FUNDAMENTAL CONCEPTS OF MANY-BODY PHYSICS

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LECTURE 1. BOSONS

1. A bit of history
   - definition(s) of "superfluidity"
2. BEC in a noninteracting gas.
   - conditions for occurrence
   - effect of conservation laws.
3. BEC in generic case
   - order parameter
   - superfluid velocity
   - "spinful" case
   - when and why?
   - (non-)existence theorems
4. BEC and "superfluidity"
   - effect of internal DOF
5. GP description of dilute gas
6. Is GP self-consistent?
   - need for Bogoliubov-level descrⁿ.
7. (if time): spin-1/2 Bose gas
   - KSA state
   - Ramsey-fringe expts

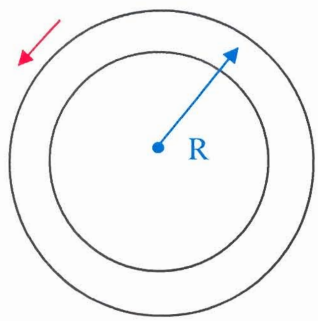
SUPERFLUIDITY IN LIQUID ^{4}HE

^{4}He liquefied:	1908	
$T < T_\lambda$ (2.17 K):	1920	~ 20 YEARS!
Frictionless flow below T_λ:	1938	

Modern point of view:

Define

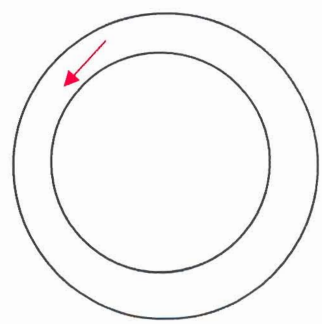
$\omega_c \equiv \hbar/mR^2 \equiv$ quantum unit of rotation (~10^{-4} Hz for R ~ 1cm)



EXPT. A
("Hess-Fairbank" effect)

walls rotate with
ang. velocity $\lesssim \omega_c$,
liquid stationary

EQUILIBRIUM
EFFECT



EXPT B
(Persistent currents)

walls at rest,
liquid rotating with
ang. velocity $\gg \omega_c$.

METASTABLE
EFFECT

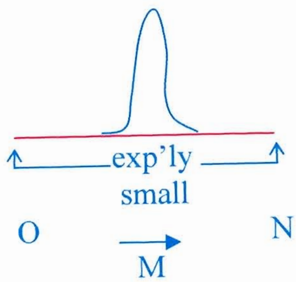
BEC IN A NONINTERACTING BOSE GAS: THE EFFECTS OF STATISTICS

I. Qualitative argument:

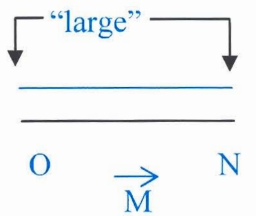
Distribute N objects between 2 boxes: what is probability P(M) of finding M in one box?

A. Objects distinguishable ($\equiv$ coin toss):

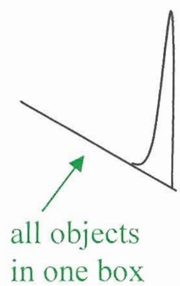
$P(M) = \dfrac{N!}{M!\ N\text{-}M!}$



B. Objects indistinguishable (bosons):



P(M) = 1/N



II. Quantitative argt. (Einstein, 1925):

chemical potential, ≤ 0

$$n_i(T) = [\exp(\varepsilon_i - \mu(T)/k_B T - 1)]^{-1}$$

$$\mu(T) \text{ fixed by: } \sum_i n_i(T: \mu(T))) = N$$ ← total no. of particles

$T \to \infty \Rightarrow \mu \to -\infty$: $T\downarrow \Rightarrow \mu\uparrow$. But what if

$$\sum_i [\exp(\varepsilon_i/k_B T) - 1]^{-1} < N?$$

Einstein: Macroscopic no. of particles occupy lowest ($\varepsilon = 0$) state!

WHEN is the condition for BEC,

$$\sum_i \left(\exp(\varepsilon_i/kT) - 1\right)^{-1} < N$$

satisfied (for noninteracting gas)?

