Critical manifolds for percolation and Potts models from graph polynomials

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Conformal Invariance, Discrete Holomorphicity and Integrability
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Collaborator: Christian R. Scullard (Lawrence Livermore Nat'l Lab)

Potts model and bond percolation

Partition function

$$Z = \sum_{\sigma} \prod_{(ij)\in E} \exp\left(K\delta_{\sigma_i,\sigma_j}\right)$$

- Spins $\sigma_i = 1, 2, \ldots, q$ with nearest-neighbour coupling K
- Planar lattice $G = (V, E)$ with vertices $i \in V$ and edges $(ij) \in E$

Fortuin-Kasteleyn representation

- Write $\exp\left(K\delta_{\sigma_i,\sigma_j}\right) = 1 + v\delta_{\sigma_i,\sigma_j}$ with $v := \mathrm{e}^K - 1$

$$Z = \sum_{A\subseteq E} v^{|A|} q^{k(A)}$$

- $q \to 1$ produces bond percolation, with $p = \frac{v}{1+v}$

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Critical manifold and percolation threshold

Solvability only on a few lattices G

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\begin{aligned}
(v^2 - q)(v^2 + 4v + q) &= 0\,, && \text{(square lattice)} \\
v^3 + 3v^2 - q &= 0\,, && \text{(triangular lattice)} \\
v^3 - 3qv - q^2 &= 0\,. && \text{(hexagonal lattice)}
\end{aligned}
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- Comes from integrability / discrete holomorphicity
- Certain inhomogeneous extensions (spectral parameter)

Percolation case

- Given q, all solutions for v are physically interesting!
- For percolation, usually only $p_c \in [0,1]$ is considered:

$$p_c^{\rm sq} = \frac{1}{2}\,, \qquad p_c^{\rm tri} = 1 - p_c^{\rm hex} = 2 \sin\left(\frac{\pi}{18}\right)$$

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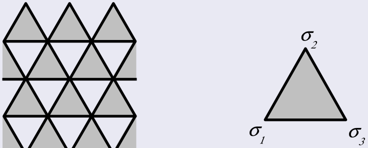
All solvable cases are of the “3-terminal form”



(a) (b)

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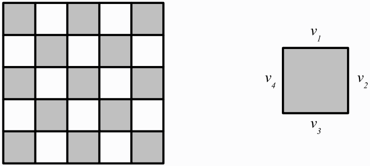

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What about other lattices?

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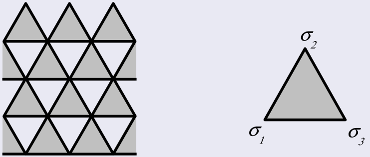

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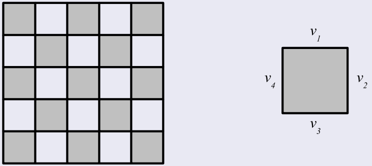

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Homogeneity assumption (F.Y. Wu)

- Inspired guesswork by analogies with 3-terminal results

Kagome lattice (Wu 1979) Benchmark for non 3-terminal case

$$v^6 + 6v^5 + 9v^4 - 2qv^3 - 12qv^2 - 6q^2v - q^3 = 0$$

- Initially conjectured exact by Wu
- For $q = 2$ correctly giving $v_c = \sqrt{3 + 2\sqrt{3}} - 1$
- For $q = 1$ it predicts $p_c = 0.524\,429\,717 \cdots$
- But numerics gives: $p_c = 0.524\,404\,978\,(5)$

- Not clear why this is so precise
- Not clear if one can make this **even more** precise
- The adaptation to other lattices is somewhat *ad hoc*

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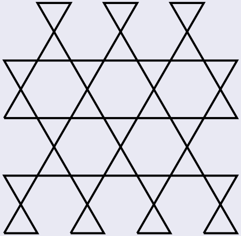

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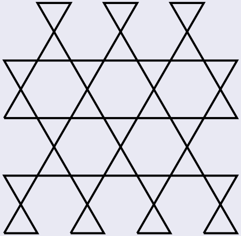

