2$$

$$= \int \frac{d^dp}{(2\pi)^d} \left[ \tilde{V}(p) \exp \left[ -\frac{p^2}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\bar{K}(k)} \right] + \frac{1}{2} \tilde{V}(p) \tilde{V}(-p) \right]$$

$$+ \frac{d}{2} \int \frac{d^Dk}{(2\pi)^D} \left( \ln \left[ \frac{\bar{K}(k)}{k^2} \right] + \frac{k^2}{\bar{K}(k)} - 1 \right) .$$  (683)

We now extremize Eq. (683) both with respect to $\bar{K}$ (variational approximation) and with respect to $\tilde{V}$ (to obtain the instanton solution). Extremizing w.r.t. $\bar{K}(k)$ yields the equation

$$\bar{K}(k) = k^2 - \frac{1}{d} \int \frac{d^dp}{(2\pi)^d} \ p^2 \tilde{V}(p) \exp \left[ -\frac{p^2}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\bar{K}(k)} \right] ,$$  (684)

which implies that the variational Hamiltonian depends just on a mass $m_{\text{var}}$

$$\bar{K}(k) = k^2 + m_{\text{var}}^2 .$$  (685)

Extremizing Eq. (683) w.r.t. $\tilde{V}(p)$ gives

$$\tilde{V}^{\text{var}}(p) = - \exp \left[ -\frac{p^2}{2} A \right]$$  (686)

with

$$A = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\bar{K}(k)} = m_{\text{var}}^{D-2} \frac{\Gamma(1 - \frac{D}{2})}{(4\pi)^{D/2}} .$$  (687)

$\Gamma$ is Euler’s Gamma function. Thus, in the variational approximation the instanton potential is Gaussian. Inserting Eq. (686) into Eq. (684) yields the self-consistent equation for $m_{\text{var}}$

$$m_{\text{var}}^2 = \frac{1}{d} \int \frac{d^dp}{(2\pi)^d} \ p^2 e^{-p^2 A} = \frac{1}{2} (4\pi)^{-d/2} A^{-1-d/2} .$$  (688)
We finally get in terms of $D$, $\varepsilon$ and $d = 2(2D - \varepsilon)/(2 - D)$

$$m_{\text{var}} = \sqrt{4\pi} \left[ 2 \Gamma\left(\frac{2-D}{2}\right)^{1+\frac{D}{D-\varepsilon}} \right].$$

(689)

The final result for $A$ reads

$$A = \frac{1}{4\pi} \Gamma\left(\frac{2-D}{2}\right)^{\frac{D}{D-\varepsilon}} 2^{\frac{D}{D-\varepsilon}}.$$  

(690)

We can now insert these results into Eq. (683), and after straightforward calculations get the variational instanton action

$$S_{\text{var}} = S_{\text{var}}[V_{\text{var}}] = \left(1 - \frac{\varepsilon}{D}\right) \left[ 2 \Gamma\left(\frac{2-D}{2}\right)^{\frac{D}{D-\varepsilon}} \right].$$

(691)

The corresponding variational estimate for the large order constant $C$ defined by Eq. (658) is

$$\frac{1}{C_{\text{var}}} = 2 \Gamma\left(\frac{2-D}{2}\right)^{\frac{D}{D-\varepsilon}}.$$  

(692)

As claimed in the previous subsection, although intermediate results are singular at $\varepsilon = D$, the final result is regular for all $\varepsilon > 0$. We shall discuss the physical significance of these results in the next subsection.

### 14.4 Discussion of the variational result

$D = 1$

It is interesting to compare the variational estimate with the exact result for polymers, i.e. for the case $D = 1$. Let us consider the LGW instanton action, as given by Eq. (672). It is equal to the inverse of the large order constant $C$. On figure 40 we have plotted the variational result for $1/C_{\text{var}}$, as given by Eq. (692) and the exact result for $1/C$ obtained by numerical solution, as a function of $0 < d < 4$. First we note that always

$$C \geq C_{\text{var}},$$

(693)

as expected from the variational inequality $E \leq E_{\text{var}}$. This implies that the variational method gives an underestimate of the large orders.

Second the variational estimate becomes good for small $d$, and exact for $d \to 0$. This is not unexpected, since in that limit the membrane $M$ has no inner degrees of freedom, and the functional integration over $V(r)$ reduces to a simple integration over $V \in \mathbb{R}$. Since this integral is Gaussian, the variational method becomes exact.

Finally, the variational estimate for $C$ is regular when $d \to 4$, and then equals $1/(2\pi^2)$; this is 50% smaller than the exact result $3/(4\pi^2)$. Thus the variational method is only qualitatively correct when $\varepsilon = 0$. This is not so surprising, since the limit $\varepsilon \to 0$ is somewhat peculiar. Indeed when $d = 4$ the ground state energy $E_0$ in the equation (665) for the wave function $\Psi_0$ is equal to 0. Then the most general solution to Eq. (665) (for $d = 4$ and $E_0 = 0$) is

$$\Psi_0(r) = \frac{2r_0}{r_0^2 + r^2},$$

(694)

with $r_0$ an arbitrary scale (the size of the instanton). $r_0$ is fixed by the normalization condition (666) which cannot be fulfilled at $d = 4$ for finite $r_0$. In fact a more careful analysis of the rotationally invariant solutions of Eqs. (665) and (666) (see Appendix A of [38]) shows that as $d \to 4$, $E_0$ should scale as $E_0 \sim 4 - d$ and that for $0 < 4 - d \ll 1$ the true solution $\Psi_0$ is well approximated by Eq. (694) (at least as long as $|r|^2(4 - d) \ll 1$) with an instanton size $r_0$ which vanishes as $d \to 4$ as

$$r_0 \sim \frac{1}{\sqrt{|\ln(4 - d)|}}.$$  

(695)
The corresponding instanton potential \( V_{\text{inst}} = -|\Psi_0|^2 \) is also singular in the limit of \( d \to 4 \) (it may be considered as a Dirac-like \( \delta \)-function), and is very poorly approximated by the Gaussian variational solution at \( D = 1, d = 4 \) for the potential

\[
V^{\text{var}} = -16\pi^2 e^{-16\pi^2 r^2},
\]

with positive width. As usual with variational methods, the approximation for the ground state energy is much better than that for the wave function.