Introduce

$$\rho(\epsilon) \equiv \sum_i \delta(\epsilon - \epsilon_i) \Leftarrow \text{single-particle DOS}$$

then condition is

$$\int_0^\infty \frac{\rho(\epsilon)\,d\epsilon}{\exp(\epsilon/kT) - 1} < N$$

If $\rho(\epsilon) \underset{\epsilon \to 0}{\sim} \epsilon^n$, $n \leq 0$, integral divergent

$\Rightarrow$ no BEC. This happens in free d-dim^l. space

($\rho(\epsilon) \sim \epsilon^{(\frac{d}{2}-1)}$) for $d \leq 2$, $\Rightarrow$

NO BEC IN FREE SPACE FOR $d \leq 2$.

In harmonic trap, $\rho(\epsilon) \sim \epsilon^{d-1} \Rightarrow$ no BEC for $d = 1$.

If BEC does occur, rough criterion is

no. of states with en. less than $k_B T < N$

Impt. special cases:

$$2\pi(\hbar^2/mk_B)\cdot(\zeta(3/2))^{-2/3} g_s^{-2/3}$$

3D free space: $T < T_c = \text{const.}\,(N/V)^{2/3}$

("$n\lambda_T^3 \sim 1$")

3D isotropic harmonic trap:

$$T < T_c = \text{const.}\, N^{1/3}(\hbar\omega_0/k_B)$$

$$(g_s\zeta(3))^{-1/3}$$

"Condensate Fraction" in a Noninteracting Gas

To get total no. right, must have

$$N_0(T) + \underbrace{\sum_{i \neq 0} (\exp(\epsilon_i / k_B T) - 1)^{-1}}_{N_n(T)} = N$$

But by df. of T_c

$$\underbrace{\sum_{i \neq 0} (\exp \epsilon_i / k_B T_c - 1)^{-1}}_{N_n(T_c)} = N$$

$\Rightarrow$

$$N_0(T) + N_n(T) = N_n(T_c) = N$$

$\Rightarrow$

"condensate fraction" $\rightarrow$ $$\frac{N_0(T)}{N} = 1 - \frac{N_n(T)}{N_n(T_c)} = \left(1 - \left(\frac{T}{T_c}\right)^n\right) \qquad \rho(\epsilon) \sim \epsilon^n$$

e.g. in 3D free gas

$$N_0(T)/N = 1 - (T/T_c)^{3/2}$$

in 3D harmonic trap

$$N_0(T)/N = 1 - (T/T_c)^3$$

Can Conservation Laws Prevent BEC?

Yes! Ex.: KSA* state

N spin-1/2 bosons, unpolarized for $T \gg T_c$ and cooled by spin-conserving mechanism.

Problem: if all condensed into $\underset{\sim}{k} = 0$ orbital state, spin state must be completely symmetric $\Rightarrow \langle S^2 \rangle \sim N^2$!

Possible ansatz:

$$\Psi = \Psi_{KSA} \equiv (a^+_{0\uparrow} a^+_{1\downarrow} - a^+_{0\downarrow} a^+_{1\uparrow})^{N/2} |vac\rangle$$

* Kuklov-Svistunov-Ashhab

DEFINITION OF BEC IN GENERAL CASE (Penrose-Onsager-Yang)

(interacting system, not nec? in thermal equilibrium)

A. "Spinless" case (e.g. ^4He):

For any pure N-particle state $\Psi_N \equiv \Psi(\underset{\sim}{r}_1 \underset{\sim}{r}_2 \ldots \underset{\sim}{r}_N : t)$

define

$$\rho_1(\underset{\sim}{r}, \underset{\sim}{r}' : t) \equiv N \int dr_2 \ldots dr_N \, \Psi_N^*(\underset{\sim}{r}, r_2 r_3 \ldots r_N : t) \cdot \Psi_N(r', r_2 r_3 \ldots r_N : t)$$

single-particle density matrix

"behavior of 1st particle averaged over N-1 others"

For a mixed N-particle state, ($\Psi_N^{(s)}$ with prob. p_s)

simply average ρ_1 over states s, i.e.

$$\rho_1(\underset{\sim}{r}, \underset{\sim}{r}' : t) \equiv \sum_s p_s \rho_1^{(s)}(\underset{\sim}{r}, \underset{\sim}{r}' : t) \quad (\equiv \langle \psi^\dagger(rt)\psi(r't)\rangle)$$

Theorem: since $\rho_1(\underset{\sim}{r}, \underset{\sim}{r}' : t)$ is Hermitian, can be diagonalized:

$$\rho_1(\underset{\sim}{r}, \underset{\sim}{r}' : t) = \sum_i n_i(t) \chi_i^*(\underset{\sim}{r}t) \chi_i(\underset{\sim}{r}'t)$$

$\langle a_i^+ a_j \rangle(t) = \delta_{ij} n_i(t)$

eigenvalues — eigenfunctions

This result is general (also for non-Bose systems).