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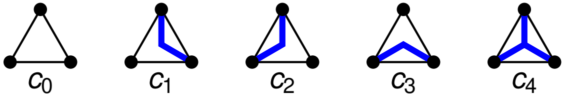
Solvability of 3-terminal lattices

- Boltzmann weight of elementary triangle
  $w_{123} = c_0 + c_1\delta_{23} + c_2\delta_{13} + c_3\delta_{12} + c_4\delta_{123}$



- FK cluster partition function $Z = \sum_{A\subseteq E} q^{k(A)} \prod_{p=0}^{4} (c_p)^{N_p}$

Solution for the critical manifold (Wu-Lin 1980)

- Cluster boundaries live on (another) triangular lattice
- Imposing invariance under $\pi/3$ rotations gives: $c_4 = qc_0$
- This provides the exact critical manifold for all 3-terminal lattices: $P(q, v) = c_4 - qc_0 = 0$

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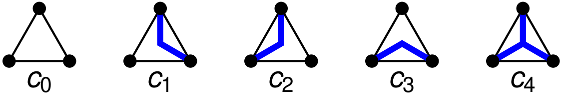

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The graph polynomial

Contraction-deletion identity for Potts model partition function

$$Z_G(q,\{v\}) = v_e Z_{G/e}(q,\{v\}) + Z_{G\backslash e}(q,\{v\})$$

Method and key hypothesis

- Let B (the "basis") be a finite portion of G with N terminals
- G is obtained by tiling space with B in a certain way (the "embedding"), gluing copies of B at the terminals
- Suppose the critical polynomial $P_B(q,v)$ satisfies the contraction-deletion identity, for any edge in B
- When reduced to 3-terminal case, replace by exact $P(q,v)$
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4-terminal examples with checkerboard embedding

Square lattice with $B =$ square of four edges

$$P_B(q,\{v_1,v_2,v_3,v_4\}) = v_4 P_{B^{\rm tri}}(q,\{v_1,v_2,v_3\}) + P_{B^{\rm hex}}(q,\{v_1,v_2,v_3\})$$

- 1. term: two terminals have been identified
- 2. term: embedding is used to flip one edge

$$P_B(q,\{v_i\}) = v_1 v_2 v_3 v_4 + (v_2 v_3 v_4 + v_1 v_3 v_4 + v_1 v_2 v_4 + v_1 v_2 v_3) - q(v_1+v_2+v_3+v_4) - q^2$$

- Just integrability of 6V model with staggered spectral parameters
- Homogeneous case: $P_B(q,v) = (v^2-q)(v^2+4v+q)$

Kagome lattice with $B =$ bow tie of six edges

- We recover precisely Wu's sixth-order polynomial
- Suggests improving the precision by increasing the size of B

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General structure of the results (1)

Factorisation property for $G =$ solvable case

- $P_B(q, v)$ factorises, shedding a “small factor”
- Small factor independent of size of B, and gives exact solution
- Checked for “all” known solutions of the Potts model

- Even when the Potts model on G is not solvable in general, we find factorisation (i.e., exact result) for the Ising model $q = 2$

Computing $P_B(q, v)$ serves to detect exact solvability

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- E.g. bond percolation threshold on the $(3, 12^2)$ lattice with $B = 9n^2$ edges

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- With extrapolation, we can attain 12 or 13-digit precision
- Same precision for other q, at least when $v > 0$

Computing $P_B(q, v)$ in practice

Getting started

- By hand, up to ~ 10 edges
- By deletion-contraction, up to ~ 40 edges

Transfer matrix method

Made possible by an equivalent definition of $P_B(q, v)$

- Naive method, up to ~ 100 edges
- Improved method (using periodic TL algebra), up to ~ 400 edges

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Equivalent definition of $P_B(q, v)$

Experimental approach

- Compute P_B by deletion-contraction with inhomogeneous $\{v_i\}$
- Various G and B, different embeddings, up to ~ 30 edges
- Terms in $P_B(q, v)$ interpreted as connectivities among B-terminals

