THE UNITARY GAS

OUTLINE:

I. Simple facts

II. How to model the interaction

III. Dynamical scaling invariance in a trap

I. SIMPLE FACTS

WHAT IS THE UNITARY GAS ?

A gas...

- **a dilute system with respect to interaction range:**

$$nb^3 \ll 1$$

- **Scattering amplitude f_k matters rather than $V(r)$**

...at unitary limit:

- **For relevant relative momentum k, f_k reaches maximal modulus: maximally interacting gas**

$$f_k = -\frac{1}{ik}$$

- **From optical theorem indeed:**

$$\mathrm{Im}\, f_k = k\,|f_k|^2 \Rightarrow f_k = -\frac{1}{u(k) + ik}, \quad u(k) \ \text{real}$$

WHAT THIS IMPLIES FOR AN ATOMIC GAS

S-wave low k expansion of scattering amplitude:

$$u(k) = \frac{1}{a} - \frac{1}{2}k^2 r_e + \dots$$

- a is scattering length
- r_e is effective range
- $\dots$ assumed negligible for $kb \ll 1$

Unitary gas as a double limit:

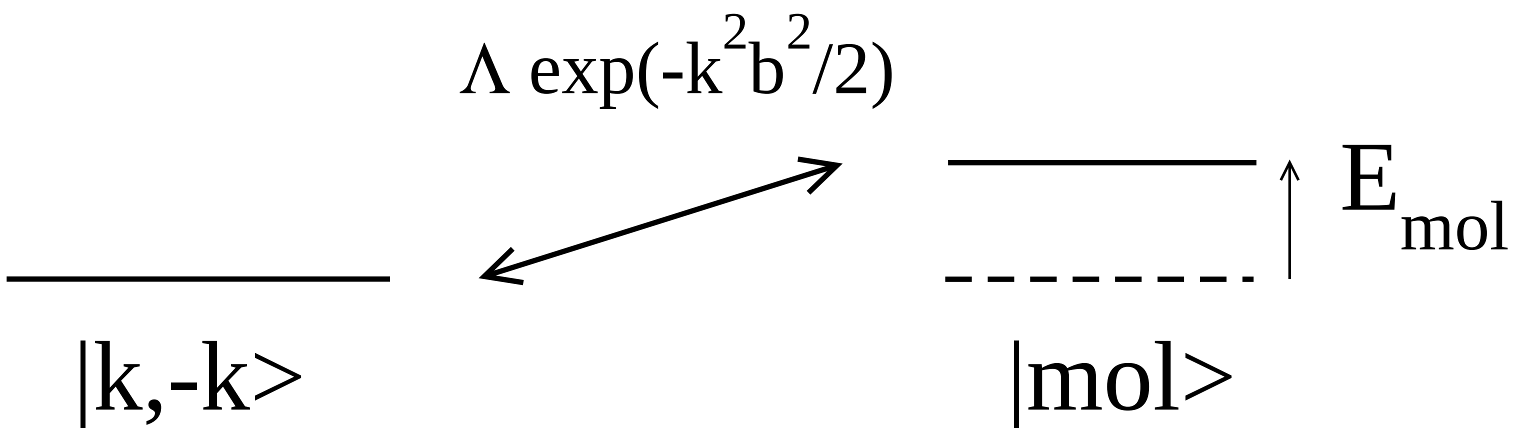
(1) zero range limit $kb \ll 1$, $k|r_e| \ll 1$

(2) infinite scattering length limit: $k|a| \gg 1$

- If one assumes $k \sim n^{1/3}$ double limit achieved in present experiments on broad Feshbach resonances ($|r_e| \sim b$).
- Assumption $k \sim n^{1/3}$ not necessarily true (effective three-body Efimov attraction, bosons or large mass ratio fermions)

CAN ONE HAVE r_e NON ZERO WITH $b \to 0$

Yes, simple two-channel model of Feshbach resonance:



- **Tune $E_{\rm mol}$ to have $|a| = \infty$**
- **Then effective range**

$$r_e = \frac{4b}{\pi^{1/2}} - \frac{8\pi\hbar^4}{m^2\Lambda^2}$$

WHY IS THE UNITARY GAS FASCINATING ?

Universality:

- no parameter left describing the interaction
- eigenenergies E_i depend on $\hbar^2/m$ and on shape of container $U(\vec{r})$: unit of length set by the container!