**Consequences for the \( \varepsilon \)-expansion**

Of course, one is interested in the consequences of these large order estimates for the \( \varepsilon \)-expansion of the scaling exponents for self-avoiding membranes and polymers. Let us recall that in renormalized perturbation theory one computes the renormalization group \( \beta \)-function \( \beta(g) \) as a power series in \( g \) of the form\(^{10} \)

\[
\beta(g) = -\varepsilon g + B_1 g^2 + \mathcal{O}(g^3).
\]

(697)

Its zero at \( g^* = \varepsilon/B_1 + \mathcal{O}(\varepsilon^2) \) is the IR fixed point which governs the scaling limit for large membranes. Other anomalous dimensions, like the dimension \( \nu(g) \) of the field \( r \) (which gives the fractal dimension of the membrane) can also be computed as a series in \( g \). Their values at the fixed point \( g^* \) give the scaling exponents of the membrane, and may be expanded as power series in \( \varepsilon \).

By analogy with the ordinary Wilson-Fisher \( \varepsilon \)-expansion for LGW field theories, let us assume that the large orders of the function \( \beta(g) \) and of the other anomalous dimensions are given by the instanton estimate, and that they can be resummed by Borel-techniques. We are not able at the moment to give any more precise argument to this last claim (which is still a conjecture even for the LGW theories). Then a simple calculation consists in estimating the “optimal” order \( n_{\text{opt}} \) beyond which the \( \varepsilon \)-expansion starts to diverge. If we only know the first \( n \) terms of the expansion, we expect that for \( n < n_{\text{opt}} \) “ordinary” resummation procedures (like Padé) will be sufficient. If \( n > n_{\text{opt}} \), or if one seeks higher precision, knowledge of the large orders and more sophisticated resummation methods are required. Assuming that for \( \varepsilon = 0 \) the \( n \)-th coefficient of \( \beta(g) \) is of order \((-C^g)^n n!\), and that we can approximate the fixed point \( g^* \) by its first order estimate \( \varepsilon/B_1 \), the term of order \( n \) in the \( \varepsilon \) expansion should behave as

\[
\left( -\frac{\varepsilon C}{B_1} \right)^n n!.
\]

(698)

The optimal order \( n_{\text{opt}} \) is obtained when the absolute value of (698) is the smallest, that is for

\[
n_{\text{opt}} \varepsilon \approx \frac{B_1}{C}.
\]

(699)

With our choice of normalizations for the coupling constant \( b \) in the Hamiltonian (638), the 1-loop coefficient of the \( \beta \)-function is

\[
B_1 = \frac{1}{2} \left[ \frac{(2-D)S_D}{4\pi} \right]^{d/2} S_D^2 \left[ 1 + \frac{1}{2-D} \frac{\Gamma\left(\frac{D}{2-D}\right)}{\Gamma\left(\frac{2D}{2-D}\right)} \right]^2,
\]

(700)

\(^{10}\)Strictly speaking \( g \) is now the renormalized coupling constant.
Large orders

Figure 39: Optimal order $n_{\text{opt}}(D')$ for the $\varepsilon$-expansion of a membrane as function of the extrapolation (dimension) parameter $D'$, as obtained from the variational estimate for the large orders.

with $S_D$ defined in Eq. (A.1). Let us replace $C$ in Eq. (699) by the variational approximant $C_{\text{var}}$ given by Eq. (692). Setting finally $\varepsilon = 0$ in $B_1/C$ (since we are interested in the expansion around $\varepsilon = 0$), we obtain the following variational estimate for the r.h.s. of (699)

$$n_{\text{opt}} \varepsilon \approx \frac{16}{(2 - D)^2} \left[ \frac{\Gamma \left( \frac{4-D}{2} \right)}{\Gamma \left( \frac{D}{2} \right)} \right]^{\frac{4}{D}} \left[ 1 + \frac{1}{2 - D} \left( \frac{\Gamma \left( \frac{D}{2-D} \right)}{\Gamma \left( \frac{2D}{2-D} \right)} \right)^2 \right].$$

(701)

Let us recall that in practice the $\varepsilon$-expansion is used as follows: in order to compute for instance the scaling exponent $\nu^*$ for a membrane with internal dimension $D = 2$ in $d$-dimensional space, one starts from some point $D' \neq D$, $\varepsilon = 0$ (i.e. $d' = 4D'/(2 - D')$), and uses an expansion in $\varepsilon$ and $D - D'$ (or some more general expansion parameters) to evaluate $\nu^*(D')$, which thus depends on the expansion point $D'$. $\nu^*$ is then taken as the best estimate $\nu^*(D'_{\text{opt}})$, as determined for instance by a minimal sensitivity criterium. Membranes ($D = 2$) always correspond to $\varepsilon = 4$, so setting $\varepsilon = 4$ and replacing $D$ by $D'$ in Eq. (701) should give an estimate of the “optimal order” $n_{\text{opt}}(D')$ for the $\varepsilon$-expansion at $D'$. The result for $n_{\text{opt}}(D')$ is plotted on Fig. 39. Some interesting comments can be made on this curve. For $D' > 1.6$, $n_{\text{opt}}(D') > 2$ and becomes large as $D' \to 2$, while for $D' < 1.6$, $n_{\text{opt}}(D') < 2$ and becomes small as $D' \to 0$. In the first regime ($D' \to 2$) we thus expect that the power series in $\varepsilon$ will behave like a convergent series, up to some quite large order $n_{\text{opt}}$. In the second regime ($D'$ small), we expect that the power series in $\varepsilon$ will be divergent from the very first terms. This is in agreement with the calculations at second order in [26, 25]. For large $d$, the 2-loop results for $\nu^*$ can neatly be resummed, and the stability of the various resummation procedures and extrapolation schemes analyzed in [26, 25] is good. The final estimates are close to the prediction of a variational approximation $4/d$ for $\nu^*$. For smaller values of $d$ stability is less good, but in all cases, the reliable extrapolations are obtained for values of the extrapolation dimension $D' \approx 1.6$ or larger. It is not possible to resum safely the 2-loop results if one starts the $\varepsilon$-expansions from $D' \leq 1.5$. Thus it seems that our rough estimates for the large order behavior may explain some general features of the calculation at second order, and corroborate the results of the estimates of [26, 25].
Limit $D \to 2$