Same result found in all cases

$$P_B(q, v) = Z_{\rm 2D} - qZ_{\rm 0D}$$

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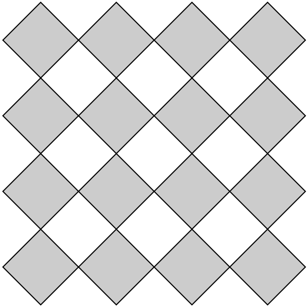

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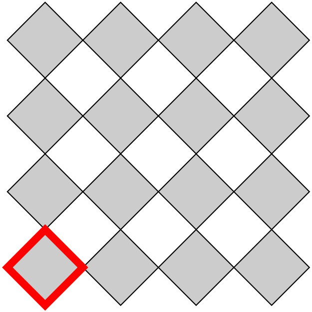

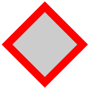

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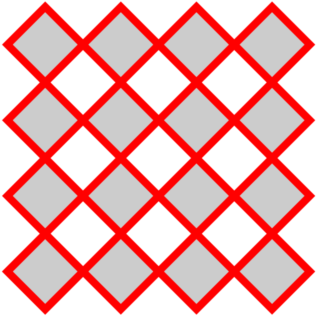

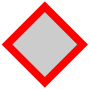

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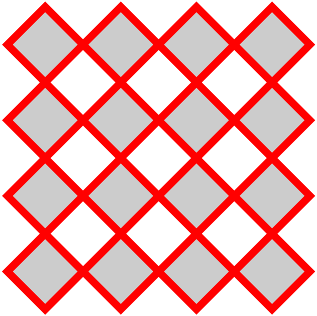

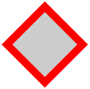

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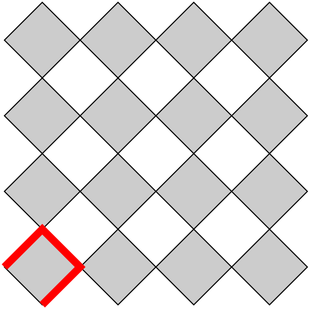

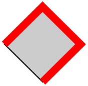

$4qv^3$

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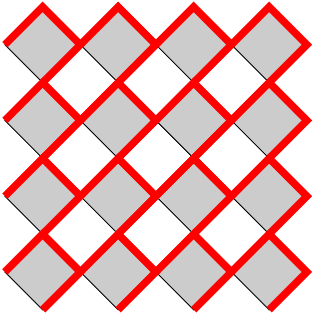

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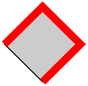

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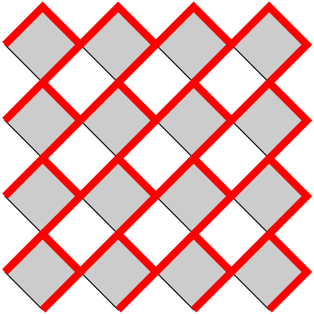

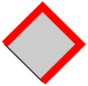

$4qv^3$

$Z_{2D} = qv^4 + 4qv^3$

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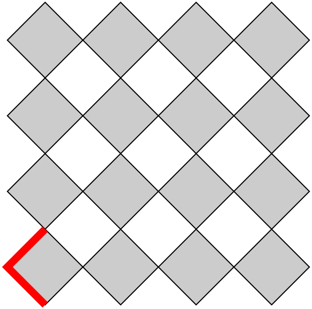

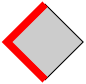

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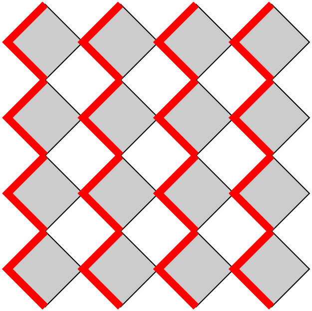

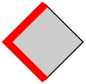

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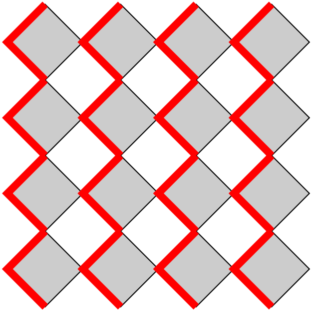

