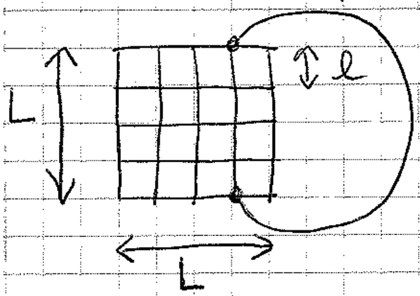
Vortex density in the 2D ideal Bose gas

Preliminary remarks:

(1) We use a lattice model (to have a momentum cutoff)

l = lattice spacing. Our 2D gas is in a square of size L, with periodic boundary conditions.



$\frac{L}{l} \in \mathbb{N}$.

(At the end we will take the limits: $l \to 0$ and then ($L \to \infty$, ρ fixed))

Reminder: $\vec{r} \in \{0, l, 2l, .., (\frac{L}{l} - 1) l\}^2$;

$\langle \vec{r} | \vec{r}' \rangle = \frac{\delta_{\vec{r}, \vec{r}'}}{l^2}$; $\langle \vec{r} | \vec{k} \rangle = \frac{e^{i \vec{k} \cdot \vec{r}}}{\sqrt{V}}$ $(V = L^2)$;

$\vec{k} \in \mathcal{D} = \left(\frac{2\pi}{L} \mathbb{Z} \cap \left(-\frac{\pi}{l} , \frac{\pi}{l} \right] \right)^2$; $\langle \vec{k} | \vec{k}' \rangle = \delta_{\vec{k}, \vec{k}'}$.

Hamiltonian of the ideal gas (for the lattice model):

$$\hat{H} = \sum_{\vec{k} \in \mathcal{D}} \frac{k^2}{2} a^+_{\vec{k}} a_{\vec{k}} .$$

[This is the non-interacting version of the lattice model used in 2.3.3 and 6.3 of the lectures.]

(2) We consider thermal equilibrium in the grand canonical ensemble. Thus

- The density operator is $\hat{\sigma} = Z^{-1} e^{-\beta(\hat{H} - \mu \hat{N})}$
- $\langle \hat{n}_{\vec{k}} \rangle = \frac{1}{e^{\beta(\frac{k^2}{2} - \mu)} - 1} \equiv n_k$.

I) Glauber - P representation

(1) Notations

- Given a family of complex coefficients $(\alpha_{\vec{k}})_{\vec{k} \in \mathcal{D}}$, we define: $$\psi(\vec{r}) = \sum_{\vec{k} \in \mathcal{D}} \alpha_{\vec{k}} \frac{e^{i \vec{k} \cdot \vec{r}}}{\sqrt{V}} . \quad (*)$$
- The notation $\int \mathcal{D}\psi \cdot F[\psi]$ means $\int \left(\prod_{\vec{k} \in \mathcal{D}} d\alpha_{\vec{k}} \right) F[\psi]$ where ψ is related to the $\alpha_{\vec{k}}$'s by $(*)$, and

$\int d\alpha_{\vec{k}}$ means $\int_{-\infty}^{+\infty} d\alpha_{\vec{k},1} \int_{-\infty}^{+\infty} d\alpha_{\vec{k},2}$ with

$\alpha_{\vec{k}} = \alpha_{\vec{k},1} + i\,\alpha_{\vec{k},2}$ ($\alpha_{\vec{k},1}$ and $\alpha_{\vec{k},2}$ real).

• Given a complex function $\psi(\vec{r})$, one can define $|\psi\rangle$ by $\langle \vec{r}|\psi\rangle = \psi(\vec{r})$, and the coherent state $|\mathrm{coh}:\psi\rangle$ by: $|\mathrm{coh}:\psi\rangle = e^{-\langle\psi|\psi\rangle/2}\, e^{a^\dagger_{|\psi\rangle}}|0\rangle$.