Of course these arguments are valid if the variational approximation for the instanton action stays (at least qualitatively) correct in the limit $D \to 2$. First let us note that, although Eqs. (691) and (692) give estimates for $S_{\text{inst}}$ and $C$ which are singular when $D \to 2$, our variational formula for $n_{\text{opt}}$ is much less singular, since according to Eq. (701) it behaves as

$$n_{\text{opt}}(D) \approx \frac{1}{\varepsilon} \frac{16 e^{-4\gamma}}{(2 - D)^2} \quad \text{as } D \to 2 \, , \, \varepsilon \text{ fixed} \, , \quad (702)$$

with $\gamma = 0.577216$ the Euler constant. If we use as in section 3 the usual coupling $b_0$, which according to Eq. (639) is related to $b$ by

$$b = b_0 \frac{2}{S_D^2} \left( \frac{4\pi}{(2 - D)S_D} \right)^{d/2} \quad (703)$$

as expansion parameter instead of $g = bL^\varepsilon$, the large order constant $C$ in Eqs. (642) and (658) is

$$C_{b_0} = C \frac{S_D^2}{2} \left[ \frac{(2 - D)S_D}{4\pi} \right]^{-d/2} \, , \quad (704)$$

which in the variational approximation reads

$$C_{\text{var}} = \frac{S_D^2}{4} \left[ \frac{(2 - D)S_D}{4\pi} \Gamma \left( \frac{2 - D}{2} \right) \right]^{-\frac{d}{2}} \approx \text{const} \left( \frac{2 - D}{2} \right)^{2 - \frac{d}{2}} \quad \text{as } D \to 2 \, , \, \varepsilon \text{ fixed} \, , \quad (705)$$

with $\text{const} = \pi^2(\pi e^{-2\gamma})^{2-\varepsilon/2}$. Therefore in this normalization also the singularities as $D \to 2$ are simply algebraic. The same remark holds for the “second virial coefficient” [21] $z$, defined as

$$z = \left[ \frac{(2 - D)S_D}{4\pi} \right]^{d/2} b L^\varepsilon \, . \quad (706)$$

### 14.5 Beyond the variational approximation and $1/d$ corrections

The variational result for the instanton-action (691) can be used as a basis for a systematic expansion in $1/d$ [38]. A straightforward $1/d$-expansion however is inconsistent with the fact that the variational method is exact for $d = 0$. In [38], the following improved $1/d$-correction was obtained:

$$\frac{S - S_{\text{var}}}{S_{\text{var}}} = \frac{D(2D - \varepsilon)}{(2 + D - \varepsilon)(D - \varepsilon)} \frac{\sin \frac{\pi D}{2}}{\pi} \times \int_0^\infty dp \, p^{d-1} \left[ \ln \left( 1 - \frac{2 + D - \varepsilon}{4} \mathbb{J}(p) \right) + \frac{2 + D - \varepsilon}{4} \mathbb{J}(p) \right] \, , \quad (707)$$

where

$$\mathbb{J}(p) = \int_0^1 dx \left[ 1 + x(1 - x)p^2 \right]^{D/2} \, . \quad (708)$$

Remember that as discussed in section 14.1, this result is only true for $\varepsilon$ positive, i.e. $d < d_c(D)$.

In the remainder, we shall focus on the case $D = 1$, for which we can most easily test Eq. (707). In $D = 1$, $\mathbb{J}(p)$ is exactly given by

$$\mathbb{J}(p) = \frac{4}{4 + p^2} \, . \quad (709)$$
Figure 40: The inverse of the large order constant $1/C$ for the Edwards model ($D = 1$) as a function of the bulk dimension $d$. The dotted curve is the $O(1)$ variational estimate, the dashed curve the estimate from Eq. (711), the continuous curve the exact result.

Eq. (707) is then integrated (using the residue calculus) with the result

$$
\frac{S - S_{\text{var}}}{S_{\text{var}}} \bigg|_{D=1} = \frac{2d}{(d+2)(d-2)} \left( \sqrt{3 - \frac{d}{2}} + \frac{d}{8} - \frac{7}{4} \right). \quad (710)
$$

The large order estimate is finally obtained as

$$
C^{-1}(D = 1) \approx 2 \pi^{d/2} \left[ 1 + \frac{d}{d+2} \left( \sqrt{3 - \frac{d}{2}} + \frac{d}{8} - \frac{7}{4} \right) + \ldots \right], \quad (711)
$$

which is plotted on figure 40. We see that this corrects 50% of the deviation of the variational result from the exact result in $d = 4$, and is even better in lower dimensions.

15 Conclusions

In this article, we gave an overview of techniques, which allow to generalize the concept of local field theories to multilocal ones. The most prominent example of such theories are self-avoiding polymers and membranes. The same techniques apply to dynamical problems and even to the motion of an extended elastic objects in the presence of quenched disorder.

Some less settled topics have not been studied here, but certainly deserve further consideration. The most urging of these questions is why simulations generically see a flat phase. In [25] it had been argued that the pure self-avoidance fixed point should become unstable for space dimension $d < d_i$, with $d_i \approx 3.8$. A solution of this problem first demands an identification of the mechanism which destabilizes the pure self-avoidance fixed point [28], and second a treatment of the full problem. Most promising seems to be the route via the functional renormalization group approach [235]. A thorough theoretical understanding would also help to design experimental tests.
It would certainly also be promising to directly work at $D = 2$, instead of calculating at $D < 2$ and then continuing analytically to $D = 2$. However, all attempts to use methods adopted to two dimensions, as e.g. conformal field theory, have failed so far. We hope that also this route will be explored in the future.

In conclusion: We have described very powerful methods to treat non-local interacting systems, and we hope that this review will help to make these techniques profitable to a broader audience.

A Appendices

A.1 Normalizations

We use peculiar normalizations in order to simplify the calculations. First of all, we normalize the integration measure of the internal space as

$$\int_x = \frac{1}{S_D} \int d^D x, \quad (712)$$

where

$$S_D = \frac{2 \pi^{D/2}}{\Gamma(D/2)}. \quad (713)$$

is the surface of the $D$-dimensional unit-ball. This provides

$$\int_x |x|^{\varepsilon - D} \Theta(L - |x|) = \frac{1}{\varepsilon} L^\varepsilon. \quad (714)$$

Consequently, the $\delta$-distribution in $x$-space is defined such that

$$\int_y f(y) \tilde{\delta}^D (x - y) = f(x). \quad (715)$$

The $\delta$-distribution in the embedding space is normalized according to

$$\tilde{\delta}^d (r(x) - r(y)) = (4\pi)^{d/2} \delta^d (r(x) - r(y)) = \int_p e^{ip[r(x)-r(y)]} \quad (716)$$

with

$$\int_p = \pi^{-d/2} \int d^d p \quad (717)$$

in order to have

$$\int_p e^{-p^2 a} = a^{-d/2}. \quad (718)$$

Using for the free Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2 - D} \int_x \left( \frac{1}{2} \left( \nabla r(x) \right)^2 \right) \quad (719)$$

yields the correlator (for a derivation see appendix A.7)

$$C^{ij}(x - y) := \left\langle \frac{1}{2} [r^i(x) - r^j(y)]^2 \right\rangle_0 = \delta^{ij} C(x - y) \quad (720)$$

with

$$C(x - y) = |x - y|^{2-D}. \quad (721)$$

It satisfies the Laplace-equation (see appendix A.7)

$$\Delta C(x - y) = (2 - D) \tilde{\delta}^D (x - y) \equiv (2 - D) S_D \delta^D (x - y). \quad (722)$$
A.2 List of symbols and notations used in the main text