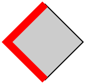

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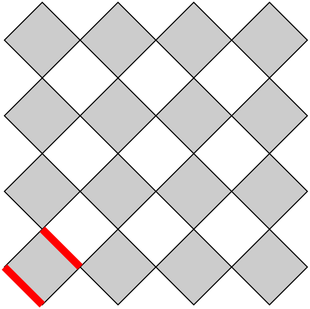

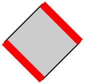

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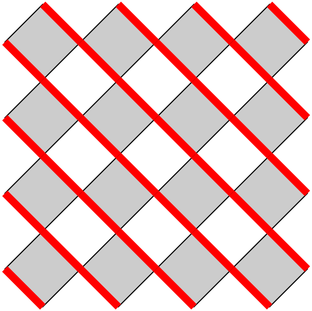

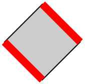

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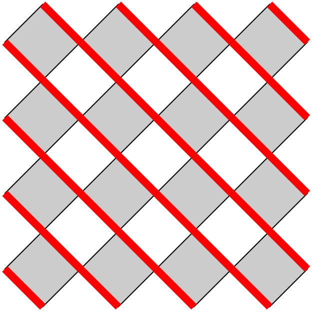

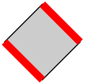

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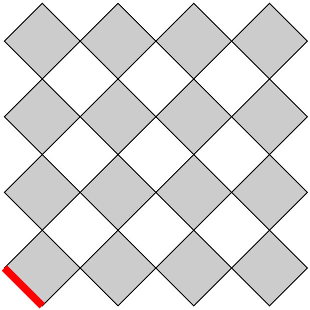

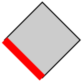

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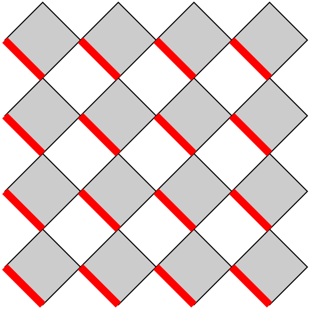

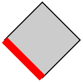

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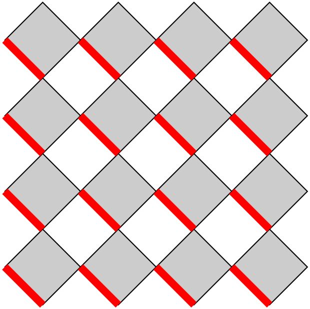

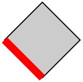

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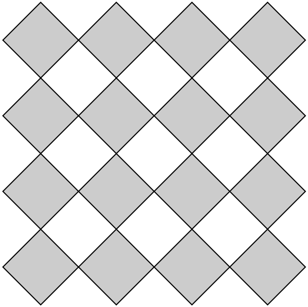

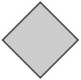

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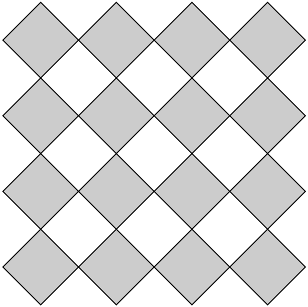

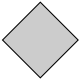

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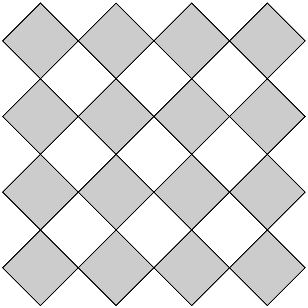

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$$Z_{\mathrm{0D}} = 4qv + q^2$$

$$P_B(q, v) = Z_{\mathrm{2D}} - qZ_{\mathrm{0D}} = q(v^2 - q)(v^2 + 4v + q)$$

Naive transfer matrix method

- Let $C_N = \frac{1}{N+1}\binom{2N}{N}$ be the Catalan numbers
- C_N partitions of the N terminals of a basis B (respecting planarity)
- Transfer matrix computes weight [polynomial in (q, v)] of each partition
- From this construct $Z_{\rm 2D}$, $Z_{\rm 1D}$ and $Z_{\rm 0D}$