[Remarks: • $\hat{\Psi}(\vec{r})|\mathrm{coh}:\psi\rangle = \psi(\vec{r})|\mathrm{coh}:\psi\rangle$

• $\langle \mathrm{coh}:\psi|\mathrm{coh}:\psi\rangle = 1$.]

2 <u>Property</u> (Glauber - P representation for the ideal gas)

$$\hat{\sigma} = \int \mathcal{D}\psi \quad P[\psi] \quad |\mathrm{coh}:\psi\rangle\langle\mathrm{coh}:\psi| \qquad (1)$$

where $P[\psi] = \mathcal{N}\cdot \exp\left[-\sum_{\vec{k}\in D} \frac{|\alpha_{\vec{k}}|^2}{n_k}\right]$

($\mathcal{N}$ being a normalisation constant)

<u>Rem</u>: With the above probability distribution $P[\psi]$, we see that the $(\alpha_{\vec{k},1})$'s and $(\alpha_{\vec{k},2})$'s are all independent gaussian random variables.

<u>Notation</u>: $\overline{F[\psi]} \equiv \int \mathcal{D}\psi \cdot P[\psi]\cdot F[\psi]$.

So, $\overline{(\alpha_{\vec{k},1})^2} = \overline{(\alpha_{\vec{k},2})^2} = \frac{n_k}{2}$.

$\boxed{\text{Ex. 1}}$ Proove (1).

Hint: • Take matrix elements of (1) in the Fock basis $|(m_{\vec{k}})\rangle$

• Use the relation:

$$|\mathrm{coh}:\psi\rangle = \prod_{\vec{k}\in D}\left[e^{-|\alpha_{\vec{k}}|^2/2}\, e^{\alpha_{\vec{k}} a^\dagger_{\vec{k}}}\right]|0\rangle$$

<u>Solution</u>: Clearly:

$$\langle (m_{\vec{k}})|\hat{\sigma}|(m'_{\vec{k}})\rangle = Z^{-1}\prod_{\vec{k}}\left[\delta_{m_{\vec{k}},m'_{\vec{k}}}\, e^{-\beta m_{\vec{k}}\left(\frac{k^2}{2}-\mu\right)}\right]. \quad (**)$$

The matrix element of the right hand side of (1) between $\langle (m_{\vec{k}})|$ and $|(m'_{\vec{k}})\rangle$ is:

$$R = \int \left(\prod_{\vec{k}} d\alpha_{\vec{k}}\right) \mathcal{N} \exp\left[-\sum_{\vec{k}} \frac{|\alpha_{\vec{k}}|^2}{n_k}\right] \langle (m_{\vec{k}}) | \mathrm{coh}: \psi \rangle \langle \mathrm{coh}: \psi | (m'_{\vec{k}}) \rangle \quad (4.3)$$

$$= \mathcal{N} \prod_{\vec{k}} \int d\alpha_{\vec{k}} \; \exp\left(-\frac{|\alpha_{\vec{k}}|^2}{n_k} - |\alpha_{\vec{k}}|^2\right) \cdot \frac{(\alpha_{\vec{k}})^{m_{\vec{k}}}}{\sqrt{m_{\vec{k}}!}} \; \frac{(\alpha_{\vec{k}}^*)^{m'_{\vec{k}}}}{\sqrt{m'_{\vec{k}}!}} .$$

Setting $\alpha_{\vec{k}} = \rho e^{i\theta}$, we have $d\alpha_{\vec{k}} = d\rho \cdot \rho \cdot d\theta$, and after integration over θ:

$$R = \mathcal{N} \cdot \prod_{\vec{k}} \delta_{m_{\vec{k}}, m'_{\vec{k}}} \cdot 2\pi \cdot \int_0^\infty d\rho \; \rho \; \frac{\rho^{2m_{\vec{k}}}}{m_{\vec{k}}!} \; e^{-\rho^2\left(\frac{1}{n_k}+1\right)}$$

$$= \mathcal{N} \cdot \prod_{\vec{k}} \delta_{m_{\vec{k}}, m'_{\vec{k}}} \cdot \pi \cdot e^{-\beta\left(\frac{k^2}{2}-\mu\right)(m_{\vec{k}}+1)} .$$