Throughout the text, we abbreviate the operators encountered by the following symbols:

\[
\begin{align*}
1 &= 1 \\
\uparrow &= \frac{1}{2} (\nabla r)^2 \\
\overline{\bullet} &= \delta^d(r(x) - r(y)) \\
\bullet\overline{\bullet} &= (-\Delta_r) \delta^d(r(x) - r(y)) \\
\overline{\bullet}^{(2n)} &= (-\Delta_r)^n \delta^d(r(x) - r(y)) \\
\bullet\overline{\bullet} &= \text{LR-interaction} \\
\overline{\bullet}^{L} &= \delta^d(r(x) - r(y)) \delta^d(r(x) - r(z)) .
\end{align*}
\]

In dynamic theories, we use the following graphical symbols

\[
\begin{align*}
\overline{\bullet}(x,t) &= \overline{\delta}(x,t) \\
\overline{\bullet}^{L} &= \delta(x,t) \\
\overline{\bullet}^{T} &= \delta(0,0) \\
\overline{\bullet}^{i,j} &= \int_k \overline{\delta}(x,t) e^{ik[r(x,t)-r(y,t')]} \overline{\delta}(y,t') \\
\overline{\bullet}^{L} &= \int_k P^{ij}_L(k) \overline{\delta}(x,t) e^{ik[r(x,t)-r(y,t')]} \overline{\delta}(y,t') \\
\overline{\bullet}^{T} &= \int_k P^{ij}_T(k) \overline{\delta}(x,t) e^{ik[r(x,t)-r(y,t')]} \overline{\delta}(y,t') .
\end{align*}
\]

When two endpoints are approached, we denote this by a dashed line. E.g. \( \overline{\delta}(x) \) for small \( x \) is denoted as

\[
\begin{align*}
\overline{\bullet}(x,y),
\end{align*}
\]

and the arguments \( x \) and \( y \) will be dropped, whenever confusion is impossible. This contraction has a MOPE, which is denoted by

\[
\begin{align*}
\overline{\bullet} = \left( \overline{\bullet} \right) 1 + \left( \overline{\bullet} \right) \uparrow + \ldots .
\end{align*}
\]

An expression like \( \left( \overline{\bullet} \right) \) is a MOPE-coefficient, i.e. a function of \( x - y \). The notation is chosen in the spirit of Feynman’s bra- and ket-notation. Furthermore, we denote a “diagram”

\[
\begin{align*}
\left( \overline{\bullet} \right) \downarrow_L := \int_{|x-y|<L} \left( \overline{\bullet} \right) \uparrow ,
\end{align*}
\]
and this is defined as the integral over the MOPE-coefficient, with all distances appearing in the MOPE-coefficient, i.e. all distances included within the dashed lines, bounded by \( L \). In general, this diagram will be diverging. Suppose the result scales like \( L^\varepsilon \), then we can set

\[
\langle \circ \star | \star \circ \rangle_L = \langle \circ \star | \star \circ \rangle_1 L^\varepsilon
\]

and we may write this as a Laurent-series in \( \varepsilon \), which in the concrete example given here, starts with a term of order \( \varepsilon^{-1} \):

\[
\langle \circ \star | \star \circ \rangle_1 = \langle \circ \star | \star \circ \rangle_{\varepsilon^{-1}} + \langle \circ \star | \star \circ \rangle_{\varepsilon^0} + \langle \circ \star | \star \circ \rangle_{\varepsilon^1} + \ldots .
\]

The first term \( \langle \circ \star | \star \circ \rangle_{\varepsilon^{-1}} \) is the term of order \( \varepsilon^{-1} \), the second the term of order \( \varepsilon^0 \), etc. In many cases, only the leading term in \( 1/\varepsilon \) is needed, and we will denote the residue by

\[
\langle \circ \star | \star \circ \rangle_{\varepsilon^{-1}} = \varepsilon \langle \circ \star | \star \circ \rangle_{\varepsilon^{-1}} .
\]

This should not be confounded with the term of order \( \varepsilon = \varepsilon^1 \), which is denoted by \( \langle \circ \star | \star \circ \rangle_{\varepsilon^1} \). (The latter does not appear in this review.)

### A.3 Longitudinal and transversal projectors

The longitudinal and transversal projectors are defined in Fourier-representation as

\[
P_L^{ij}(k) := \frac{k^i k^j}{k^2},
\]

\[
P_T^{ij}(k) := \delta^{ij} - \frac{k^i k^j}{k^2} .
\]

They are projectors, i.e. they obey the rules (summing over repeated indices is understood)

\[
P_L^{ij}(k) + P_T^{ij}(k) = \delta^{ij}
\]

\[
P_L^{ij}(k) P_T^{ij}(k) = P_T^{ij}(k)
\]

\[
P_T^{ij}(k) P_T^{ij}(k) = P_T^{ij}(k)
\]

\[
P_T^{ij}(k) P_L^{ij}(k) = P_L^{ij}(k) P_T^{ij}(k) = 0 .
\]

We will mostly use the projectors to separate a matrix \( A^{ij}(k) \) into its longitudinal and transversal part. Such a separation is possible, if the matrix is symmetric and rotationally invariant. Setting

\[
A^{ij}(k) = l(k) P_L^{ij}(k) + t(k) P_T^{ij}(k) ,
\]

we can apply the longitudinal and transversal projectors onto \( A^{ij}(k) \) and then take the trace, to obtain (\( d = \delta_{ii} \) is the dimension)

\[
l(k) = P_L^{ij}(k) A^{ij}(k) , \quad t(k) = \frac{1}{d-1} P_T^{ij}(k) A^{ij}(k) .
\]
For a two-dimensional matrix $A_{ij}(k)$, such a separation is always possible, as we show now. Chose as a basis $k$ and $o$ orthogonal to $k$, i.e. $ok = 0$. In this basis, $A_{ij}(k)$ can be decomposed as follows:

$$A_{ij}(k) = \frac{(kAk)k_i k_j}{(k^2)^2} + \frac{(kAo)k_i o_j + (oAk) o_i k_j}{k^2 o^2} + \frac{(oAo) o_i o_j}{(o^2)^2}$$

$$= k_i \varphi_j + k_j \varphi_i + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \Phi$$

with

$$\varphi_j = \frac{kAo}{k^2 o^2} o_j + \frac{1}{2} \frac{kAk}{(k^2)^2} k_j$$

$$\Phi = \frac{oAo}{(o^2)^2} = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) A_{ij}.$$  

(The latter equations are used in section 2.4.)