Spatial scaling invariance:

- Remains unitary if one changes volume of container.
- Not true for fixed finite value of a: $n^{1/3}a$ changes.
- If one applies to container a similarity of factor λ:

$$E_i \longrightarrow \frac{E_i}{\lambda^2}$$

$$\psi_i(\vec{X}) \longrightarrow \frac{\psi_i(\vec{X}/\lambda)}{\lambda^{3N/2}}$$

DIRECT CONSEQUENCES

In harmonic isotropic trap:

$$\frac{E_i}{\hbar\omega} = \mathrm{function}_i(N).$$

In free space:

- No bound state can be at unitarity.

In a box at thermodynamic limit:

- Assume that $E_0/N = e_0$, $F/N = f$ are intensive.

$$e_0(n/\lambda^3) = e_0(n)/\lambda^2 \rightarrow e_0(n) = \eta e_0^{\text{ideal Fermi gas}}(n).$$

$$f(n/\lambda^3, T/\lambda^2) = f(n, T)/\lambda^2.$$

- Taking the derivative in $\lambda = 1$:

$$\frac{5}{3}E - \mu N = TS \qquad \text{(Zwerger)}$$

IS THERE UNITARITY IN LOWER DIMENSIONS ?

In 1D:

- Tonks-Girardeau Bose gas.
- Mappable to an ideal Fermi gas.

In 2D:

- Low-k scattering characterized by a_{2D}:

$$-\frac{1}{f_k} = -\ln(ka_{2D}/2) - \gamma + i\pi/2 + \ldots$$

$$\psi_0(r) = \ln(r/a_{2D}) \quad \text{for} \quad r > b.$$

- No scale invariance for finite a_{2D}.
- $a_{2\mathrm{D}} \longrightarrow +\infty$: ideal gas.
- Have $n^{1/2} a_{2D} \sim 1$ to maximize interactions.

IS THERE UNITARITY IN OTHER PARTIAL WAVES ?

P-wave interaction for fully polarized fermions:

$$u(k) = \frac{1}{k^2 \mathcal{V}_s} + \alpha + \dots$$

- Tune $\mathcal{V}_s$ to infinity with Feshbach resonance.
- Can one have $\alpha = 0$ at resonance ?
- Lower bound for compact support potential of radius b:

$$\alpha_{\mathrm{res}} b \geq 1. \qquad \text{(Pricoupenko)}$$

- For $\mathcal{V}_s$ large and negative, $|u(k)| \ll k$ around

$$k_0 = \frac{1}{\sqrt{\alpha |\mathcal{V}_s|}}.$$

IS THE UNITARY GAS ATTRACTIVE OR REPULSIVE ?

Common sayings:

- $a > 0$: **effective repulsive interaction.**
- $a < 0$: **effective attractive interaction.**
- $|a| = \infty$: **gas properties do not depend on the sign of a.**

Naive way out of this paradox: (Kokkelmans)

- **mean field with k-dependent coupling constant $-\mathrm{Re}\, f_k$**
- **unitary gas would then be non-interacting.**

IS THE UNITARY GAS ATTRACTIVE OR REPULSIVE ?

Answer to paradox in short:

- **Start from weakly interacting gas.**
- **Two adiabatic procedures**

$$a = 0^+ \rightarrow a = +\infty \quad \text{and} \quad a = 0^- \rightarrow a = -\infty$$

lead to different states, that is they follow different branches.

Illustration on a toy model for fermions (Pricoupenko, Castin):

