Disordered Quantum Systems

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Part 1: Introduction

Part 2: BCS + disorder

INSTITUT HENRI POINCARE

Centre Emile Borel

Gaz quantiques

23 avril - 20 juillet 2007

Finite size quantum physical systems

Atoms

Nuclei

Molecules

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.

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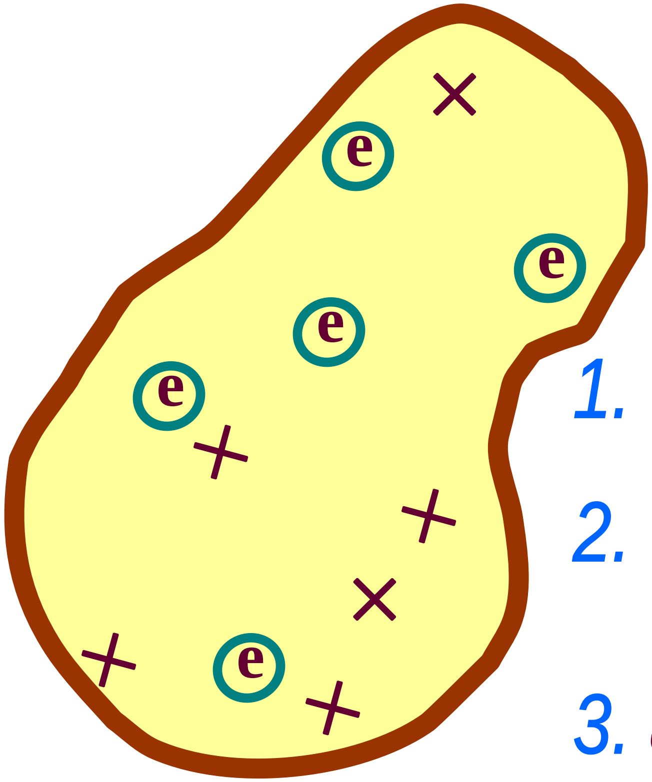
Quantum Dots

Cold gas in a trap ?



Quantum Dot

1. *Disorder (× – impurities)*
2. *Complex geometry*
3. ***e-e*** *interactions*

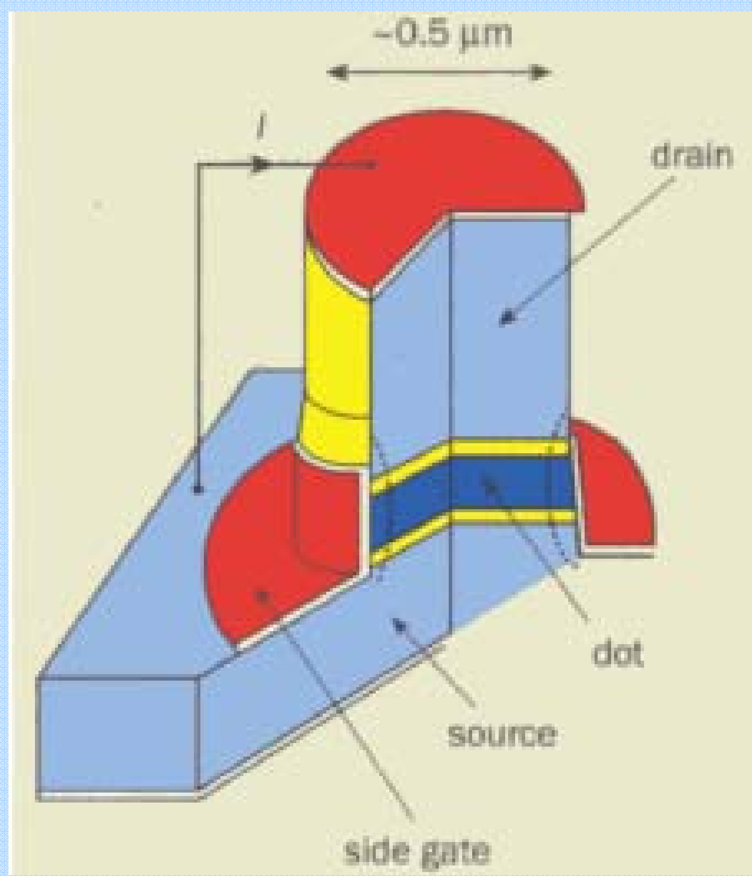
Realizations:

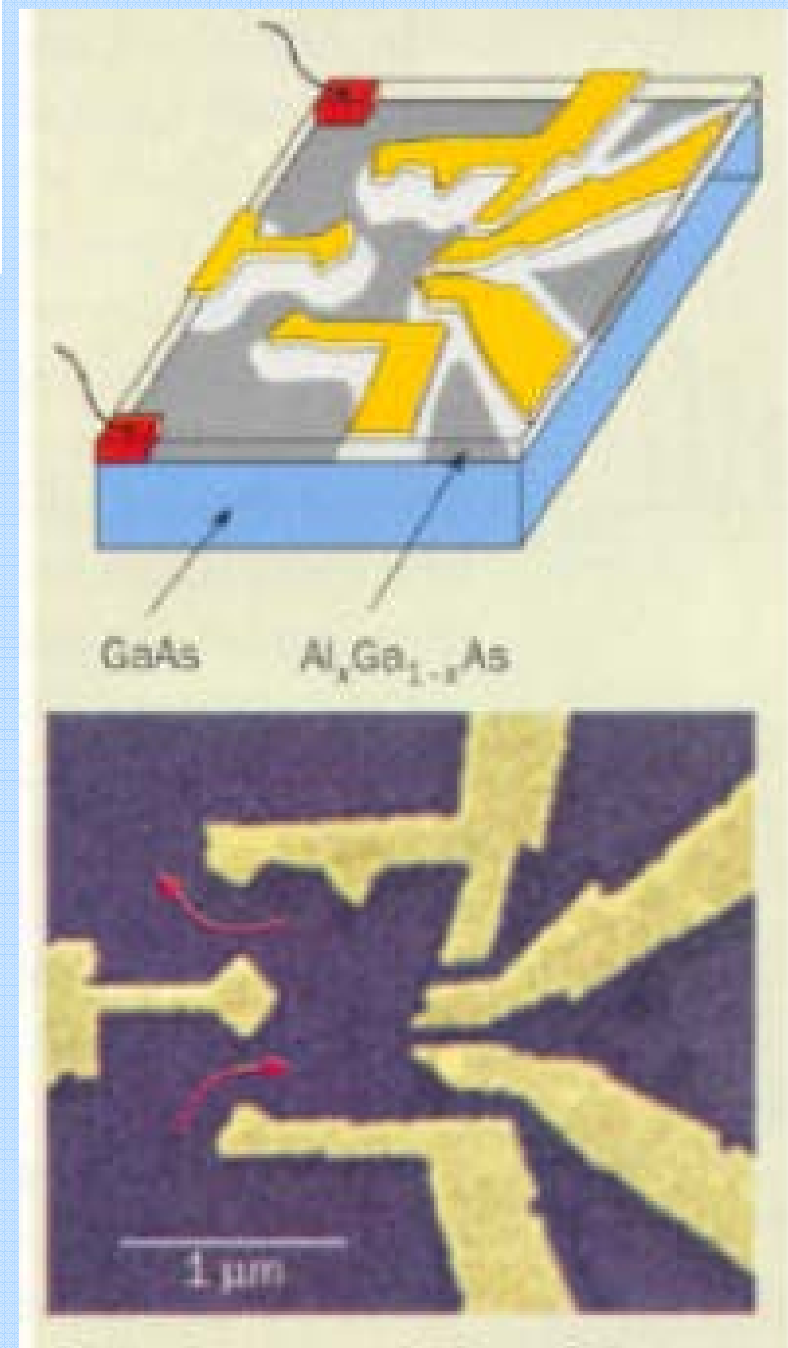
- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes
-
-

Quantum dots

Leo Kouwenhoven and Charles Marcus

PHYSICS WORLD JUNE 1998



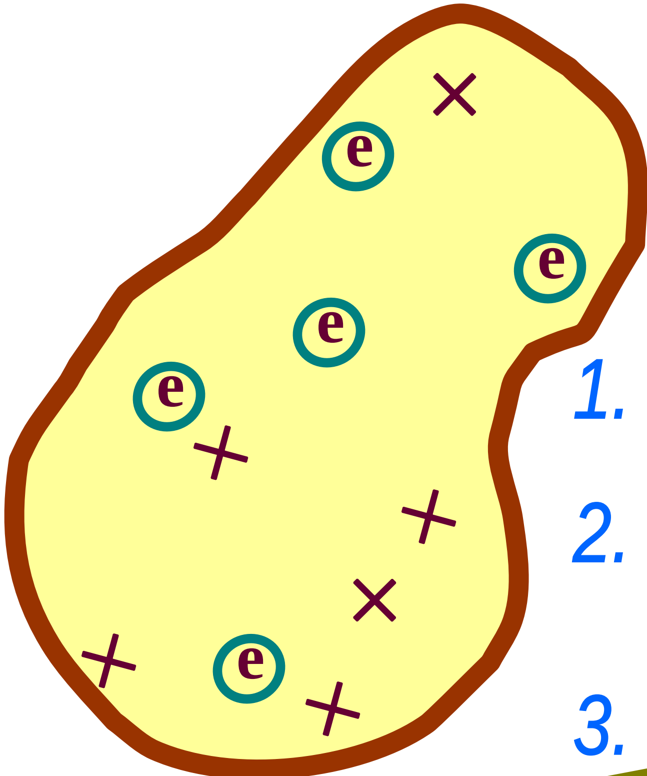


Finite number N of electrons:

$$\hat{H}\Psi_{\alpha} = E_{\alpha}\Psi_{\alpha}$$

No interactions between electrons →
Shrodinger eqn in d dimensions

In the presence of the interactions
between electrons →
Shrodinger equation in dN dimensions



Quantum Dot

1. *Disorder (× – impurities)*

2. *Complex geometry*

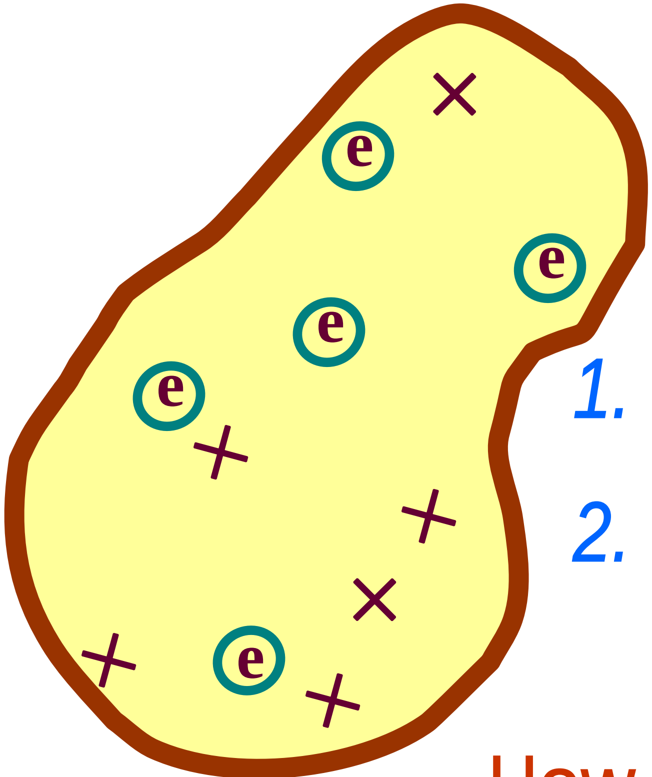
3. ~~*e-e interactions*~~ for a while

Realizations:

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes
-
-

I. Without interactions

Random Matrices,
Anderson Localization
Quantum Chaos



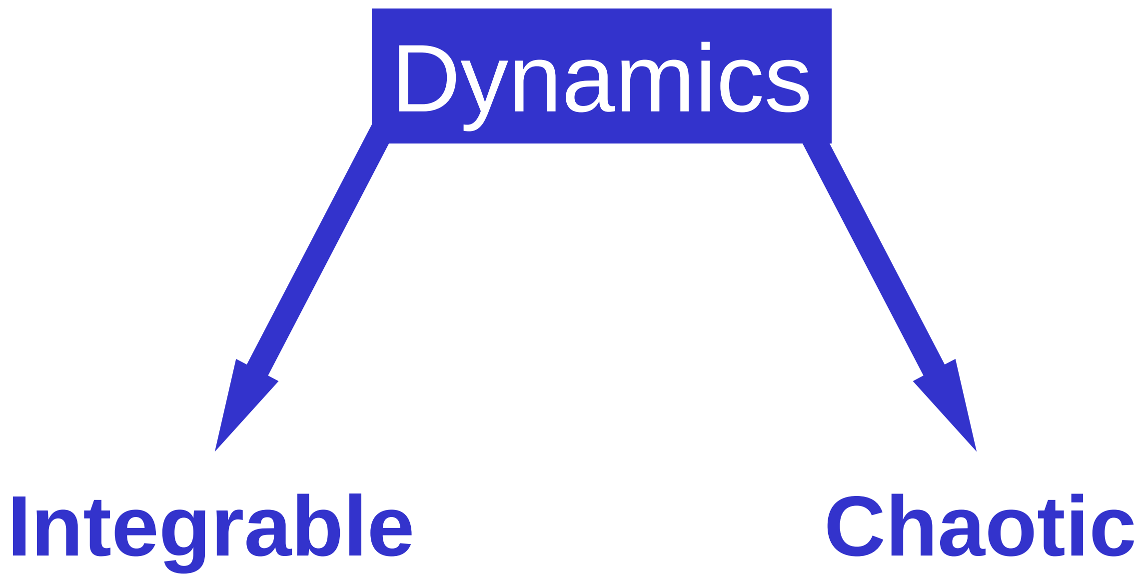
1. Disorder (× – impurities)

2. Complex geometry

How to deal with disorder?

- ~~Solve the Shrodinger equation exactly~~
- Start with plane waves, introduce the mean free path, and . . .

How to take quantum interference into account?


Dynamics
Integrable
Chaotic

Classical ($\hbar$ =0) Dynamical Systems with *d* degrees of freedom

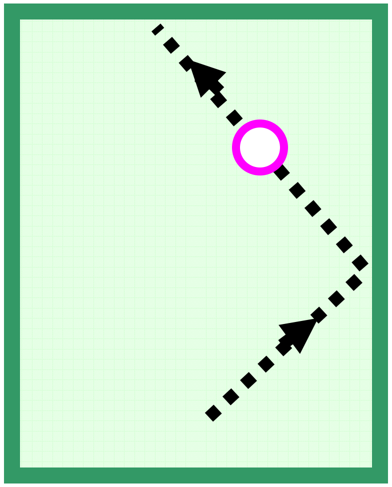
Integrable Systems

The variables can be separated and the problem reduces to *d* one-dimensional problems → *d* integrals of motion

Examples

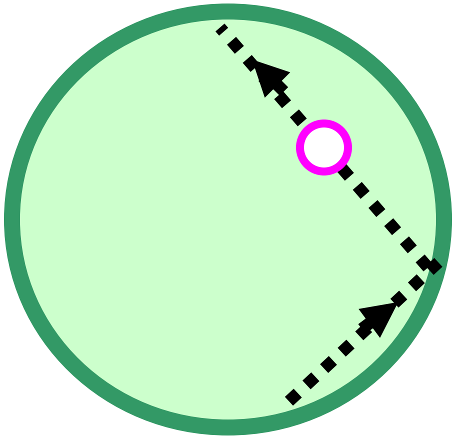
1. A ball inside rectangular billiard; *d=2*

- Vertical motion can be separated from the horizontal one
- Vertical and horizontal components of the momentum, are both integrals of motion



2. Circular billiard; *d=2*

- Radial motion can be separated from the angular one
- Angular momentum and energy are the integrals of motion



Classical Dynamical Systems with *d* degrees of freedom

Integrable Systems

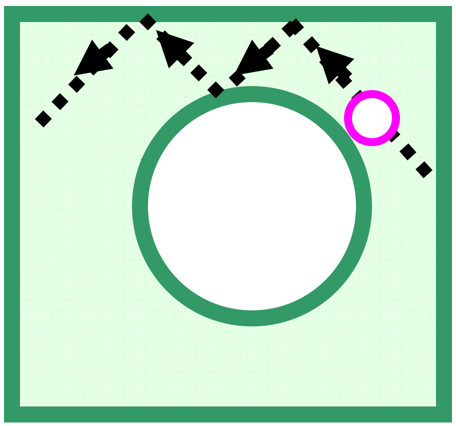
The variables can be separated ⇨ *d* one-dimensional problems ⇨ *d* integrals of motion

Rectangular and circular billiard, Kepler problem, . . . , 1d Hubbard model and other exactly solvable models, . .