This indeed equals $(**)$, if we take

$$\mathcal{N} = \left[Z \prod_{\vec{k}} \pi \, e^{-\beta\left(\frac{k^2}{2}-\mu\right)} \right]^{-1} . \qquad \square$$

3 Physical interpretation.

In quantum mechanics, a system is fully characterized by its density operator. From (1), we can thus imagine that the gas is in the coherent state $|\mathrm{coh}: \psi\rangle$, with probability distribution $P[\psi]$.

For example, let us consider the following Gedankenexperiment, which mimics an experiment where an image of the atomic positions is recorded on a camera.

- First, measure the atom number $\hat{N}$.

Call N the result (which is random since the gas is at equilibrium in the grand canonical ensemble).

- Then, measure the positions $\vec{r}_1, \dots, \vec{r}_N$ of the N atoms.

One can check easily, using (1), that the result of such an experiment can be simulated exactly by the following algorithm:

- Generate a random field $\psi(\vec{r})$ according to the probability distribution $P[\psi]$.

– Generate a random value $N \in \mathbb{N}$ according to the probability: $P(N) = e^{-\langle \psi|\psi\rangle} \cdot \dfrac{\langle \psi|\psi\rangle^N}{N!}$

– Generate random positions $\vec{r}_1, \ldots, \vec{r}_N$ according to the probability density $\dfrac{|\psi(\vec{r}_1)|^2}{\langle\psi|\psi\rangle} \times \ldots \times \dfrac{|\psi(\vec{r}_N)|^2}{\langle\psi|\psi\rangle}$.

II Vortex density

Def

- We shall say that the field $\psi(\vec{r})$ has a vortex at $\vec{R}$ if $\psi(\vec{R}) = 0$
- The charge q of a vortex is $q \equiv \int_\gamma d\theta$, where $\psi(\vec{r}) \equiv A(\vec{r})\, e^{i\theta(\vec{r})}$ [A and θ being real and smooth] and γ is a small contour around $\vec{R}$.

One then easily checks that $q \in \mathbb{Z}$.

Notations: • $\vec{r} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$; $\psi(\vec{r}) = \psi_1(\vec{r}) + i\,\psi_2(\vec{r})$ [$\psi_1, \psi_2 \in \mathbb{R}$]

$\vec{\psi}(\vec{r}) \equiv \begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{pmatrix}$

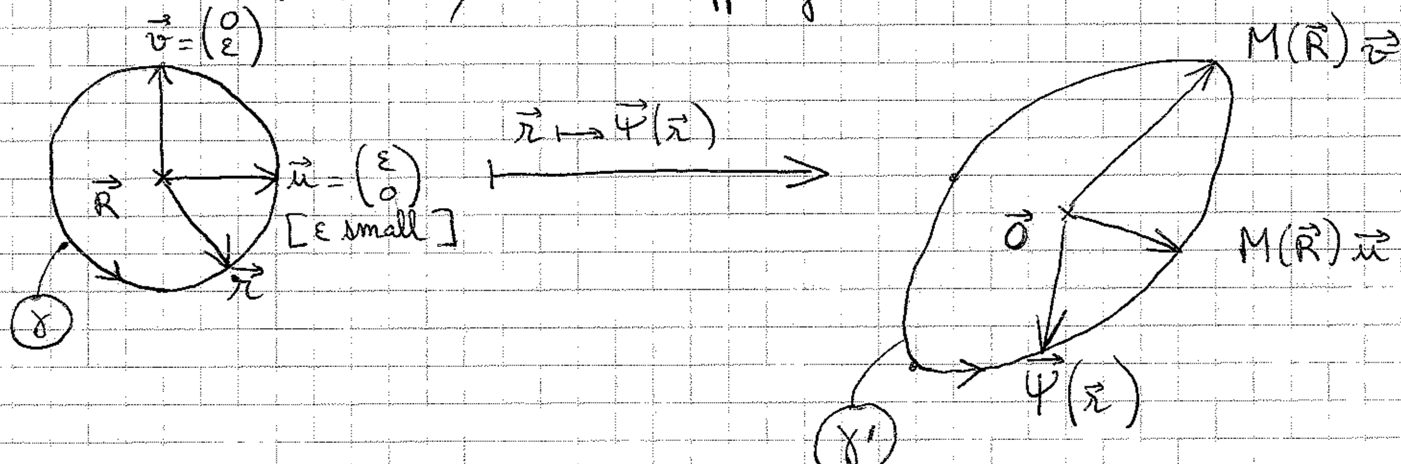
• $M_{ij}(\vec{r}) \equiv \partial_j \psi_i(\vec{r}) \equiv \dfrac{\partial \psi_i}{\partial x_j}(\vec{r})$.