- **A matter wave in hard wall spherical cavity of radius R**

$$\phi(R) = 0 \qquad R \sim n^{-1/3}$$

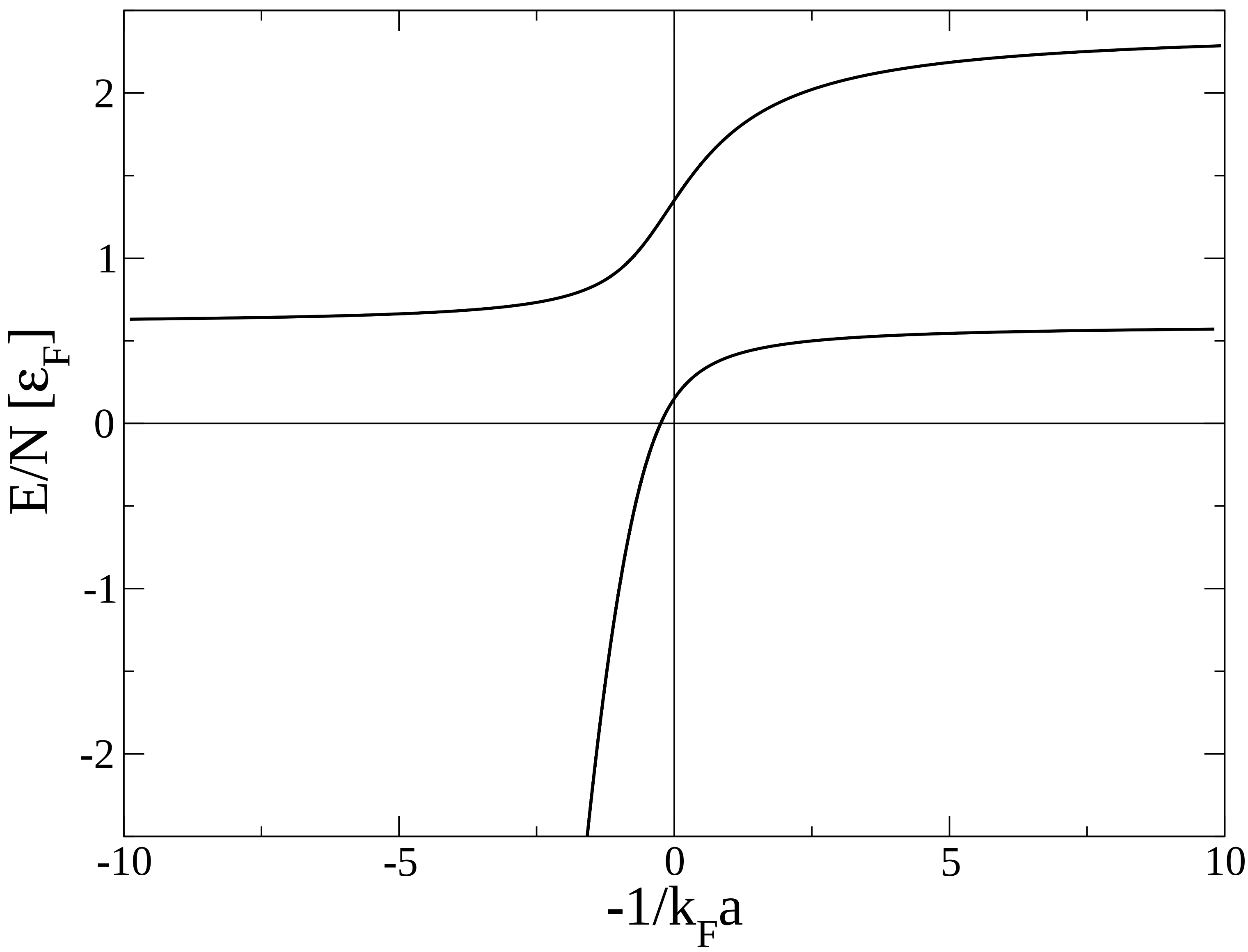
to mimick Pauli exclusion principle.

- **In presence of a scattering center at the origin:**

$$\phi(r) = A\left(\frac{1}{r} - \frac{1}{a}\right) + o(1).$$

to mimick nearest neighbour interaction.

THE LOWEST ENERGY BRANCHES OF TOY MODEL



II. HOW TO MODEL THE INTERACTION

APPROACH 1

A finite range model:

- potential with finite range b and infinite a
- calculate eigenenergies, thermodynamic properties, ...
- go to $b = 0$ limit at the end of the calculation

Non-trivial question: universality

- Eigenstate universal, i.e. reaches unitary limit, if (E_i, ψ_i) converge for $b \to 0$.
- Typical non-universal state: $E_i \to -\infty$
- To use $\rho = \exp(-\beta H)$, avoid models with non-universal states: negative $V(r)$ not good for large N (Seiringer, Lobo)
- Favor models solvable by Quantum Monte Carlo.