Note: eigenfunctions $\chi_i(\underset{\sim}{r}t)$ not necessarily eigenfunctions of single-particle Hamiltonian. If they are, in particular for tr-inv^t system in eq^m, then

$$n_i(t) \to n_k(t) \equiv \langle a_k^+ a_k \rangle(t)$$

Df. of BEC in General Case (cont.)

Recap: quite generally,

$$\rho_1(\underset{\sim}{r}, \underset{\sim}{r}' : t) = \sum_i n_i(t)\,\chi_i^*(\underset{\sim}{r}t)\,\chi_i(\underset{\sim}{r}'t)$$

1. If no n_i is $O(N)$, "normal"
2. If several n_i are $O(N)$, "fragmented" BEC.
3. If one and only one of the n_i is $O(N)$, ("simple") BEC.

In case of simple BEC, call relevant value of i 0: then

$$n_0(t) \equiv N_0(t) \equiv \text{"condensate number"}$$

$$\chi_0(rt) \equiv \text{"wave function of condensate"}$$

Order Parameter:

$$\Psi(\underset{\sim}{r}, t) \equiv \sqrt{N_0(t)}\,\chi_0(\underset{\sim}{r}t)$$

no need to invoke "spontaneous breaking of U(1) gauge symmetry"

Superfluid Velocity:

$$\chi_0(\underset{\sim}{r}t) \equiv |\chi_0(\underset{\sim}{r}t)| \exp i\varphi(rt)$$

$$\underset{\sim}{v}_s(\underset{\sim}{r}t) \equiv \frac{\hbar}{m}\underset{\sim}{\nabla}\varphi(rt) \qquad \leftarrow \text{irrotational}$$

(contrast: $v_n(rt) \equiv j(rt)/\rho(rt) \equiv \dfrac{\hbar}{m}\dfrac{\sum_i n_i(t)\,|\chi_i(rt)|^2\,\underset{\sim}{\nabla}\varphi(rt)}{\sum_i n_i(t)\,|\chi_i(rt)|^2}$)

not irrotational

B. Df. of BEC ("Spinful" Case)

$$\Psi_N(t) = \Psi(r_1\alpha_1, r_2\alpha_2 \dots r_N\alpha_N : t)$$

$$\rho_1(r\alpha, r'\alpha' : t) \equiv \sum_{\{\alpha_2 \dots \alpha_N\}} \int dr_2 \dots dr_N \, \Psi^*(r\alpha : r_2\alpha_2 \dots r_N\alpha_N : t) \, \Psi(r'\alpha' : r_2\alpha_2 \dots r_N\alpha_N : t)$$

$$\left(\text{or } \sum_s p_s \rho_1^{(s)}(rr't)\right)$$

$$\rho_1(r\alpha, r'\alpha' : t) = \sum_i n_i(t) \chi_i^*(r\alpha : t) \chi_i(r'\alpha' : t)$$

("Simple") BEC if one and only one of $n_i(t)$ is $O(N)$. If so, then df N_0, χ_0 in obvious way, and

$$\Psi(r, \alpha : t) \equiv \sqrt{N_0(t)} \, \chi_0(r, \alpha : t)$$

Can df. superfluid velocity (in simple way) only if

$$\chi_0(r, \alpha : t) = \chi_0(r, t) \, \psi(\alpha : t)$$

spin "orientation" constant in space.

(then $\chi_0(rt) = |\chi| \exp i\varphi$, $v_s = \frac{\hbar}{m} \nabla \varphi$ as in spinless case)

Note: generally speaking, "fragmentation" more likely to occur in "spinful" case. E.g. KSA state

$$\Psi_N = (a^+_{0\uparrow} a^+_{1\downarrow} - a^+_{0\downarrow} a^+_{1\uparrow})^{N/2} |vac\rangle$$

has a ρ_1 with 4 eigenvalues of $N/4$.

WHY BEC ?