Prop: $\det M(\vec{R}) \neq 0 \Rightarrow q = \pm 1$.

Justification: For $\vec{r} \simeq \vec{R}$, $\psi_i(\vec{r}) \simeq \vec{\nabla}\psi_i(\vec{R}) \cdot (\vec{r} - \vec{R})$

$\Rightarrow \vec{\psi}(\vec{r}) \simeq M(\vec{R}) \cdot (\vec{r} - \vec{R})$.

For $\det M(\vec{R}) \neq 0$, the mapping $\vec{r} \mapsto \vec{\psi}(\vec{r})$ looks like:



Thus we see: $q = \dfrac{1}{2\pi}\displaystyle\int_\gamma d\theta = \dfrac{1}{2\pi}\int_{\gamma'} d\theta = \pm 1$

"Clearly" one has with probability 1 that:

- The vortices of $\psi(\vec{r})$ are isolated points $\vec{R}_\ell$
- $\det M(\vec{R}_\ell) \neq 0$, $\forall \ell$; and

thus all vortices have a charge $q = \pm 1$.

<u>Def</u>: • Vortex density for a given realisation $\psi(\vec{r})$ of the field: $\rho_v(\vec{r}) \equiv \sum_\ell \delta^2(\vec{r} - \vec{R}_\ell)$

• Mean vortex density: $\overline{\rho_v(\vec{r})}$ $\left[= \int \mathcal{D}\psi \, P[\psi] \, \rho_v(\vec{r}) \right]$

Let us calculate this. [We follow: B.I. Halperin, in "Physics of defects", Les Houches Lecture Series, Vol. 35 (1981), page 813; M.V. Berry and M.R. Dennis, Proc. R. Soc. London, Ser. A <u>456</u>, 2059 (2000).]

<u>Prop</u>: $\sum_\ell \delta^2(\vec{r} - \vec{R}_\ell) = \delta^2(\vec{\psi}(\vec{r})) \cdot |\det M(\vec{r})|$

<u>Justif</u>: Both sides of the equation vanish, except for $\vec{r}$ "infinitely close" to $\vec{R}_\ell$, in which case the equation becomes: $\delta^2(\vec{X}) = \delta^2(N\vec{X}) \cdot |\det N|$

where $\vec{X} \equiv \vec{r} - \vec{R}_\ell$, $N \equiv M(\vec{R}_\ell)$.

Since $\det N \neq 0$, this is indeed true, because for any function $f(\vec{Y})$,

$$f(\vec{0}) = \int d^2\vec{Y} \, f(\vec{Y}) \, \delta^2(\vec{Y}) \underset{(\vec{Y} = N\vec{X})}{=} \int d^2\vec{X} \cdot |\det N| \cdot f(N\vec{X}) \, \delta^2(N\vec{X})$$

and on the other hand $f(\vec{0}) = \int d\vec{X} \, f(N\vec{X}) \, \delta^2(\vec{X})$. ☒

So, $\overline{\rho_v(\vec{r})} = \overline{\delta^2(\vec{\psi}(\vec{r})) \cdot |\det M(\vec{r})|}$.