APPROACH 2

Replace interaction by Bethe-Peierls contact conditions:

- **Hamiltonian is the one of the ideal gas**

$$H = -\frac{\hbar^2}{2m}\Delta_{\vec{X}} + \frac{1}{2}m\omega^2 X^2$$

- **The domain $D(H)$ is not the ideal gas one!**
- **Contact cond. for $r_{ij} \to 0$ at fixed centroid $\vec{R}_{ij} \neq \vec{r}_k$:**

$$\psi(\vec{X}\,) = A_{ij}(\vec{R}_{ij}; \{\vec{r}_k, k \neq i, j\}) \left[\frac{1}{r_{ij}} - \frac{1}{a}\right] + o(1)$$

- **Scale invariance of $D(H)$ to ensure universality**

  if $\psi \in D(H), \psi_\lambda \in D(H)\, \forall \lambda > 0$ with $\psi_\lambda(\vec{X}\,) = \psi(\vec{X}/\lambda)$.

NB. Here we exclude $\vec{r}_i = \vec{r}_j$. Otherwise a regularized delta interaction pseudo-potential appears.

REMINDER: DOMAIN OF A HAMILTONIAN

Practical definition:

- $D(H)$ is the set of wave functions over which the action of Hamiltonian is represented by differential operator H.
- If one does not care, paradoxes ... due to errors.

Simple example:

- One particle in 1D in a box:

$$H = -\frac{1}{2}\frac{d^2}{dx^2}$$

with boundary conditions $\psi(0) = \psi(1) = 0$.

- A wavefunction in the domain:

$$\psi(x) = x(1-x).$$

$$\langle H \rangle_\psi = 5 \quad ; \quad \langle H^2 \rangle_\psi = 0?!$$

This last result is wrong: $H\psi \notin D(H)$. Right value: 30.

NON-TRIVIAL QUESTION IN APPROACH 2

Is the Hamiltonian self-adjoint ?

- **This amounts to proving the unitarity of the gas.**
- **For $N = 2$: answer is yes. (book by Albeverio)**
- **For $N = 3$ bosons: no. See later.**
- **For $N = 3$ equal mass fermions: probably yes.**
- **For $N \geq 4$ equal mass fermions: ?**

Partial universality:

- **Restrict H to subspace where it is hermitian.**
- **This means: A non-complete family of universal states.**
- **For $N = 3$ bosons: all universal states determined. See later. (Jonsell, Heiselberg and Pethick; Werner and Castin)**

- For arbitrary number N of bosons, trivial universal states (common to ideal gas) with $A_{ij} \equiv 0$:

$$\psi(\vec{X}) \to 0 \quad \text{for} \quad r_{ij} \to 0.$$

- These trivial states dominate the ideal gas density of states at high energy. (Werner and Castin)

A TRIVIAL QUESTION IN APPROACH 2

I see no interaction energy in $\boldsymbol{H}$, is the energy of kinetic nature only ?

Answer: no.

$$E_{\mathrm{kin}} = \int \frac{\hbar^2}{2m} |\partial_{\vec{X}} \psi|^2 = +\infty.$$

$$E_{\mathrm{kin}} + E_{\mathrm{int}} = -\int \frac{\hbar^2}{2m} \psi^* \Delta_{\vec{X}} \psi.$$

OUR CANDIDATE FOR APPROACH 1

A Hubbard-type lattice model (here for spin 1/2 fermions):

- cubic lattice of step b.
- "tunneling": one-body eigenstates are plane waves with dispersion relation ϵ_k

$$\vec{k} \in \mathcal{D} \equiv \left[-\frac{\pi}{b}, \frac{\pi}{b}\right[^3 \quad \text{and} \quad \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

- on-site interaction with coupling constant g_0

$$H = \sum_{\vec{k}\in\mathcal{D}} \sum_{\sigma=\uparrow,\downarrow} \epsilon_k a^\dagger_{\vec{k},\sigma} a_{\vec{k},\sigma} + g_0 \sum_{\vec{r}} b^3 \hat{\psi}^\dagger_\uparrow(\vec{r}) \hat{\psi}^\dagger_\downarrow(\vec{r}) \hat{\psi}_\downarrow(\vec{r}) \hat{\psi}_\uparrow(\vec{r})$$

- Field commutation relations mimicking continuous space ones:

$$\{\hat{\psi}_\sigma(\vec{r}), \hat{\psi}^\dagger_{\sigma'}(\vec{r}\,')\} = \delta_{\sigma\sigma'} \frac{\delta_{\vec{r},\vec{r}\,'}}{b^3}.$$

HOW TO CHOOSE THE COUPLING CONSTANT g_0

To have the correct scattering length: (Mora, Castin)