(a) statistical argt. (Einstein)

(b) for repulsive interaction, BEC is intrinsically energetically advantageous!

Consider problem of 2 identical spinless bosons with effective contact interaction $V_0 \delta(r)$:

$$\Delta E = V_0 |\psi(0)|^2$$

$V_0 = \frac{4\pi\hbar^2 a_s}{m}$ (C.C-T.)

$|\psi(0)|^2 \equiv |\psi(\underset{\sim}{r}_1, \underset{\sim}{r}_2)|^2_{\underset{\sim}{r}_1 = \underset{\sim}{r}_2} \equiv$ prob. of finding particles at same point

What is $|\psi(0)|^2$?

A. Particles in same single-particle state $\chi(r)$:

$$\psi(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = \chi(r_1)\chi(r_2)$$

$$\Rightarrow |\psi(0)|^2 = |\chi(r)|^4.$$

B. Particles in different (orthogonal) single-particle states $\chi_1(r), \chi_2(r)$:

$$\psi(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = \frac{1}{\sqrt{2}}\left(\chi_1(r_1)\chi_2(r_2) + \chi_2(r_1)\chi_1(r_2)\right)$$

symmetriz^n

norm^n

$$\Rightarrow |\psi(0)|^2 = \frac{1}{2} \cdot |2\chi_1(r)\chi_2(r)|^2$$

$$= 2|\chi_1(r)|^2 \cdot |\chi_2(r)|^2$$

hence eg for plane-wave states, particles in different states interact twice as strongly as in same state!

WHY BEC? (cont.)

Suppose for some reason we need to have two different single-particle states $\chi_1(r), \chi_2(r)$ macroscopically occupied ($\langle N_1\rangle \sim \langle N_2\rangle \sim N$). (ex: decay of superflow in annular geometry) Two obvious extreme possibilities:

(a) $\Psi_N^{(F)} \equiv (a_1^+)^{N_1}(a_2^+)^{N_2}|vac\rangle$ Fock state, "fragmented"

$(N_1 \cong \langle N_1\rangle,\ N_2 \cong \langle N_2\rangle)$

(b) $\Psi_N^{(GP)} \equiv (\alpha a_1^+ + \beta a_2^+)^N|vac\rangle$ GP (coherent) state, simple BEC

$(|\alpha|^2 = \langle N_1\rangle/N,\ |\beta|^2 = \langle N_2\rangle/N)$

note: $\Psi_N^{(GP)} \equiv \Psi_N(\Delta\varphi)$, $\Delta\varphi \equiv \arg(\alpha/\beta)$

$$\Psi_N^{(F)} = \frac{1}{2\pi}\int_0^{2\pi} d(\Delta\varphi)\, \Psi(\Delta\varphi) \exp i(N_1 - N_2)\Delta\varphi$$

If we assume $\langle\Delta\varphi|\hat{H}|\Delta\varphi'\rangle$ is negligible, then

$$\langle H\rangle_{Fock} = \frac{1}{2\pi}\int_0^{2\pi} \langle H\rangle_{\Delta\varphi}\, d(\Delta\varphi)$$

$$\Rightarrow \quad \exists\ \Delta\varphi : \langle H\rangle_{\Delta\varphi} \leq \langle H\rangle_{Fock}$$

in fact, for single-particle potential $V(r)$

$$\langle H\rangle_{\Delta\varphi} - \langle H\rangle_{Fock} = 2N|\alpha|\cdot|\beta|\, \mathrm{Re}\left\{e^{i\Delta\varphi}\int V(r)\chi_1(r)\chi_2^*(r)\,dr\right\}$$

and for 2-particle int^n $U_0\delta(r_1 - r_2)$,

$$\langle H\rangle_{\Delta\varphi} - \langle H\rangle_{Fock} = 2NU_0\, \mathrm{Re}\{Ae^{i\Delta\varphi} + Be^{2i\Delta\varphi}\}$$

$$A \equiv 2|\alpha|\cdot|\beta|\int(|\alpha|^2|\chi_1(r)|^2 + |\beta|^2|\chi_2(r)|^2)\cdot\chi_1(r)\chi_2^*(r)\,dr$$

CONCLUSION:

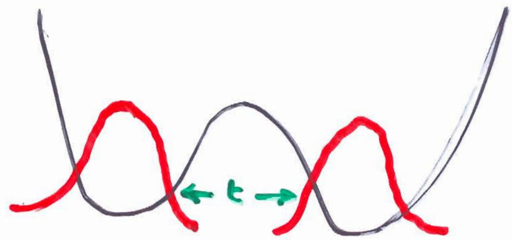
NATURE LIKES SIMPLE BEC!