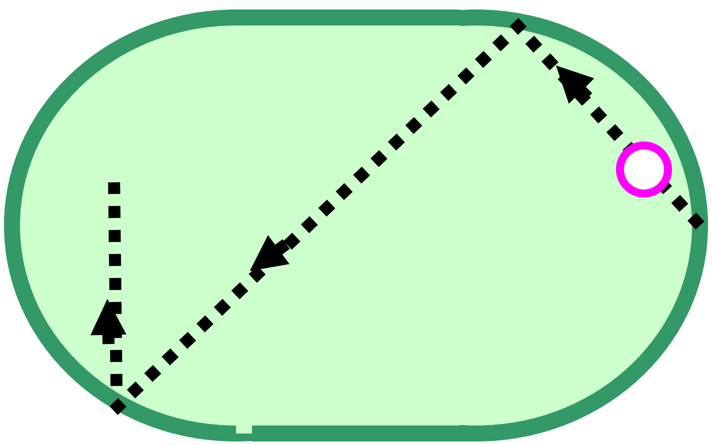
Chaotic Systems

The variables can not be separated ⇨ there is only one integral of motion - energy

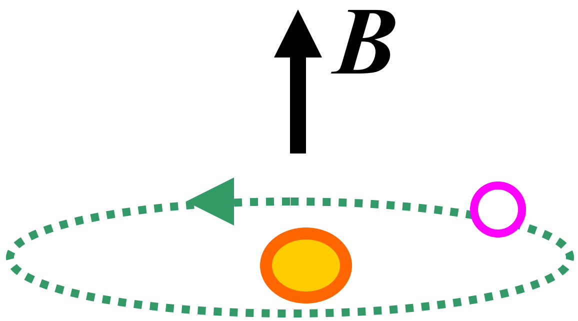
Examples



Sinai billiard



Stadium

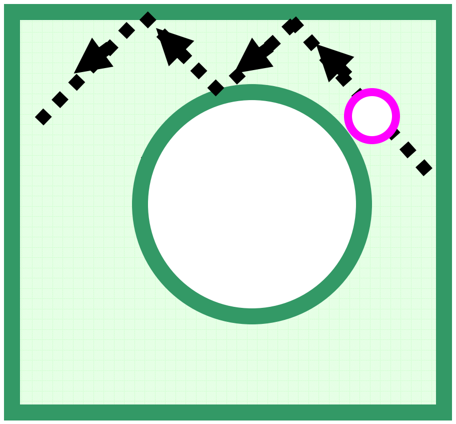


Kepler problem in magnetic field

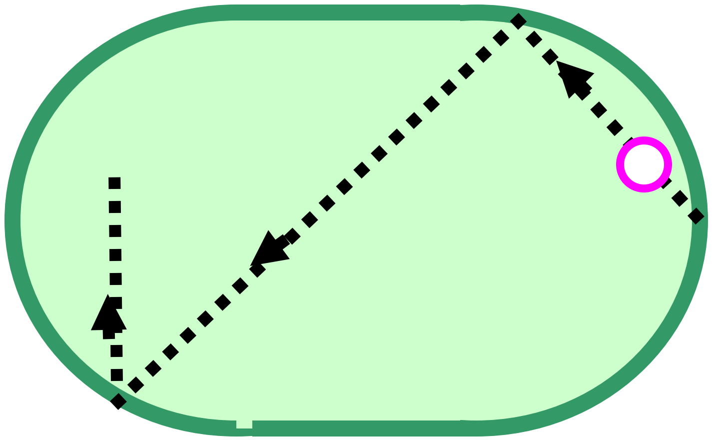
Chaotic Systems

The variables can not be separated ⇨ there is only one integral of motion - energy

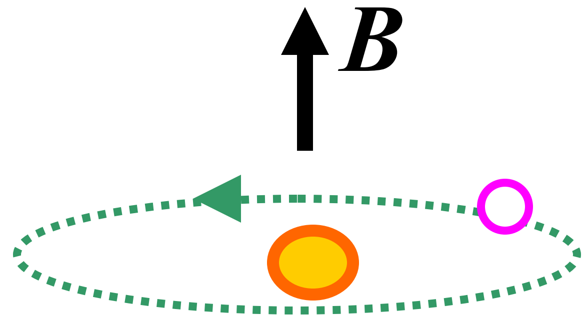
Examples



Sinai billiard



Stadium



Kepler problem in magnetic field

Yakov Sinai

Leonid Bunimovich

Johnnes Kepler

Integrable d-dimensional systems

d integrals of motion, d quantum numbers

$$I_k \qquad k = 1,2,\ldots,d$$

Chaotic d-dimensional systems

The only conserved quantity is the energy
Each eigenstate is characterized only by the eigenvalue of the Hamiltonian

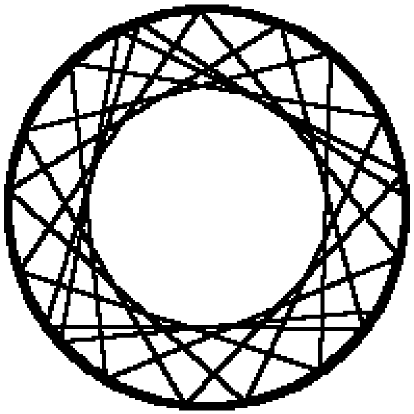
Connection with the inverse problem:

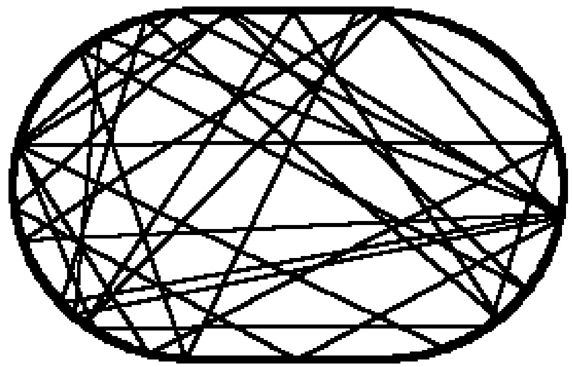
Q: Why original conditions can not be used as the integrals of motion ?

A: Not stable

Classical Chaos
$\hbar = 0$

- *Nonlinearities*
- *Lyapunov exponents*
- *Exponential dependence on the original conditions*
- *Ergodicity*





Quantum description of any System with a finite number of the degrees of freedom is a linear problem – Shrodinger equation

Q: What does it mean Quantum Chaos ?

RANDOM MATRICES

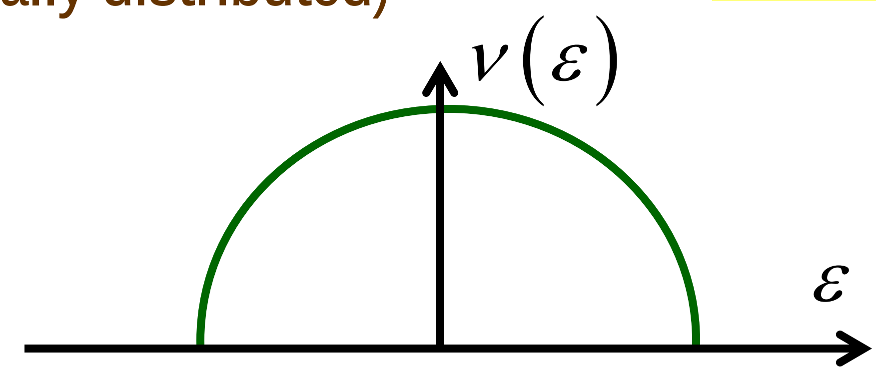
$N \times N$ ***ensemble of Hermitian matrices with random matrix element*** $N \to \infty$

E_α **- spectrum (set of eigenvalues)**

$$\nu(\varepsilon) \equiv \left\langle \sum_\alpha \delta(\varepsilon - E_\alpha) \right\rangle$$ **- density of states**

$\langle \ldots\ldots \rangle$ **- ensemble averaging**

Gaussian ensembles (matrix elements are normally distributed) $\xrightarrow{N \to \infty}$ Wigner Semicircle



RANDOM MATRICES

Spectral statistics

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle = \frac{1}{\nu}$ - mean level spacing

$\langle \ldots\ldots \rangle$

$s_\alpha \equiv \frac{E_{\alpha+1} - E_\alpha}{\delta_1}$ arest

$P(s)$ - distributi ction of spacings between th earest neighbors

Noncrossing rule

Spectral Rigidity

Level repulsion

$P(s=0) = 0$

$P(s \ll 1) \propto s^\beta \qquad \beta = 1,2,4$

Noncrossing rule (theorem)

Suggested by Hund (*Hund F. 1927 Phys. v.40, p.742*)

Justified by von Neumann & Wigner (*v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467*) . . . . .

Usually textbooks present a simplified version of the justification due to Teller (*Teller E., 1937 J. Phys. Chem 41 109).*

Arnold V. I., 1972 Funct. Anal. Appl.v. 6, p.94

Mathematical Methods of Classical Mechanics (Springer-Verlag: New York), Appendix 10, 1989

RANDOM MATRICES

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β
real	orthogonal	1
complex	unitary	2
2×2 matrices	simplectic	4

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(H_{22} - H_{11})^2 + |H_{12}|^2}$$

small small small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ($(H_{22}- H_{11})$ and H_{12}) should be small and thus $P(s) \propto s$ $\quad \beta = 1$