- scattering of two particles in the infinite lattice
- for a zero total momentum:

$$H_{\mathrm{rel}} = \frac{p^2}{m} + V \quad \text{with} \quad V = g_0 |\vec{r} = \vec{0}\,\rangle\langle \vec{r} = \vec{0}\,|$$

- calculate the T-matrix on the grid

$$T(E + i0^+) = V + V G_{\mathrm{rel}}(E + i0^+) V$$

- expand at low energy, setting $E = \hbar^2 q^2/m$, $q \geq 0$:

$$\langle \vec{k}\,|T(E + i0^+)|\vec{k}\,'\rangle = \frac{4\pi\hbar^2/m}{a^{-1} + iq + O(q^2 b)}$$

HOW TO CHOOSE THE COUPLING CONSTANT g_0 (2)

Result and discussion:

$$g_0 = \frac{4\pi\hbar^2 a/m}{1 - Ca/b} \quad \text{with} \quad C = 2.442\,749\ldots$$

- Born regime: $|a| \ll b$
- impenetrable regime: $g_0 = +\infty$ gives $a = b/C$
- infinite scattering length:

$$g_0 = -\frac{4\pi}{C}\frac{\hbar^2 b}{m}$$

so an attractive Hubbard-type model with $g_0 \to 0^-$ in unitary limit.

ADVANTAGES OF THIS LATTICE MODEL

For fermions, link with condensed matter physics:

- Unitary limit = zero filling factor limit of Hubbard model with

$$\frac{U}{J} = \frac{g_0/b^3}{\hbar^2/(2mb^2)} = \text{well chosen constant}$$

- Quantum Monte Carlo possible with no sign problem:

$$T_c^{\text{Svistunov}} \simeq 0.15 T_F \quad T_c^{\text{Bulgac}} \simeq 0.2 T_F$$

$$\eta \simeq 0.44 \ \text{ and gap } \ \Delta \simeq 0.44 E_F \quad \text{(Juillet)}$$

From a theoretical point of view:

- no tricky $D(H)$, standard variational methods apply:

$$\eta \leq \eta_{\text{BCS}} = 0.5906\ldots \quad \text{(Randeria)}$$

From an experimental point of view in a lattice:

- For bosons: $|a| = \infty$ without a Feshbach resonance

$b \to 0$ LATTICE MODEL $\Longleftrightarrow$ BETHE-PEIERLS ?

(Pricoupenko, Castin)

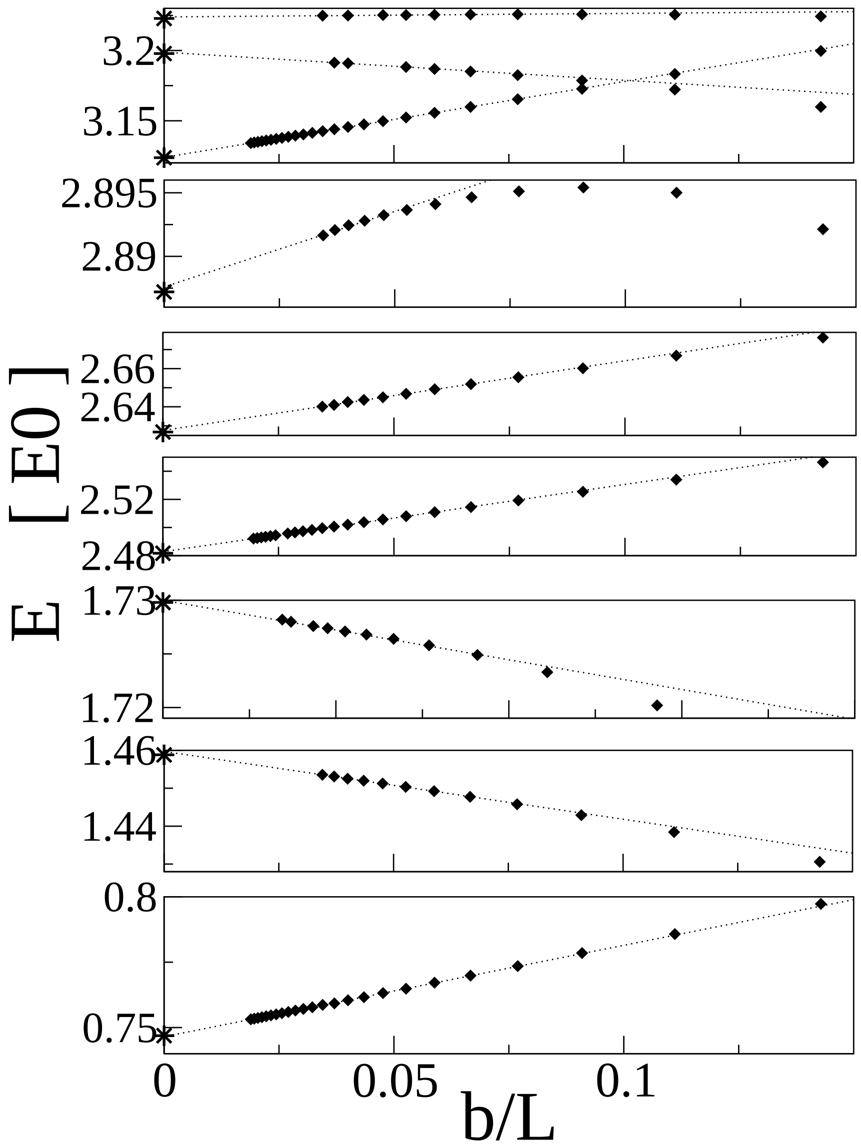
Case of two particles:

- **Proof of equivalence for the eigenergies E_i**

Case of three equal mass fermions:

- **numerically, coincidence.**
- **analytically: if finite limit of $E_i(b)$ exists, coincidence.**
- **all $E_i > 0$ checked up to $b/L = 1/81$ (diagonalisation of a matrix $531\,441 \times 531\,441$)**

THE TWO MODELS FOR 3 EQUAL MASS FERMIONS



CASES OF $b \to 0$ LATTICE MODEL $\neq$ BETHE-PEIERLS

Case of $|a| = \infty$ bosons:

- Variational calculation with $|N : \vec{r} = \vec{0}\rangle$

$$E_0(b) \leq g_0 N[N - 2.92]/(2b^3) \stackrel{b\to 0}{\to} -\infty.$$

- Approaches 1 and 2 are then not equivalent.

Same result for 2 massive fermions and a light particle:

- Variational calculation with the 2 fermions localized on neighboring sites:

$$E_0(b) \leq -0.2\frac{\hbar^2}{mb^2}\left(1 - 42\frac{m}{M}\right)$$

- For a large enough mass ratio M/m, Pauli principle not sufficient to prevent 3-body deeply bound states (see lecture by Petrov).

III. DYNAMICAL SCALING INVARIANCE IN A TRAP

FIRST MOMENT OF THE TRAPPING ENERGY: VIRIAL THEOREM

We consider a normalized eigenstate of H:

$$H\psi = E\psi$$

then one has the virial theorem: (exp. check: Thomas)

$$\langle\psi|H|\psi\rangle = 2\langle\psi|H_{\rm trap}|\psi\rangle$$

with $H_{\rm trap} = \frac{1}{2}m\omega^2 X^2$.

Proof: for a Hermitian H, an eigenstate is a stationary point of the mean energy

$$E(\lambda) \equiv \frac{\langle\psi_\lambda|H|\psi_\lambda\rangle}{\langle\psi_\lambda|\psi_\lambda\rangle} = \lambda^{-2}\langle\psi|H-H_{\rm trap}|\psi\rangle+\lambda^2\langle\psi|H_{\rm trap}|\psi\rangle.$$

$$\left(\frac{dE}{d\lambda}\right)(\lambda = 1) = 0.$$

SCALING SOLUTION IN A TIME DEPENDENT TRAP

Isotropic trap is time dependent for $t > 0$:

- **Free Schrödinger equation over manifold $r_{ij} \neq 0$:**

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\Delta_{\vec{X}} + \frac{1}{2}m\omega^2(t)X^2\right]\psi$$

- **plus contact conditions for $r_{ij} \to 0$:**

$$\psi(\vec{r}_1, \ldots, \vec{r_N}) = \frac{A_{ij}(\vec{R}_{ij}, \{\vec{r_k}, k \neq i, j\})}{r_{ij}} + o(1).$$

- **Initially, stationary state in static trap $\omega(t = 0) = \omega$ with energy E.**

- **Relevant for experiments: time of flight, collective modes.**

Ansatz: gauge plus scaling transform:

$$\psi(\vec{X}, t) = \frac{e^{-i\theta(t)}}{\lambda^{3N/2}(t)} \exp\left[\frac{im\dot{\lambda}}{2\hbar\lambda} X^2\right] \psi(\vec{X}/\lambda(t), 0).$$