**A.4 Derivation of the RG-equations**

In this section, we give a derivation of the renormalization group functions. Starting from

$$b = b_0 \frac{Z^{-d/2}}{Z^{d/2}} \mu^{-\epsilon}, \quad (734)$$

where $b$ is the dimensionless renormalized coupling and $b_0$ the dimensionfull bare one, the $\beta$-function is given through the variation of the renormalized coupling, at fixed bare quantities, as

$$\beta(b) := \frac{\partial}{\partial \mu} \bigg|_{\mu=0} b. \quad (735)$$

From the derivative of Eq. (734) with respect to $\mu$, we obtain

$$\beta(b) \left(1 + b \frac{\partial}{\partial b} \ln(Z b Z^{d/2})\right) = -\epsilon b,$$  

leading to

$$\beta(b) = \frac{-\epsilon b}{1 + b \frac{\partial}{\partial b} \ln(Z b Z^{d/2})}.$$  

(737)

For infinite membranes, the scaling exponent $\nu^*$ is obtained from the large-distance behavior of the correlation-function

$$C(|x - y|, b, \mu) := \frac{1}{2d} \left\langle (r(x) - r(y))^2 \right\rangle_{b, \mu}, \quad (738)$$

as

$$C(\ell, b, \mu) \sim \ell^{2\nu^*}.$$  

(739)

To calculate $\nu^*$, we first observe that the bare correlator $C_0(\ell, b_0) = Z C(\ell, b, \mu)$ is independent of the renormalization-scale $\mu$. Therefore

$$0 = \mu \frac{d}{d\mu} C_0(\ell, b_0) = \mu \frac{d}{d\mu} \left[Z(b) C(\ell, b, \mu)\right]$$

$$= Z(b) \left[\beta(b) \frac{\partial}{\partial b} \ln Z(b) + \mu \frac{d}{d\mu}\right] C(\ell, b, \mu).$$  

(740)
Since $C(\ell, b, \mu)$ has canonical dimension $D - 2$ in units of $\mu$, it can be written as
\[ C(\ell, b, \mu) = \mu^{D-2} f(b, \mu \ell) . \tag{741} \]

We thus obtain
\[ 0 = \left[ \beta(b) \frac{\partial}{\partial b} \ln Z(b) + \mu \frac{d}{d\mu} \right] \mu^{D-2} f(b, \mu \ell) = \left[ \beta(b) \frac{\partial}{\partial b} \ln Z(b) + D - 2 + \ell \frac{\partial}{\partial \ell} + \beta(b) \frac{\partial}{\partial b} \right] \mu^{D-2} f(b, \mu \ell) . \tag{742} \]

Denoting
\[ \nu(b) = \frac{2 - D}{2} - \frac{1}{2} \beta(b) \frac{\partial}{\partial b} \ln Z(b) , \tag{743} \]
and reinserting Eq. (741), Eq. (742) is written as
\[ \ell \frac{\partial}{\partial \ell} C(\ell, b, \mu) = \left[ 2\nu(b) - \beta(b) \frac{\partial}{\partial b} \right] C(\ell, b, \mu) . \tag{744} \]

At the IR-fixed point $b^*$ with $\beta(b^*) = 0$, the dimension of the field is thus
\[ \nu^* = \nu(b^*) . \tag{745} \]

Let us now consider the case of a grand canonical ensemble of tethered membranes as introduced in section 13. There, the scaling exponent $\nu^*$ relates the chemical potential $t$ and radius of gyration $R$ through
\[ R \sim t^{-\nu^*/D} . \tag{746} \]

To obtain $\nu^*$, we first observe that the dimensionless combination $R^2 t^{2-D/\nu}$ is a function of $b$ and $t/\mu^D$ only, i.e.
\[ R^2 t^{2-D/\nu} = f(b, t/\mu^D) . \tag{747} \]

Since in addition, $t$ can be expressed as a function of $t_0$ and $b$ only, Eq. (747) implies that
\[ \mu \frac{d}{d\mu} \left. \left[ R^2 t^{2-D/\nu} \right] \right|_{t_0} = \left( \beta(b) \frac{\partial}{\partial b} - Dt \frac{\partial}{\partial t} \right) \left[ R^2 t^{2-D/\nu} \right] . \tag{748} \]

Next observe that $R^2 t^{2-D/\nu}$ is independent of the renormalization scale $\mu$. Replacing bare quantities by their renormalized counterparts, we obtain a relation for the total derivative with respect to $\mu$, as
\[ \mu \frac{d}{d\mu} \left. \left[ R^2 t^{2-D/\nu} Z(b) Z_t(b) \right] \right|_{t_0} = 0 . \tag{749} \]

Combining the latter relation with Eq. (748) gives
\[ \left( Dt \frac{\partial}{\partial t} - \beta(b) \frac{\partial}{\partial b} - \beta(b) \frac{\partial}{\partial b} \ln \left( ZZ_t^{2-D/\nu} \right) \right) \left[ R^2 t^{2-D/\nu} \right] = 0 . \tag{750} \]

The scaling of the membrane is thus given by
\[ \nu(b) = \frac{2 - D}{2} - \frac{1}{2} \beta(b) \frac{\partial}{\partial b} \ln \left( ZZ_t^{2-D/\nu} \right) , \tag{751} \]
and the scaling exponent $\nu^*$ is as in Eq. (745) given by $\nu^* = \nu(b^*)$. 

A.5 Reparametrization invariance

In this appendix, we shall explore the consequences of a reparametrization

\[ x \rightarrow x' = x Z^{-1/D}_\alpha, \]  

(752)

on the (renormalized) Hamiltonian of the grand-canonical ensemble (see section 13)

\[ \mathcal{H} = \frac{Z}{2-D} \int_x \frac{1}{2} (\nabla r(x))^2 + b \mu^\varepsilon Z_b \int_x \int_y \delta^d(r(x) - r(y)) + t Z_\Omega, \]  

(753)

for a self-avoiding membrane \((D = 1\) for polymers). The Hamiltonian in Eq. (753) is in fact not invariant under this rescaling because of the cutoff implicit in the interaction. In order to achieve scale invariance, the cutoff, or equivalently the renormalization scale \(\mu\), must also be rescaled to

\[ \mu \rightarrow \mu' = \mu Z^{1/D}_\alpha. \]  

(754)

The Hamiltonian then changes to

\[ \mathcal{H} = \frac{Z Z^{2-D}_\alpha}{2-D} \int_x \frac{1}{2} (\nabla r(x))^2 + b \mu^\varepsilon Z_b Z^{\varepsilon/D - 2}_\alpha \int_x \int_y \delta^d(r(x) - r(y)) + t Z_t Z^{-1}_\alpha \Omega. \]  

(755)

Comparing to the original Hamiltonian then identifies the new renormalization group factors

\[ Z' = Z Z^{2-D}_\alpha, \]
\[ Z'_t = Z_t Z^{-1}_\alpha, \]
\[ Z'_b = Z_b Z^{2+\varepsilon/D}_t. \]  

(756)

As discussed in section 13.3, see Eqs. (600) ff., the renormalization group-functions are left unchanged by the transformations in Eqs. (756); the most useful case being \(Z_\alpha = Z_t\).