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small

small

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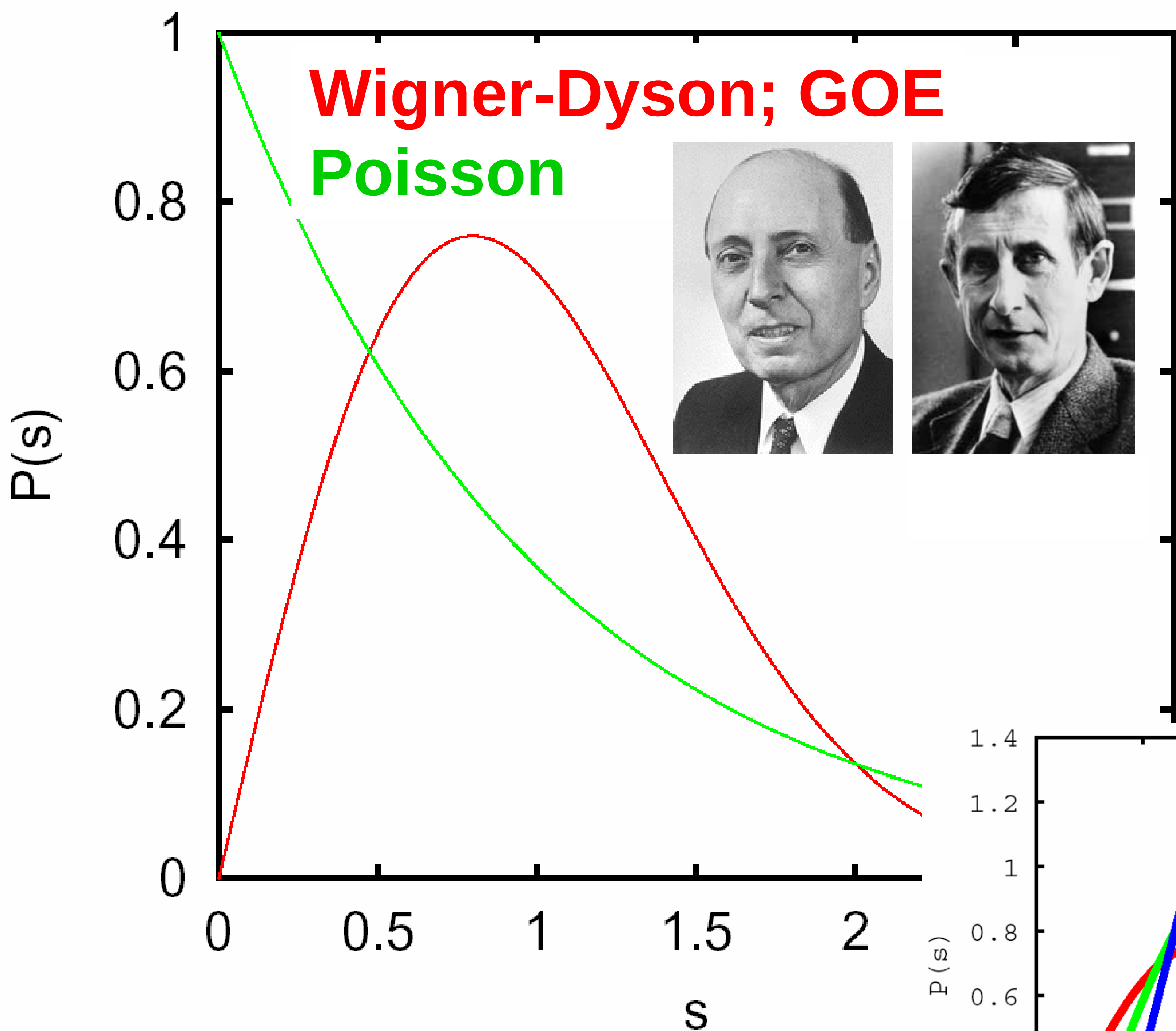
3. Complex H_{12} (unitary ensemble) $\Rightarrow$ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent $\Rightarrow$ three independent random variables should be small $\Rightarrow$ $P(s) \propto s^2 \qquad \beta = 2$

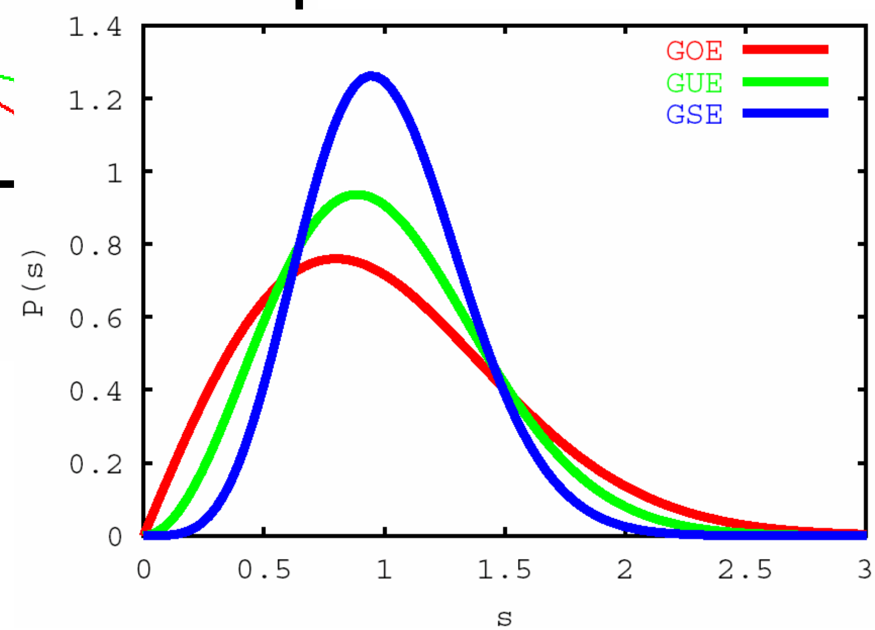


~~Gaussian~~
Orthogonal
Ensemble

Orthogonal
$\beta=1$

Unitary
$\beta=2$

Simplectic
$\beta=4$



Poisson – completely uncorrelated levels

RANDOM MATRICES

$N \times N$ ***ensemble of Hermitian matrices with random matrix element*** $N \to \infty$

No conservation laws ⇨ no quantum numbers except the energy !