- No need to consider all of G to distinguish between 2D, 1D and 0D
- Follows from B and the embedding by using the Euler relation

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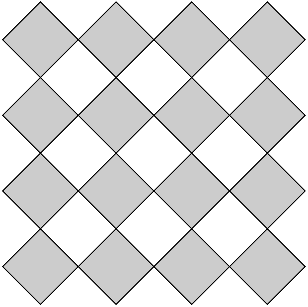
Improved transfer matrix method

- We are mainly interested 4-terminal lattices with embeddings that correspond to imposing toroidal boundary conditions on B



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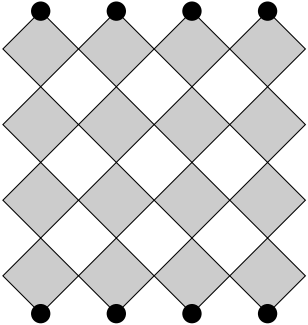

● $=$ terminal of B

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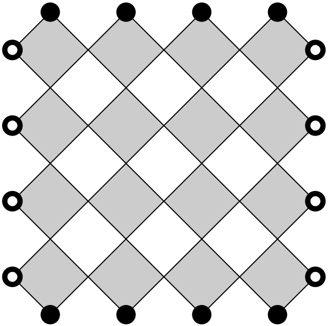

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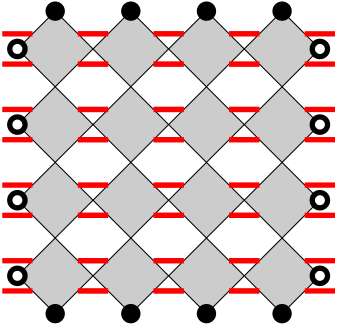

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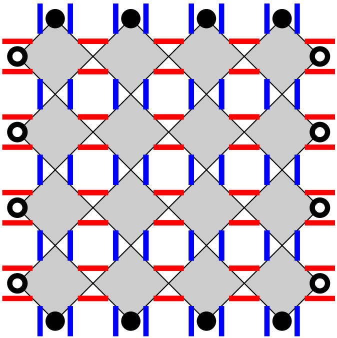

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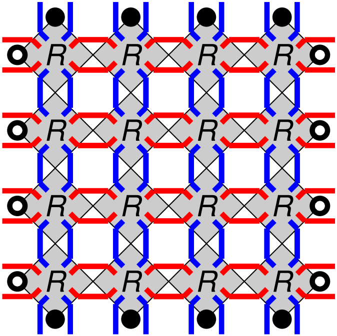

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R-matrix (in terms of TL algebra)

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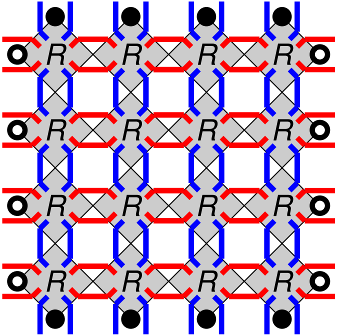

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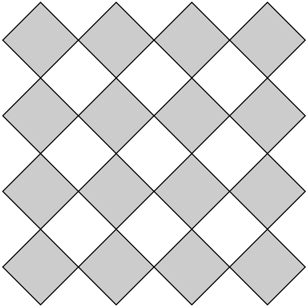
R-matrix (in terms of TL algebra)

- Divides by 2 the number of terminals, but requires some thoughts about the correct elimination of Z_{1D} diagrams (periodic TL algebra)

Archimedean lattices (examples)



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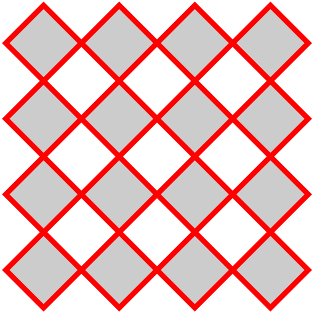
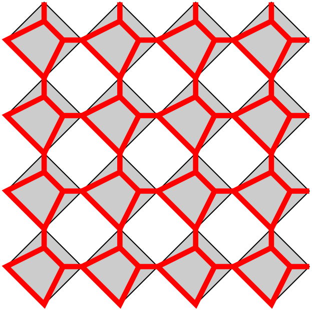