One and only one $n_i \sim O(N)$

Some exceptions:

1. "Coulomb blockade"

   ($\langle \Delta\varphi | H | \Delta\varphi' \rangle$ not negligible)



2. Fragmentation in internal DOF: nb. no "factor-of-2" in this case to prevent it!

   due to interactions

   ex: LPB state of spin-1 bosons, all in $\underset{\sim}{k} = 0$ orbital state but

$$\Psi^{(spin)}_{LPB} = (a^+_1 a^+_1 + a^+_{-1} a^+_{-1} - a^+_0 a^+_0)^{N/2} |vac\rangle$$

3. Fragmentation in internal DOF due to conservⁿ law:

   ex: KSA state

$$\Psi_{KSA} = (a^+_{0\uparrow} a^+_{1\downarrow} - a^+_{0\downarrow} a^+_{1\uparrow})^{N/2} |vac\rangle$$

lowest orbital — next lowest orbital

4. Rotation close to instability

RIGOROUS THEOREMS ON BEC

(interacting system)

1. EXISTENCE AT T=0

3D free space, pert^n theory starting from noninteracting gas convergent: Gavoret + Nozières 1964

hard-core lattice gas at half filling: Kennedy et al. 1988.

2. EXISTENCE AT T≠0

infinite-range interaction : Toth, Penrose 1992

no proof for short-range interactions
(?: Lieb + Seiringer Dec. 01)

3. NONEXISTENCE AT T≠0

free space, $d \leq 2$: Hohenberg 1967

many extensions to partially finite geometries, etc.

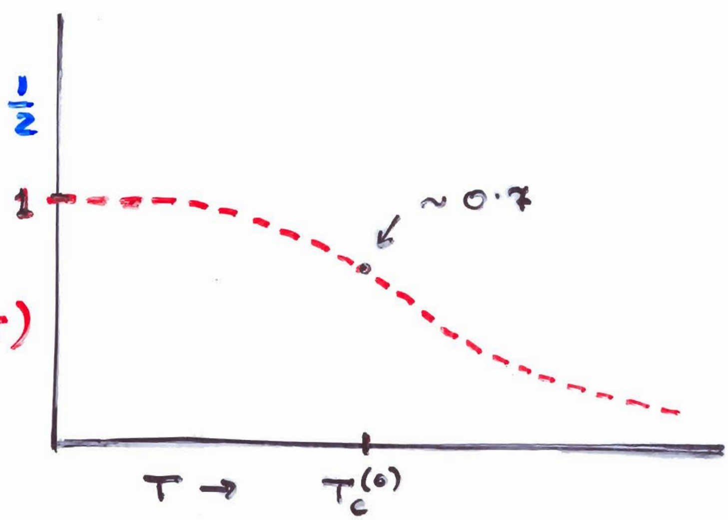
4. UPPER BOUND ON $f \equiv N_0/N$

Hohenberg's lemma: (general for velocity-ind^t interactions):

$$\langle n_k \rangle \geq \left(\frac{m k_B T}{\hbar^2 k^2}\right) f - \frac{1}{2}$$

3D free space: ⇒

$$\frac{f}{(1-f)^{2/3}} \leq \gamma (T_c^{(0)}/T)$$



(Roepstorff (1978): $\gamma = 2$)

(Rigorous) Theorems on BEC, cont.

5. Pertⁿ theory (nonrigorous!) suggests repulsive interactions increase T_c : specifically (3D free space)

$$\Delta T_c / T_c^{(0)} = \text{const.} \, (n a_s^3)^{1/3}$$

(e.g. Baym et al. 2000).

QUESTION: Can we derive an upper bound on f which is tighter than the Hohenberg-derived one, and in particular tends to the free-gas value for interaction $\Rightarrow 0$?

ANSWER: Yes, at least for a simple model of interactions. (A.J.L., New Journ. of Physics 3, 23 (2001))

Model: N spinless bosons in vol. Ω, $N, \Omega \to \infty$,

$N/\Omega \to$ const. $\equiv n$.