- scaling preserves contact conditions
- gauge transform preserves contact conditions:

$$r_i^2 + r_j^2 = 2R_{ij}^2 + \frac{1}{2} r_{ij}^2.$$

- solves Schrödinger equation if

$$\ddot{\lambda} = \frac{\omega_0^2}{\lambda^3} - \omega^2(t)\lambda$$

$$\theta(t) = E \int_0^t \frac{d\tau}{\hbar\lambda^2(\tau)}.$$

Y. Castin, Comptes Rendus Physique 5, 407 (2004).

PRACTICAL INTEREST OF SCALING SOLUTION

Ballistic expansion is a perfect lens:

- **For mean density** $n(\vec{r},t) = \frac{1}{\lambda^3(t)} n_0[\vec{r}/\lambda(t)]$
- **but also for higher order density correlation functions:**

$$g^{(2)}(\vec{r}_1,\vec{r}_2,t) = \frac{1}{\lambda^6(t)} g_0^{(2)}[\vec{r}_1/\lambda(t),\vec{r}_2/\lambda(t)].$$

- **Applies even at** $T > T_c$ **and for all gas polarisations.**
- **But requires** $|a| = \infty$ **and an isotropic harmonic trap.**

Can one relax these two conditions ?

- **At first sight, no:**
  - **finite** $|a|$ **breaks scaling invariance.**
  - **anisotropic trap expected to lead to anisotropic expansion, but anisotropic scaling does not preserve** $D(H)$

- **However there is a clever way to lift the two conditions (Lobo).**

APPLICATION: RAISING/LOWERING OPERATORS

Gedanken experiment: weak change of ω for $0 < t < t_f$:

- Resulting change for the scaling parameter:

$$\lambda(t) - 1 = \epsilon\, e^{-2i\omega t} + \epsilon^*\, e^{2i\omega t} + O(\epsilon^2).$$

  An undamped mode of frequency 2ω (Pitaevskii, Rosch).

- Resulting change for the wavefunction:

$$\psi(\vec{X}, t) = \Big[e^{-iEt/\hbar} - \epsilon e^{-i(E+2\hbar\omega)t/\hbar} L_+ + \epsilon^* e^{-i(E-2\hbar\omega)t/\hbar} L_-\Big]\, \psi(\vec{X}, 0) + O(\epsilon^2)$$

- Raising and lowering operators:

$$L_\pm = \pm \left[\frac{3N}{2} + \vec{X} \cdot \partial_{\vec{X}}\right] + \frac{H}{\hbar\omega} - m\omega X^2/\hbar$$

- Repeated action of $L_\pm$: ladder of eigenenergies with equal spacing $2\hbar\omega$.

LINK WITH SO(2,1) LIE ALGEBRA (Pitaevskii, Rosch)

Trapped unitary gas has SO(2,1) hidden symmetry:

- **Energy ladders directly from commutation relations:**

$$[H, L_\pm] = \pm 2\hbar\omega\, L_\pm \qquad [L_+, L_-] = -4H/(\hbar\omega)$$

- **Do not forget to check that $L_\pm$ preserve domain.**
- **Introduce what will be the generators of the group:**

$$T_1 \pm iT_2 = \frac{L_\pm}{2} \qquad T_3 = \frac{H}{2\hbar\omega}$$

- **Then commutation relations of SO(2,1) Lie algebra:**

$$[T_1, T_2] = -iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

- **Casimir operator, which commutes will all the elements of the algebra**

$$C = -4\left[T_1^2 + T_2^2 - T_3^2\right] = H^2 - (\hbar\omega)^2(L_+L_- + L_-L_+)/2$$

EXISTENCE OF A BOSONIC DEGREE OF FREEDOM

Key point: the ladders are semi-infinite

- Virial theorem: $E \geq 3\hbar\omega/2$. Action of L_- terminates:
$$L_-\psi_g = 0,$$
so one can define the ground energy step operator H_g.
- In terms of Casimir operator:
$$C = H_g(H_g - 2\hbar\omega) \quad \text{so that} \quad H_g = \hbar\omega + \left[C + (\hbar\omega)^2\right]^{1/2}.$$
- From SO(2,1) algebra to creation/annihilation operators
$$b = [2(H+H_g)/\hbar\omega]^{-1/2}L_-, \quad b^\dagger = L_+[2(H+H_g)/\hbar\omega]^{-1/2}$$
$$[b, b^\dagger] = 1.$$
- Unitary gas has a decoupled bosonic degree of freedom, the breathing mode:
$$H = H_g + 2\hbar\omega b^\dagger b \quad \text{with} \quad [b, H_g] = 0.$$