Translational invariance implies that $\overline{\rho_v(\vec{r})} = \overline{\rho_v(\vec{0})} \equiv \rho_v$.

We also set $\psi(\vec{0}) \equiv \psi$, $\vec{\nabla}\psi(\vec{0}) \equiv \vec{\nabla}\psi$, $M(\vec{0}) \equiv M$.

So, $\rho_v = \overline{\delta^2(\vec{\psi}) \cdot |\det M|}$.

We recall: $\psi = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \alpha_{\vec{k}}$, $\vec{\nabla}\psi = \frac{1}{\sqrt{V}} \sum_{\vec{k}} i\vec{k} \, \alpha_{\vec{k}}$

Therefore:

Prop: The (ψ_β)'s and the $(\partial_\gamma \psi_\delta)$'s $[\beta, \gamma, \delta = 1 \text{ or } 2]$ are gaussian random variables. (i)

Ex 2 S.t. $\overline{\psi_\beta \cdot \partial_\gamma \psi_\delta} = 0$ (ii)

Solution: $\overline{\psi_\beta \cdot \partial_\gamma \psi_\delta} = \frac{1}{V} \sum_{\vec{k}} \underbrace{\overline{(\alpha_{\vec{k}})_\beta \cdot (i\alpha_{\vec{k}})_\delta}}_{\text{depends only on } \|\vec{k}\|} \cdot k_\gamma = 0$. ☐

From (i) and (ii) it follows that ψ and $\vec{\nabla}\psi$ are independent, so that: $\boxed{\rho_0 = \overline{\delta^2(\vec{\psi})} \cdot \overline{|\det M|}}$

$$\overline{\delta^2(\vec{\psi})} = \overline{\delta(\psi_1)\delta(\psi_2)} = \overline{\delta(\psi_1)} \cdot \overline{\delta(\psi_2)}$$

$\uparrow$ because $\alpha_{\vec{k},1}$ and $\alpha_{\vec{k},2}$ are independent

$$\overline{\delta(\psi_1)} = \int d\psi_1 \cdot \delta(\psi_1) \cdot e^{-(\psi_1)^2/(2\overline{\psi_1^2})} \Big/ \sqrt{2\pi\,\overline{(\psi_1)^2}}$$

$$= 1 \Big/ \sqrt{2\pi\,\overline{\psi_1^2}}$$

$$\overline{\psi_1^2} = \frac{1}{V} \sum_{\vec{k}} \overline{(\alpha_{\vec{k},1})^2} = \frac{1}{2V} \sum_{\vec{k}} n_k .$$

$$\boxed{\overline{\delta^2(\vec{\psi})} = \frac{1}{\frac{\pi}{V} \sum_{\vec{k}} n_k}}.$$

$$|\det M| = \left| \det \begin{bmatrix} \partial_1 \psi_1 & \partial_1 \psi_2 \\ \partial_2 \psi_1 & \partial_2 \psi_2 \end{bmatrix} \right| = \| \vec{\nabla}\psi_1 \times \vec{\nabla}\psi_2 \| .$$

Now,

$$\overline{\partial_\beta \psi_\gamma \cdot \partial_{\beta'} \psi_{\gamma'}} = \frac{1}{V} \sum_{\vec{k}} k_\beta k_{\beta'} \underbrace{\overline{(i\alpha_{\vec{k}})_\gamma (i\alpha_{\vec{k}})_{\gamma'}}}_{= \delta_{\gamma,\gamma'} \cdot \frac{n_k}{2}}$$

$$= \frac{1}{V} \sum_{\vec{k}} \frac{k^2}{2} \frac{n_k}{2} \delta_{\beta,\beta'} \delta_{\gamma,\gamma'} .$$

Thus the $(\partial_\beta \psi_\gamma)$'s are 4 independent gaussian random variables, of variance $\boxed{\sigma^2 = \frac{1}{4V} \sum_{\vec{k}} k^2 n_k}$.