Note that a similar transformation can also be performed in the case of a single (infinite) membrane; however the space-coordinate \(x\) then acquires an anomalous dimension. Also note that the MOPE is incompatible with such a transformation.

A.6 Useful formulas

Momentum space integrals

The momentum space integrals over \(n\) vectors \(k_1, \ldots, k_n\) are

\[ \int_{k_1} \ldots \int_{k_n} e^{-k_ik_j A_{ij}} = \det(A_{ij}^s)^{-d/2}, \]  

(757)

where \(A_{ij}^s = \frac{1}{2} (A_{ij} + A_{ji})\) is the symmetrized version of \(A_{ij}\). Deriving with respect to \(A_{ij}\) then leads to

\[ \int_{k_1} \ldots \int_{k_n} k_ik_je^{-k_ik_j A_{ij}} = -\frac{\partial}{\partial A_{ij}} \det(A_{ij}^s)^{-d/2} \]  

(758)

and the same for higher moments.
In the 1-dimensional case where $\mathbb{A}_{ij}$ is a number $a$ this gives
\[
\int_k k^2 e^{-k^2 a} = \frac{d}{2} a^{-d/2-1} \quad (759)
\]
\[
\int_k (k^2)^2 e^{-k^2 a} = \frac{d(d + 2)}{4} a^{-d/2-2} . \quad (760)
\]

For a $2 \times 2$-matrix $\mathbb{A}_{ij} = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix}$, this simplifies to:
\[
\int_{k_1} \int_{k_2} k_1 k_2 e^{-k_1 k_2 \mathbb{A}_{ij}} = \frac{d}{2} \mathbb{A}_{22} \det(\mathbb{A}_{ij})^{-d/2-1} \quad (761)
\]
\[
\int_{k_1} \int_{k_2} k_1 k_2 e^{-k_1 k_2 \mathbb{A}_{ij}} = -\frac{d}{2} \mathbb{A}_{12} \det(\mathbb{A}_{ij})^{-d/2-1} \quad (762)
\]
\[
\int_{k_1} \int_{k_2} k_2^2 e^{-k_1 k_2 \mathbb{A}_{ij}} = \frac{d}{2} \mathbb{A}_{11} \det(\mathbb{A}_{ij})^{-d/2-1} . \quad (763)
\]

**Calculable manifold-integrals**

The three basic solvable manifold-integrals are:
\[
\int_{|x| < L} x^{2D-\nu d} = \frac{L^{2D-\nu d}}{2D - \nu d} \quad (764)
\]
\[
\int_{xy, \max(x,y) < L} \left( |x|^{2\nu} + |y|^{2\nu} \right)^{-d/2} = \frac{1}{2D - \nu d} \frac{\Gamma\left(\frac{D}{2-D}\right)^2 L^{2D-\nu d}}{2D - \nu d} 
\left[ 1 + O\left( (2D - \nu d)^0 \right) \right] \quad (765)
\]
\[
\int_{x,y,z, \max(x,y,z) < L} \left( |x|^{2\nu} |y|^{2\nu} + |y|^{2\nu} |z|^{2\nu} + |z|^{2\nu} |x|^{2\nu} \right)^{-d/2} = \frac{1}{(2-D)^2} \frac{\Gamma\left(\frac{1}{2} \frac{D}{2-D}\right)^3 L^{3D-2\nu d}}{\Gamma\left(\frac{3}{2} \frac{D}{2-D}\right) 3D - 2\nu d} 
\left[ 1 + O\left( (3D - 2\nu d)^0 \right) \right] . \quad (766)
\]

**Distributions**

In our conventions listed in appendix A.1
\[
\Delta_x |x-y|^{2-D} = (2-D) \delta^D (x-y) . \quad (767)
\]

Therefrom we deduce:
\[
\int_x \left[ \nabla_x \left( |x-y|^{2-D} - |x-z|^{2-D} \right) \right]^2 = 2(2-D) |y-z|^{2-D} . \quad (768)
\]

A useful formula is ($\vec{a} = \vec{c} - \vec{b}$ = fixed):
\[
\frac{2-D}{2} \int_b \left( \vec{b}^{2-D} - \vec{c}^{2-D} \right)^2 = a^{2-D} . \quad (769)
\]
Another distribution which appears frequently is \((a = \text{fixed})\):

\[
(2 - D) \int_c (a^2 - D (\bar{a}c)^2 c^{-2}) c^{-D} = \int_c a \nabla_x (\bar{a} \nabla_x c^{2-D})
= a^2 \frac{1}{D} \int_c \Delta_x c^{2-D}
= \frac{2 - D}{D} a^2 \tag{770}
\]

### A.7 Derivation of the Green function

It shall be shown that

\[
g(x) := -|x|^{2-D} \tag{771}
\]

is in the sense of distributions a solution of

\[
-\Delta_x g(x) = (2 - D) \delta^D(x) \tag{772}
\]

**Proof**

Let

\[
g_\eta(x) := -\frac{(2 - D)}{(2 - D + \eta)} |x|^{2-D+\eta}, \tag{773}
\]

with \(r := |x|\) and \(f \in C^\infty_\mathbb{C}(\mathbb{R}^D)\). Then

\[
I := \int_x (-\Delta g_\eta(x)) f(x)
= \frac{(2 - D)}{(2 - D + \eta)} \int_0^\infty dr r^{D-1} \frac{1}{S_D} \int d\Omega \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{2-D+\eta} f(x)
= \eta(2 - D) \int_0^\infty dr \frac{1}{S_D} \int d\Omega \eta^{-1} f(x)
\]