Interaction: $\frac{1}{2}\sum_{ij} V(\underset{\sim}{r}_i - \underset{\sim}{r}_j)$, $V(\underset{\sim}{r}) \geq 0$, $\forall \underset{\sim}{r}$

Method:

Consider free energy $F(N)$ $(\equiv -k_B T \ln \mathrm{Tr}_N \exp - \hat{H}/kT)$

(i) Derive (f-independent) upper limit on F. (F_{max})

(ii) Derive (f-dependent) lower limit on F. $(F_{min}(f))$

(iii) Then $F_{min}(f) \leq F_{max} \Rightarrow$ upper bound on f.

(Assume, for simplicity only, that condensation is "simple" + occurs in $\underset{\sim}{k} = 0$ state)

RIGOROUS THEOREMS ON BEC, cont.

Step 1: Upper bound on $F(N)$

From "Hartree-Fock" variational ansatz,

$$\hat{\rho}_N = Z^{-1} \exp - \beta \underset{\substack{\uparrow \\ \text{KE only}}}{\hat{H}_0} \equiv \hat{\rho}_N^{(0)}$$

$$\Rightarrow \quad F(N) \leq F_0(N) + \tfrac{1}{2}(\underset{\substack{\uparrow \\ \text{"Hartree"}}}{NnV_0} + \Omega^{-1} \sum_{k \neq k'} \underset{\substack{\uparrow \\ \text{"Fock"}}}{V_{\underset{\sim}{k}-\underset{\sim}{k}'} \langle n_k \rangle \langle n_{k'} \rangle}$$

since $V(r) \geq 0$, $V_k \leq V_0$, $\forall \underline{k}$ $\qquad V_0 \equiv \int V(\underset{\sim}{r})\, d^3\underset{\sim}{r}$

$\Rightarrow$

$$F(N, \Omega, T) \leq F_0(N, \Omega, T) + NnV_0 \equiv F_{max}$$

Step 2: Lower bound on $F(N)$

Principle: $F(N, \Omega, T)$ cannot be less than $F_0(N(1-f), \Omega, T)$, otherwise we could construct a density matrix $\hat{\rho}_{N(1-f)}^{(0,\text{trial})}$ which does better for the noninteracting gas with $N(1-f)$ particles than the standard one $\hat{\rho}_{N(1-f)}^{(0)}$!

Proof: apply to true density matrix $\hat{\rho}_N$ operator $\hat{Y} \equiv (a_0)^{\hat{N}_0} (\hat{N}_0 !)^{-1/2}$, i.e. ...

Rigorous Theorems on BEC, cont.

To create trial density matrix for noninteracting gas of $N(1-f) \equiv N - N_0$ particles, start with exact density matrix of N particles and remove all the particles in the condensate, leaving rest unchanged.*

(Technically: $\hat{\rho}_{trial} \equiv \hat{\Upsilon}\hat{\rho}_N\hat{\Upsilon}^\dagger$)

$$\left\{ \begin{array}{l} \text{KE unchanged (since } \epsilon_0 \equiv 0) \\ \text{Entropy unchanged } (1 \to 1 \text{ mapping}) \\ \text{PE, originally } \geq 0 \text{, is identically zero for} \\ \quad \text{noninteracting system} \end{array} \right.$$

$$\Rightarrow \quad F_0^{trial}(N(1-f), \Omega, T) \leq F(N, \Omega, T)$$

But, if $F_0^{trial} < F_0(N(1-f), \Omega, T)$, we have found a better density matrix for $N(1-f)$ particles than the "trivial" one $\hat{\rho}_{N(1-f)} \equiv Z^{-1} \exp - \beta \hat{H}_0$!

Thus,

$$F(N, \Omega, T) \geq F_0(N(1-f), \Omega, T) \equiv F_{min}(f)$$

* Technical complication: $[\hat{\rho}_N, \hat{N}_0] \neq 0$. See paper.

Rigorous Theorems on BEC, cont.

We have proved:

(i) $F(N,\Omega,T) \leq F_0(N,\Omega,T) + NnV_0$

(ii) $F(N,\Omega,T) \geq F_0(N(1-f),\Omega,T)$

Thus,

$$\boxed{F_0(N(1-f),\Omega,T) - F_0(N,\Omega,T) \leq NnV_0}$$

free energy of noninteracting gas.

This is an implicit limit on f. To make it explicit, need to bound LHS below by an explicit function of f (messy but straightforward).