So, setting $\vec{a} = \vec{\nabla}\psi_1$ and $\vec{b} = \vec{\nabla}\psi_2$,

$$\overline{|\det M|} = \int d\vec{a}\, d\vec{b}\, \|\vec{a} \times \vec{b}\| \exp\left(-\frac{a^2+b^2}{2\sigma^2}\right) \Big/ (2\pi\sigma^2)^2$$

$$= 2\pi \int_0^\infty da \cdot a \int_0^\infty db\, b \int_0^{2\pi} d\theta \cdot ab|\sin\theta| \exp\left(-\frac{a^2+b^2}{2\sigma^2}\right) \Big/ (2\pi\sigma^2)^2$$

$|\det M| = \sigma^2$.

Finally,
$$\boxed{\rho_v = \frac{1}{4\pi} \frac{\sum_{k} k^2 \, n_k}{\sum_{k} n_k}}$$

That is,
$$\boxed{\rho_v = \frac{1}{2\pi} \frac{\langle \hat{H} \rangle}{\langle \hat{N} \rangle}}$$

We can now take the continuum limit $l \to 0$, and the thermodynamic limit.

Limiting regimes:

- <u>Non-degenerate regime</u> ($\rho \lambda^2 \ll 1$, where $\lambda = \sqrt{\frac{2\pi}{T}}$)

  Then (see Part 1, Ex. 8): $\langle \hat{H} \rangle \simeq \langle \hat{N} \rangle \cdot T$.

  So $\rho_v \simeq \frac{T}{2\pi} = \frac{1}{\lambda^2}$, and the mean number of vortices N_v divided by the mean number of atoms is: $\frac{N_v}{\langle \hat{N} \rangle} = \boxed{\frac{\rho_v}{\rho} \simeq \frac{1}{\rho \lambda^2} \gg 1}$

- <u>Degenerate regime</u> ($\rho \lambda^2 \gg 1$)

One can show that $\beta\mu \to 0^-$ in this regime.

Thus $\langle \hat{H} \rangle \simeq \sum_{\vec{k}} \frac{1}{e^{\beta \frac{k^2}{2}} - 1} \frac{k^2}{2}$

and
$$\frac{\langle \hat{H} \rangle}{\langle \hat{N} \rangle} \underset{t.l.}{\sim} \frac{1}{\rho} \int \frac{d^2 \vec{k}}{(2\pi)^2} \frac{k^2/2}{e^{\beta k^2/2} - 1}$$
$$= \frac{T^2}{\rho} \cdot \frac{\pi}{3},$$

$$\boxed{\frac{\rho_v}{\rho} = \left(\frac{T}{\rho}\right)^2 \frac{\pi}{3} = \frac{1}{[\rho \lambda^2]^2} \frac{4}{3} \pi^3 \gg 1}$$

<u>Physical consequence</u>:

Can one see the vortices by taking an image of the gas as in the Gedankenexperiment of I.3 ?

That is, can one see the zeroes of $\Psi(\vec{r})$, from one realisation where $N (\simeq \langle \hat{N} \rangle)$ atomic positions are generated with the probability density

$$\frac{|\psi(\vec{r}_1)|^2}{\langle\psi|\psi\rangle} \times \dots \times \frac{|\psi(\vec{r}_N)|^2}{\langle\psi|\psi\rangle} \quad ?$$

- Non-degenerate regime: No, because $\rho_0 \gg \rho$
- Degenerate regime: Yes, because $\rho_0 \ll \rho$.

[See also: Y. Castin, Z. Hadzibabic, S. Stock, J. Dalibard, S. Stringari, Phys. Rev. Lett. 96, 040405 (2006).]