Second-chance exam of Quantum Mechanics

M2 of quantum physics 2012-2013

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For each exercise, please write your answer in the box provided for this purpose. No explanation or intermediate result is required. The exercises are independent one from the other; each exercise brings 3 points for a correct answer, and 0 points for a wrong answer.

1. One considers four identical bosons that can occupy two orthonormal modes $|\alpha\rangle$ and $|\beta\rangle$. What is the state vector $|\Psi\rangle$ that represents the state with three particles in the mode $|\alpha\rangle$ and one particle in the mode $|\beta\rangle$? $|\Psi\rangle$ must be given in first quantization in reduced form (with the minimal number of terms) and normalised. For conciseness, one may omit the symbol $\otimes$ in the tensorial products.

$|\Psi\rangle =$

2. One considers a one-dimensional quantum harmonic oscillator, therefore of Hamiltonian $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$. One assumes that the system is at thermal equilibrium at temperature T. One sets $k = (2m\omega/\hbar)^{1/2}$ and $\theta = \tanh[\hbar\omega/(2k_BT)]$. What is, as a function of θ, the expectation value of the operator $\exp(ikX)$ in the state of the system?

$\langle\exp(ikX)\rangle =$

3. One considers a degree of freedom a of a quantum bosonic field (with $[a, a^\dagger] = 1$), that has a non-linear dynamics with the Hamiltonian $H = \frac{\hbar\chi}{2}(a^\dagger a)^2$. The initial state is the Glauber coherent state $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n\geq 0}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, where $|n\rangle$ is the Fock state with n bosons. What is the minimal value (minimised over time $t \geq 0$) of the modulus $|\langle a(t)\rangle|$, where $a(t)$ is the operator a in Heisenberg picture?

$\inf_t |\langle a(t)\rangle| =$

4. One considers, in dimension three, a spinless spatially homogeneous Bose gas, with weakly repulsive interactions, at thermal equilibrium in the regime of a quasi-pure condensate. According to Bogoliubov theory, what is the first-order coherence function $g_1(\mathbf{r}) = \langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{0})\rangle$, where $\hat{\psi}$ is the usual bosonic field operator? One is

required to give the result for $g_1(\mathbf{r})$ in terms of the real amplitudes U_k and V_k, and of the occupation numbers n_k of the Bogoliubov modes of wavevectors $\mathbf{k}$, as well as in terms of the gas total density ρ. Moreover, the result will be given in the thermodynamic limit (which leads to an integral over the wavevectors $\mathbf{k}$).

$g_1(\mathbf{r}) =$

5. In three dimensions, one scatters a quantum particle of mass μ on a square well potential, $V(r) = -\frac{\hbar^2 k_0^2}{2\mu}$ for $r < b$, $V(r) = 0$ for $r > b$. What is the s-wave scattering length a in terms of b and k_0?

$a =$

6. In three dimensions, one considers a distinguishable particle of mass m coupled to a spatially homogeneous ideal gas of spinless fermions of same mass m and of density ρ. Interaction of the impurity with the fermions is described by the cubic lattice model of lattice constant b introduced in the lecture, with a bare coupling constant g_0 linked to the effective coupling constant $g = 4\pi\hbar^2 a/m$ by $g_0 = g/(1 - gI)$, with $I = \int_{\mathrm{FZB}} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}}$. Here $E_{\mathbf{k}} = \hbar^2 k^2/(2m)$ and "FBZ" is the first Brillouin zone $[-\pi/b, \pi/b)^3$. Also the notation "FS" stands for the Fermi sea, that is the ball centered at the origin and with a radius equal to the Fermi wavevector k_F. Treating the interaction to second order of perturbation theory, one has already found the following expression for the correction to the ground state energy:

$$\Delta E = \int_{\mathrm{FS}} \frac{d^3q}{(2\pi)^3} \left[g_0 - g_0^2 \int_{\mathrm{FBZ}\setminus\mathrm{FS}} \frac{d^3k}{(2\pi)^3} \frac{1}{E_{\mathbf{k}} + E_{\mathbf{k}-\mathbf{q}} - E_{\mathbf{q}}} + O(g_0^3) \right]$$

that reads in the limit $k_F b \to 0$, after an appropriate rewriting:

$$\Delta E = \rho g (1 + \eta k_F a) - g^2 \int_{\mathrm{FS}} \frac{d^3q}{(2\pi)^3} \int_{\mathbb{R}^3\setminus\mathrm{FS}} \frac{d^3k}{(2\pi)^3} \left(\frac{1}{E_{\mathbf{k}} + E_{\mathbf{k}-\mathbf{q}} - E_{\mathbf{q}}} - \frac{1}{2E_{\mathbf{k}}} \right) + O(g^3)$$

What is the value of the constant η?

$\eta =$

7. One considers a degree of freedom a of a bosonic quantum field ($[a, a^\dagger] = 1$). This constitutes the system S, that one couples to a zero-temperature reservoir R, so that the density operator ρ_S of S evolves according to a master equation of the Lindblad form, characterized by the hermitian Hamiltonian $H_S = \hbar\omega a^\dagger a$, $\omega > 0$, and by the single jump operator $C = \gamma^{1/2} a$. At time zero, the system S is prepared in the ground state of H_S. According to the quantum regression theorem, what is at time $t \geq 0$ the correlation function $\langle a(t) a^\dagger(0) \rangle$, where $a(t)$ is the operator a in Heisenberg picture for the hamiltonian evolution of the whole $S + R$ ensemble?

$\langle a(t) a^\dagger(0) \rangle =$