$N \times N$ matrices with random matrix elements. $N \to \infty$

Spectral Rigidity
Level repulsion

$$P(s << 1) \propto s^{\beta} \qquad \beta = 1, 2, 4$$

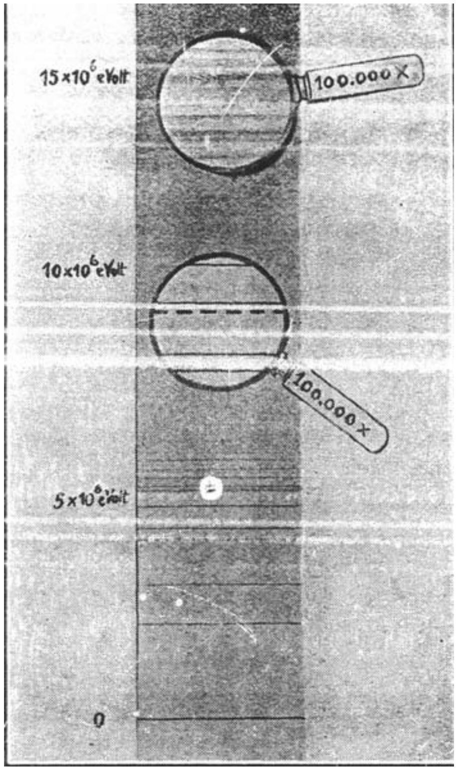
Dyson Ensembles

Realizations

Matrix elements	Ensemble	β	Realizations
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

ATOMS Main goal is to classify the eigenstates in terms of the quantum numbers

NUCLEI For the nuclear excitations this program does not work



N. Bohr, Nature 137 (1936) 344.

ATOMS

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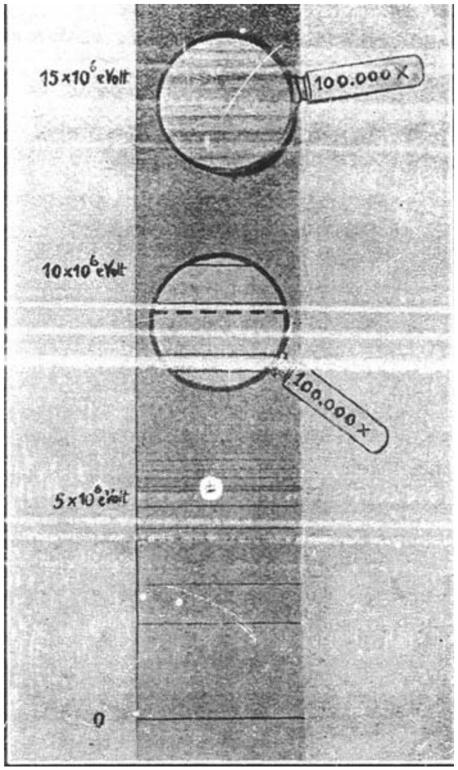
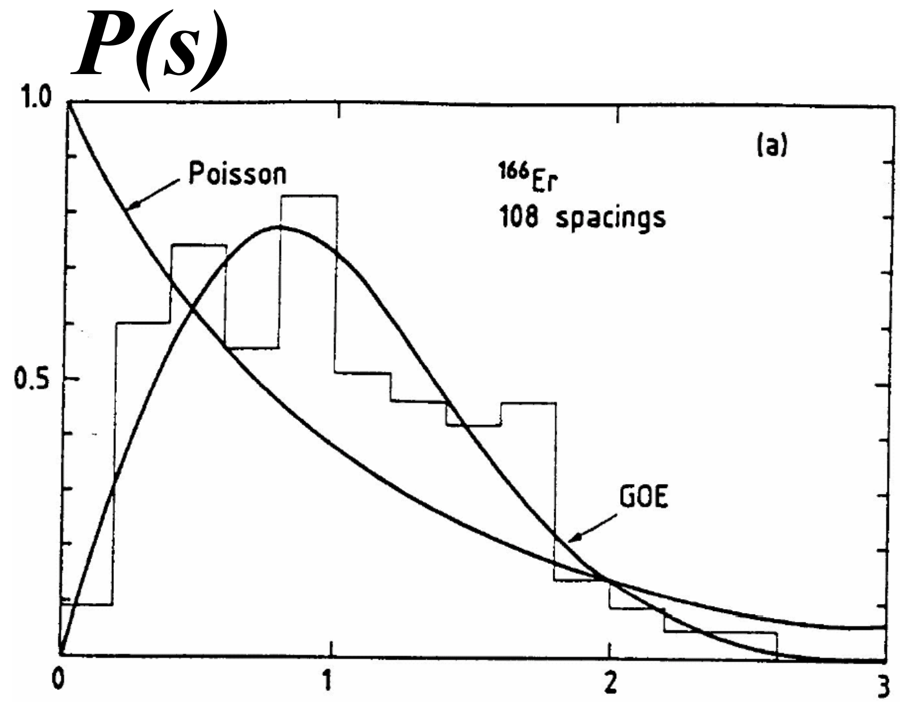
E.P. Wigner
(Ann.Math, v.62, 1955)

Study spectral **statistics** of a **particular** quantum system – a given nucleus