Square lattice

Archimedean lattices (examples)



$(4, 8^2)$ lattice

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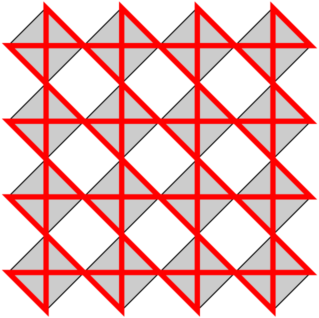
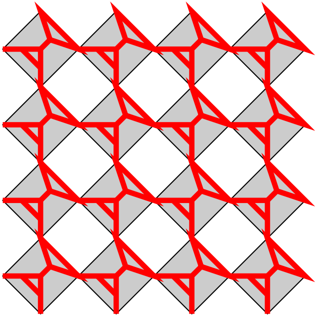


Kagome lattice

Archimedean lattices (examples)



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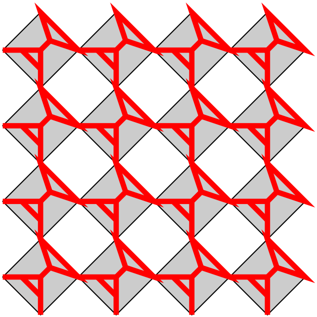
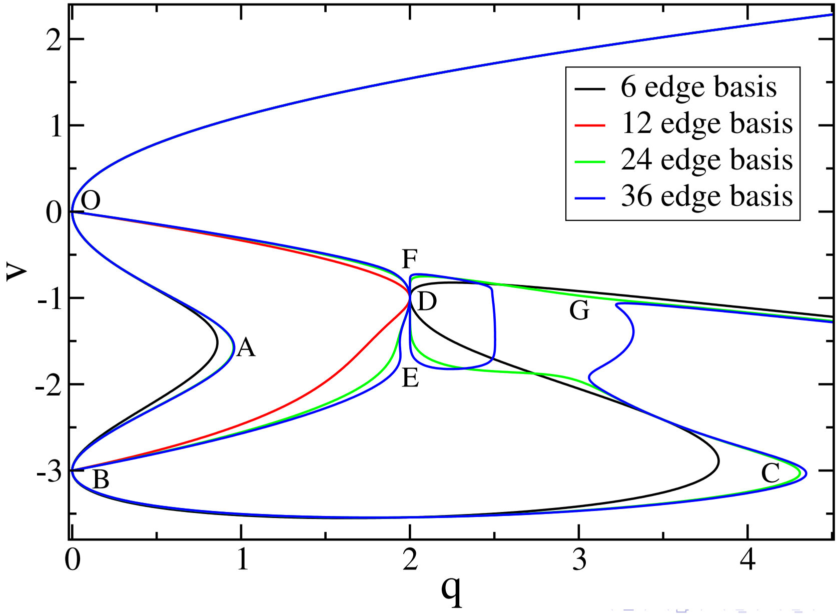


$(3, 12^2)$ lattice

- In this way we can construct all 11 Archimedean lattices
- "Small factor" gives exact result for all solvable cases
- In other cases, precision on v_c exceeds that of any other method (Monte Carlo, transfer matrix, series expansion,. . .)

Example of complete phase diagram: Kagome lattice



Conclusion

Summary

- $P_B(q, v)$ provides new method of determining critical manifolds
- Easy to compute by hand for small bases
  - Provides exact results if model is solvable
- Efficient computer algorithm for larger bases
  - Factorisation of small factor confirms exact solvability
  - High accuracy (12–13 decimal digits) for non-solvable cases ($v > 0$)
  - Intricate phase diagrams in antiferromagnetic regime ($v < 0$)

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- Relation to integrability / discrete holomorphicity must be clarified
- Applications to other types of models (including quantum)?

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