As \(f \in C^\infty_\mathbb{C}(\mathbb{R}^D)\) there are constants \(a, b\) and \(l\) such that

\[
|f(r) - f(0)| \leq a \sqrt{\eta} \quad \forall r \leq \sqrt{\eta}
|f(r) - f(0)| \leq b \quad \forall r > \sqrt{\eta}
\]

\[
f(r) = 0 \quad \forall r \geq l \tag{774}
\]

We conclude:

\[
\left| I - (2 - D) \int_0^l dr \frac{\eta}{r^{1-\eta}} f(0) \right| \leq (2 - D) \int_0^{\sqrt{\eta}} dr \frac{\eta}{r^{1-\eta}} |f(r) - f(0)| +
+ (2 - D) \int_{\sqrt{\eta}}^l dr \frac{\eta}{r^{1-\eta}} |f(r) - f(0)|
\leq (2 - D) \left( \eta^{\frac{1}{2}(1+\eta)} a + \eta^{\frac{1}{2}(1+\eta)} b l \right)
\]

So for all \(f \in C^\infty_\mathbb{C}\) there is the following diagram:

\[
\int dx (-\Delta g_\eta(x)) f(x) = \int dx g_\eta(x) (-\Delta) f(x)
\]

\[
\begin{array}{c}
\eta \to 0 \\
\eta \to 0
\end{array}
\]

proving Eq. (771).
E Exercises with solutions

E.1 Example of the MOPE

Calculate the MOPE-coefficient \( \left( \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ x \\ x \\ x \\ y \\ y \\ y \end{array} \right) \).

**Solution of exercise 1:**

Start from

\[
\begin{align*}
\int_k \int_p \int_q & :e^{ikr(x_1)} :e^{ipr(x_2)} :e^{ipr(x_3)} :e^{-ikr(y_1)} :e^{-ipr(y_2)} :e^{-ipr(y_3)} :. \\
& = \int_k \int_p \int_q :e^{ikr(x_1)+ipr(x_2)+iqr(x_3)} :e^{kp(a^2+q^2)+kq(b^2+pq)c^2}.
\end{align*}
\]  

These dipoles shall be contracted like

\[
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This can still be factorized as is known from ancient Heron:

\[
\left(\begin{array}{c}
\bullet \\
\end{array}\right) = \\
\left[\frac{1}{4} (\sqrt{a^{2\nu} + d^{2\nu}} + \sqrt{b^{2\nu} + e^{2\nu} + f^{2\nu}}) \left(\sqrt{b^{2\nu} + c^{2\nu} + f^{2\nu}} - \sqrt{a^{2\nu} + d^{2\nu}}\right) \left(\sqrt{a^{2\nu} + d^{2\nu}} + \sqrt{b^{2\nu} + e^{2\nu}} - \sqrt{c^{2\nu} + f^{2\nu}}\right)\right]^{-d/2}.
\]

(781)

### E.2 Impurity-like interactions

Show that there is no term proportional to \(\tilde{d}^{d}(r(0))\) in the MOPE of \(n\) dipoles with \(\tilde{d}^{d}(r(0))\). Show also that this implies that the full dimension of \(\tilde{d}^{d}(r(0))\) is \(-\nu^* d\).

**Solution of exercise 2:**

The MOPE of \(\tilde{d}^{d}(r(0)) = \int_k \epsilon^{ikr(0)}\) with \(n\) \(\delta\)-interactions is

\[
\int_k \epsilon^{ikr(0)} \int_{p_1} \epsilon^{ip_1[r(y_1)−r(z_1)]} \cdots \int_{p_n} \epsilon^{ip_n[r(y_n)−r(z_n)]} = \\
= \int_k \int_{p_1} \cdots \int_{p_n} \epsilon^{ikr(0)} \epsilon^{ip_1[r(y_1)−r(z_1)]} \cdots \epsilon^{ip_n[r(y_n)−r(z_n)]} \times \\
\times e^{\sum_{j=1}^n kp_j[C(y_j)−C(z_j)]} \epsilon^{1/2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j [C(y_i−y_j)+C(z_i−z_j)−C(y_i−z_j)−C(y_j−z_i)].
\]

(782)

The leading term proportional to \(\tilde{d}^{d}(r(0)) = \int_k \epsilon^{ikr(0)}\) is

\[
\int_k \epsilon^{ikr(0)} \int_{p_1} \cdots \int_{p_n} \epsilon^{1/2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j [C(y_i−y_j)+C(z_i−z_j)−C(y_i−z_j)−C(y_j−z_i)].
\]

(783)

Note that from \(e^{\sum_{j=1}^n kp_j[C(y_j)−C(z_j)]}\) only the leading term 1 has to be taken, since subleading terms generate derivatives of the \(\delta\)-distribution \(\tilde{d}^{d}(r(0))\). However, Eq. (783) is nothing but the MOPE \(\left(\begin{array}{c}
\bullet \\
\end{array}\right) \times \cdots \times \left(\begin{array}{c}
\bullet \\
\end{array}\right)\) times \(\tilde{d}^{d}(r(0))\), and the divergence is subtracted by the counter-term proportional to the volume, or equivalently by normalizing expectation values by the full partition function. Therefore, the only renormalization to \(\tilde{d}^{d}(r(0))\) comes from the anomalous dimension of the field \(r\), leading to the above stated dimension of \(-\nu^* d\).

### E.3 Equation of motion

Show that the factor of \(Z^{d/2}\) which intervenes in the renormalization of the coupling \(b_0 = b \mu^Z Z_0 Z^{d/2}\) and thus in the renormalization group \(\beta\)-function can also be understood with the help of the equation of motion or redundant operators.

**Solution of exercise 3:**

Two kinds of counter-terms are needed: Counter-terms proportional to \(\leftarrow\) (taken care of in the renormalization factor \(Z_0\)), and counter-terms proportional to \(\rightarrow\), (taken care of in the renormalization
factor $Z$). The latter appear in the form
\[ \frac{1}{2-D} \int_x \phi^x e^{-b \mu^x Z_b \int_y \phi^y}, \] (784)
and using the equation of motion can be interpreted as
\[ -\frac{d}{2} Z_b \mu^x \int_x \phi^x \phi^y e^{-b \mu^x Z_b \int_y \phi^y}. \] (785)
Exponentiation then leads to
\[ e^{-\frac{1}{2-D} \int_x \phi^x \phi^y + b \mu^x Z_b \int_y \phi^y}, \] (786)
as stated above.

The same is achieved by using the concept of redundant operators.

### E.4 Tricritical point with modified 2-point interaction

Study the tricritical point, which appears when one demands that the renormalized coupling proportional to self-avoidance, $\phi^x \phi^y$, vanishes. Show that for $D > 4/3$, the next-to-leading operator dominating the tricritical point, is $\phi^x \phi^y$. Determine the MOPE-coefficients to leading order, following the lines of section 3. Show that the $\beta$-function has an IR-stable fixed point and calculate the anomalous scaling exponent $\nu^\star$.