Final result: $(T \geq T_c^{(0)})$

Limiting cases:

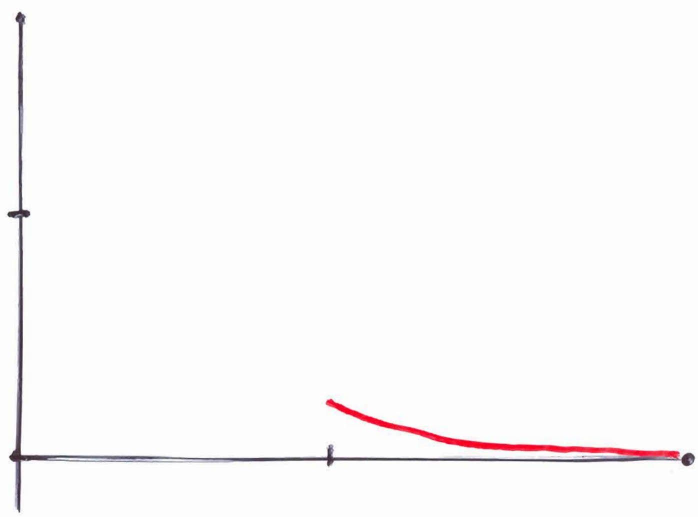
$T = T_c^{(0)}$: $f \leq \text{const.} \left(\dfrac{nV_0}{kT_c^{(0)}}\right)^{1/3}$

(const. $\cong$ 2.2)

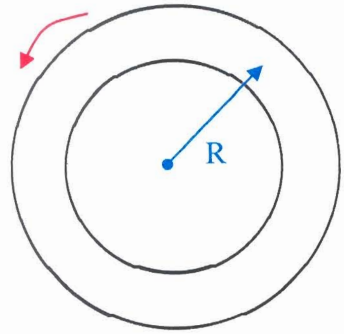
$\dfrac{nV_0}{kT_c^{(0)}} \ll 1 - T_c^{(0)}/T \ll 1$:

$f \leq \text{const.} \dfrac{nV_0}{kT_c}\left(1 - \left(\dfrac{T_c^{(0)}}{T}\right)^{3/2}\right)^{-2}$

(const. $\cong$ 3.3)



EXPLANATION OF HESS-FAIRBANK EFFECT IN TERMS OF BEC:



Walls rotating with ang. velocity

$\omega \lesssim \omega_c \Leftarrow \equiv \hbar/m R^2$

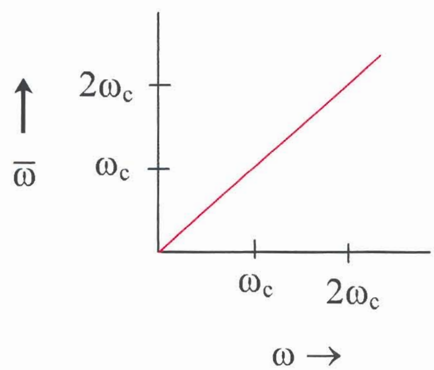
What does liquid do?

General principle: Average ang. velocity of atoms ($\overline{\omega}$) as close as possible to ω

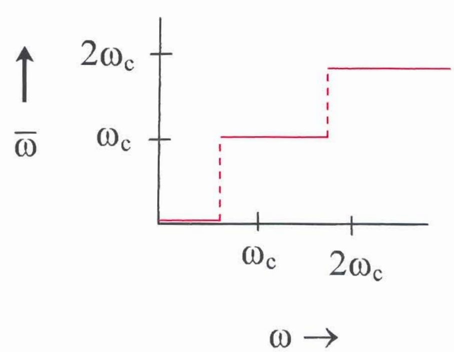
↑ : Single-atom states must obey quantization condition: $\omega = n\omega_c$ ($\ell = n\hbar$)

A. "Normal" (non-BEC) system: many different single-particle states occupied (typical value of $n \sim (kT/\hbar\omega_c)^{1/2} \sim 10^7$)

$\Rightarrow$ to get $\overline{\omega} = \omega$, just shift atoms slightly between states.