SECOND MOMENT OF THE TRAPPING ENERGY

In principle, fluctuations of trapping energy $H_{\rm trap} = m\omega^2 X^2/2$ are measurable:

- At thermal equilibrium in canonical ensemble:

$$4\langle H_{\rm trap}^2\rangle = \langle H^2\rangle + \langle H\rangle\hbar\omega\left[2\langle b^\dagger b\rangle + 1\right].$$

- Proof:

$$H_{\rm trap} = \frac{1}{2}H - \frac{\hbar\omega}{4}(L_+ + L_-)$$

$$H_{\rm trap} = \frac{\hbar\omega}{2}A^\dagger A \quad \text{with} \quad A = \left[\frac{H_g}{\hbar\omega} + b^\dagger b\right]^{1/2} - b.$$

$$\langle H_g\, b^\dagger b\rangle = \langle H_g\rangle\langle b^\dagger b\rangle.$$

- Thermometry: measuring fluctuations of the breathing mode of the unitary gas

SEPARABILITY IN HYPERSPHERICAL COORDINATES

- Hyperspherical coordinates $(X, \vec{n} \equiv \vec{X}/X)$
- Integrate $L_- \psi_g = 0$:

$$[3N/2 + X\partial_X + E_g/(\hbar\omega) - m\omega X^2/\hbar]\psi_g(\vec{X}) = 0.$$

$$\psi_g(\vec{X}\,) = e^{-m\omega X^2/2\hbar}\, X^{E_g/(\hbar\omega)-3N/2} f(\vec{n}).$$

- Mapping to scale invariant zero energy free space eigenstates (Tan)
- Gives excited ladders in terms of Laguerre polynomials.
- The hyperangular problem was solved by Efimov for $N = 3$.
- This gives the solution to the trapped 3-body unitary problem (Werner, Castin).

MORE DETAILS ON SEPARABILITY FOR $N > 2$

Form of the N-body wavefunction:

$$\psi(\vec{X}\,) = \psi_{CM}(\vec{C}\,)\phi(\vec{\Omega}\,)R^{(5-3N)/2}F(R)$$

- uses separability of the center of mass $\vec{C}$
- uses separability in internal spherical coordinates $(R, \vec{\Omega})$
- contact conditions put a constraint on $\phi(\vec{\Omega}\,)$ only, for Laplacian on unit sphere of dimension $3N-4$:

$$\Delta_{\vec{\Omega}}\phi = -\Lambda\phi.$$

- Effective 2D Schrödinger equation for the radial part:

$$-\frac{\hbar^2}{2m}\Delta_{2D}F(R)+\left(\frac{\hbar^2 s^2}{2mR^2}+\frac{1}{2}m\omega^2R^2\right)F(R) = E_{\text{int}}F(R)$$

with $s^2 = \Lambda + [(3N-5)/2]^2$.

PHYSICAL DISCUSSION FOR $N = 3$

$$-\frac{\hbar^2}{2m}\Delta_{2D}F(R) + \left(\frac{\hbar^2 s^2}{2mR^2} + \frac{1}{2}m\omega^2 R^2\right) F(R) = E_{\text{int}} F(R)$$

Efimov: s is a root of transcendental equations.

A good case: equal mass fermions

- Proof that all $s^2 \geq 0$. (Werner and Castin)
- Then one chooses $s \geq 0$. Numerically $s > 1$.
- Spectrum $E = (s + 1 + 2q)\hbar\omega$, $q \in \mathbb{N}$.

A bad case: bosons

- There is a negative s^2: $s_0 = i \times 1.00624\ldots$ then Whittaker functions are square integrable solutions $\forall E_{\text{int}} \in \mathbb{R}$: Hamiltonian not hermitian.
- Proof that all other s^2 are ≥ 0. (Werner and Castin)

SUGGESTIONS OF EXPERIMENTS

For fermions:

- measure gap Δ
- check/use of scaling transform: breathing mode, measure $g^{(2)}$
- mixture of fermions with different masses

For bosons:

- 3-body universal states for $|a| = \infty$ at the node of an optical lattice
- $|a| = \infty$ bosons in an optical lattice far from a Feshbach resonance.