This problem was first discussed in [27, 57].

**Solution of exercise 4:**

Let us start from the bare Hamiltonian
\[ \mathcal{H}[r_0] = \frac{1}{2-D} \int_x \left( \nabla r_0(x) \right)^2 + b_0 \int_x \int_y (\Delta r_0)(r_0(x) - r_0(y)). \] (787)
The canonical dimension of the coupling constant $b_0$ is
\[ \varepsilon' := [b_0]_\mu = 3D - 2 - \nu d \] (788)
and the model has UV-divergences, i.e. poles in $\varepsilon'$, for $\varepsilon' = 0$. As in section 3 two renormalizations are needed, namely
\[ r(x) = Z^{-1/2} r_0(x) \] (789)
\[ b = Z^{-d/2-1} Z_{b}^{-1} \mu^{-\varepsilon'} b_0. \] (790)
At 1-loop order, the counter-terms are in analogy to Eqs. (172) and (173)
\[ Z = 1 - \frac{b}{\varepsilon'} (2-D) \left( \phi^x \phi^y \right) + O(b^2), \] (791)
\[ Z_b = 1 + \frac{b}{\varepsilon'} \left( \phi^x \phi^y \right) + O(b^2). \] (792)
However note, that the leading term of the MOPE of $\phi^x \phi^y$ is proportional to $\phi^x \phi^y$, such that (scale-dependent) fine-tuning is necessary in order to stay at the tri-critical point.
The residues \( \langle \epsilon | + \rangle_{t'} \) and \( \langle \epsilon | - \rangle_{t'} \) are analytic functions of \( D \) for \( 0 < D < 2 \) [27]

\[
\langle \epsilon | - \rangle_{t'} = -\frac{1}{2 - D} \\
\langle \epsilon | + \rangle_{t'} = \frac{10D - D^2 - 8\frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)}}{2(2 - D)^3}.
\] (793)

The \( \beta \)-function and the scaling dimension \( \nu(b) \) of the field \( r \) are as in Eqs. (175) and (176) defined as

\[
\beta(b) := \mu \frac{\partial}{\partial \mu} \bigg|_{b_0} b, \quad \nu(b) := \frac{2 - D}{2} - \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z.
\] (794)

The new \( \beta \)-function has a non-trivial IR-fixed point \( b^* > 0 \) for \( \varepsilon' > 0 \) and the scaling dimension of the membrane at the \( \Theta \)-point becomes

\[
\nu^* = \frac{2 - D}{2} \left[ 1 - \varepsilon' \frac{\langle \epsilon | + \rangle_{t'} - \nu(d+2)\langle \epsilon | + \rangle_{t'} + O(\varepsilon^2)}{\langle \epsilon | - \rangle_{t'} - \nu(d+2)\langle \epsilon | - \rangle_{t'} + O(\varepsilon^2)} \right]
\]

\[
\nu^* = \frac{2 - D}{2} \left[ 1 + \frac{\varepsilon'}{\frac{10D - D^2 - 8\frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)}}{2(2 - D)^3} + 2D} + O(\varepsilon^2) \right].
\] (795)

### E.5 Consequences of the equation of motion

Show that [25]

\[
\langle \epsilon | - \rangle_{t-1} = -\frac{\nu d}{2} \langle \epsilon | - \rangle_{t-1} + O(\varepsilon^0) \] (796)

\[
\langle \epsilon | + \rangle_{t-1} = -\nu(d+2) \langle \epsilon | + \rangle_{t-1} + \langle \epsilon | + \rangle_{t-1} \langle \epsilon | + \rangle_{t-1} + O(\varepsilon^0).
\] (797)

### Solution of exercise 5:

Details are given in appendix C of [25]. The idea is to use the equation of motion (194) to write

\[
\langle \int\int\int\int \rangle_0 = -\nu d \langle \int\int\int\int \rangle_0.
\] (798)

Identifying divergences proportional to \( \langle \int\int\int\int \rangle_0 \) on both sides and again using Eq. (194) then yields Eq. (796).

The second identity (797) is proven along the same lines by starting from

\[
\langle \int\int \int \int \rangle_0 = -\nu d \langle \int\int \int \int \rangle_0.
\] (799)

\[\text{Note a misprint in Eq. (12.32) of [25]. Using the extrapolations for } \nu^*(d + 2) \text{ (expansion about the mean-field result } \nu_{MF} = \frac{2D}{2+d} \text{) or } \nu^*(d + 4) \text{ (expansion about the Flory result } \nu_{Flory} = \frac{2+d}{4+d} \text{) in three dimensions yields results very comparable to the Flory estimate of } \nu^* = 0.57.\]
E.6 Finiteness of observables within the renormalized model

Show that within the renormalized model (153) and to first order in $b$, the expectation value of $O = e^{ijk[r(s)−r(t)]}$ as given for the bare model in Eq. (87) is UV- and IR-finite.

Solution of exercise 6:

Two counter-terms are relevant at first order in the renormalized coupling $b$, namely $\Delta H_1$, Eq. (159) and $\Delta H_2$, Eq. (161). The third 1-loop term, $\Delta H_3$, Eq. (161), will only show up at order $b^2$. Also note, that the first one, $\Delta H_1$, has already been taken into account in Eq. (87), due to the normalization introduced in Eq. (72).

At first order in the renormalized coupling $b$, we thus only have to take care of the counter-term (161), with the full tensorial structure, resulting in

$$\langle O \rangle_b = e^{-k^2C(s−t)} \times$$

$$\left\{ 1 + b\mu^e \int_\mathbb{R} \int_\mathbb{R} \left[ 1 - \exp \left( \frac{1}{4} k^2 \left[ C(s−x) + C(t−y) - C(s−y) - C(t−x) \right]^2 \right) \right]$$
$$+ \frac{1}{4} k^2 \left[ (x−y) \nabla_z (C(s−z) - C(t−z)) \right]^2 \Theta(|x−y| < L) C(x−y)^{-d/2} \right\} \quad (800)$$

where $z$ may either be chosen as $x$ or $y$, or the symmetrized version may be used. The terms in the big square brackets exactly cancel the divergence for small $x−y$. Note that there is a possible IR-divergence for $|x−y| \to \infty$. In the original term, this is regularized by the observable-positions $s$ and $t$, which effectively cut off the integral at $|x−y| \approx |s−t|$. In the counter-term, the IR-cutoff is explicit.

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