B. BEC system ($T \ll T_c$) (almost) all atoms in condensate $\longrightarrow$ must have same value of n (n_o) $\Rightarrow \overline{\omega} \cong n_o\, \omega_c$



INTERACTIONS "OPTIONAL"

^{4}He: PERSISTENT CURRENTS

Initially, after walls stopped,

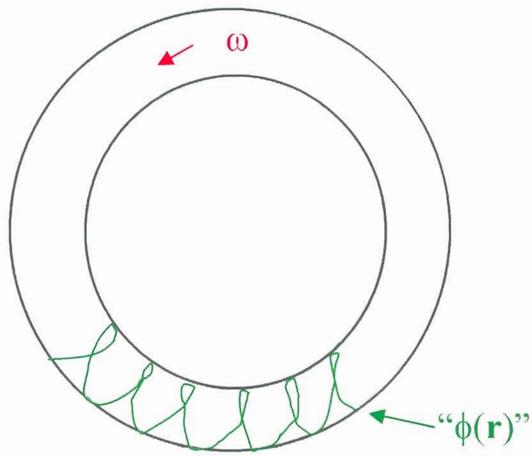
$\langle L \rangle = N_o \ell_o \hbar, \quad \ell_o \gg 1 \quad (\bar{\omega} \gg \omega_c)$

But groundstate has $\langle L \rangle = 0$. $(\omega = 0)$

Why no relaxation?

$\chi_o(\mathbf{r}) = |\chi_o(\mathbf{r})| \exp i\, \phi(\mathbf{r})$

condensate w.f.



Df: "winding no." $n \equiv \oint \dfrac{\nabla\phi \cdot d\mathbf{l}}{2\pi}$

Initially, $n = \ell_o$: eq^m state has $n = 0$.

To change n, must depress $|\chi_o(\mathbf{r})|$ to zero somewhere!

(a) Electron in atom:

Schrödinger eqn. linear $\Rightarrow$ nodes cost no extra energy, e.g.

$$\psi(t) = a(t)\,\psi_p + b(t)\,\psi_s \qquad \begin{cases} t \to -\infty: & a=1,\ b=0 \\ t \to +\infty: & a=0,\ b=1 \end{cases}$$

$\langle E \rangle(t) = |a(t)|^2 E_p + |b(t)|^2 E_s$ = monotonically decreasing

(b) BEC (^{4}He):

Extra term in energy: $\langle V \rangle = \underbrace{V_o}_{>0} \int |\chi_o(\mathbf{r}t)|^4 \, d\mathbf{r}$

$\Rightarrow$ energy NOT monotonically decreasing!

(REPULSIVE) INTERACTIONS ESSENTIAL!

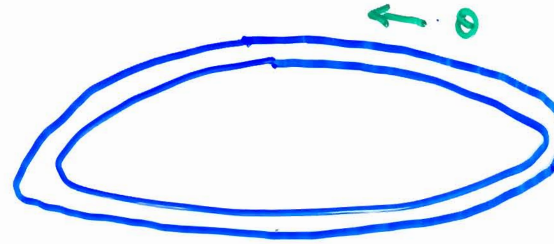
4: DOES "TOPOLOGICAL CONSERVATION LAW" HOLD FOR ANY SYSTEM WITH BEC?

No!

Argt. above assumed scalar condensate wave function (OP) (no internal DOF)

What if condensate has internal (eg hyperfine) DOF? e.g. $S = 1/2$:

$\leftarrow \theta$



suppose initially

$$\psi \equiv \psi_p = \exp i\theta \, |\uparrow\rangle$$

If at all times $\psi(\theta : \sigma) = f(\theta)|\uparrow\rangle$, then topological argt. still holds $\Rightarrow$ no decay possible. But:

rot^n of $|\uparrow\rangle$ to $|\downarrow\rangle$ around axis $\underset{\sim}{n}$ in xy-plane making $\angle\, \varphi$ with x-axis gives $\underline{n} \cdot \hat{\underline{\sigma}} \, |\uparrow\rangle = e^{-i\varphi} |\downarrow\rangle$

Hence, if we choose $\varphi = \theta$, can rotate so that

$$\exp i\theta \, |\uparrow\rangle \longrightarrow |\downarrow\rangle$$

$(L = N\hbar)$ $\qquad$ $(L = 0)$

without anywhere depressing $\rho(\theta) \equiv |\psi(\theta)|^2$!

$\Rightarrow$ BEC (even with repulsive interactions) IS NOT A SUFFICIENT CONDITION FOR METASTABILITY OF SUPERFLOW!