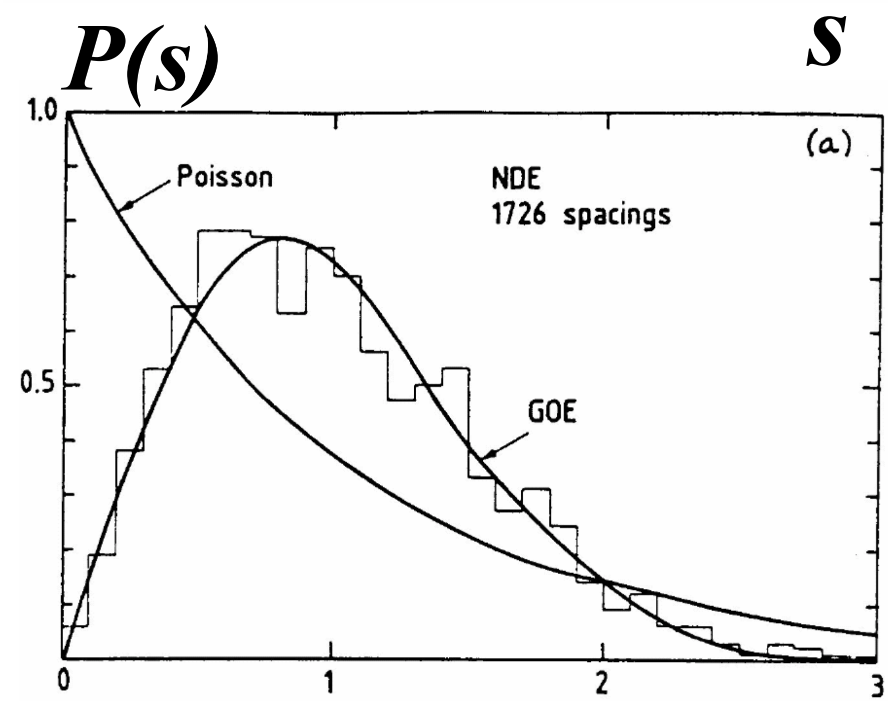


N. Bohr, Nature 137 (1936) 344.



Particular nucleus

^{166}Er



Spectra of several nuclei combined (after spacing) rescaling by the mean level

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Random Matrices	Atomic Nuclei
• *Ensemble*	• ***Particular quantum system***
• *Ensemble averaging*	• ***Spectral averaging (over α)***

Nevertheless Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics

Q: ***Why the random matrix theory (RMT) works so well for nuclear spectra*** **?**

Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

Original answer:

These are systems with a large number of degrees of freedom, and therefore the "complexity" is high

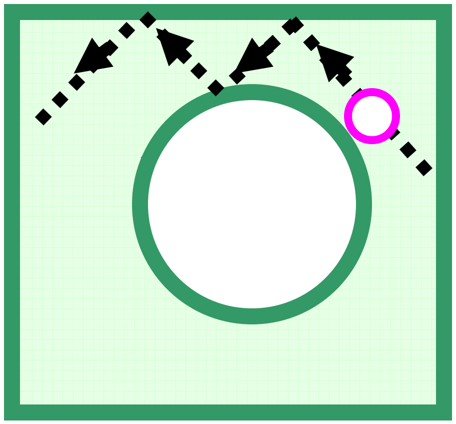
Later it became clear that

there exist very "simple" systems with as many as 2 degrees of freedom (d=2), *which demonstrate RMT - like spectral statistics*

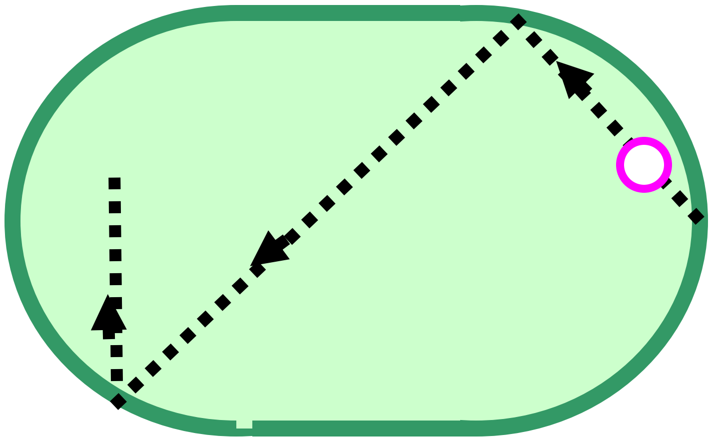
Chaotic Systems

The variables can not be separated ⇨ there is only one integral of motion - energy

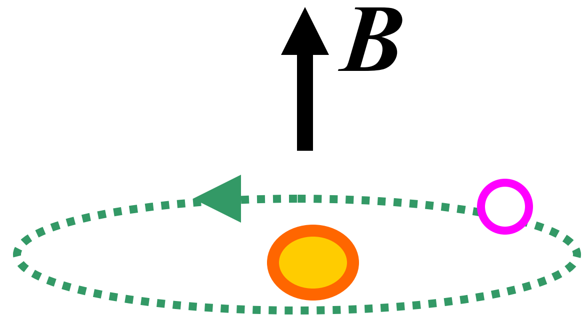
Examples



Sinai billiard



Stadium



Kepler problem in magnetic field

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Connection with the inverse problem:

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A: Not stable

$\hbar \neq 0$ Bohigas – Giannoni – Schmit conjecture

VOLUME 52 2 JANUARY 1984 NUMBER 1

Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France

(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal-invariant systems whose classical analogs are *K* systems show the same fluctuation properties as predicted by GOE

$\hbar \neq 0$ Bohigas – Giannoni – Schmit conjecture

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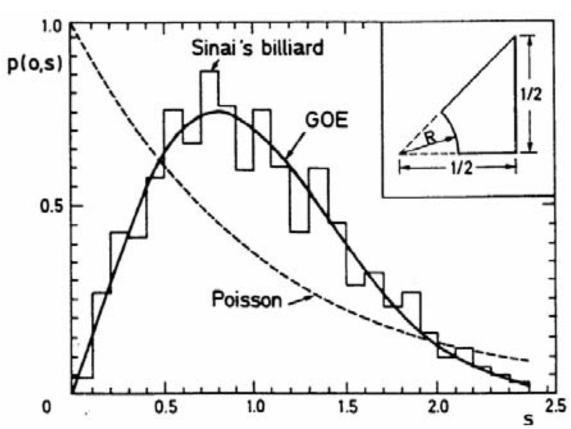
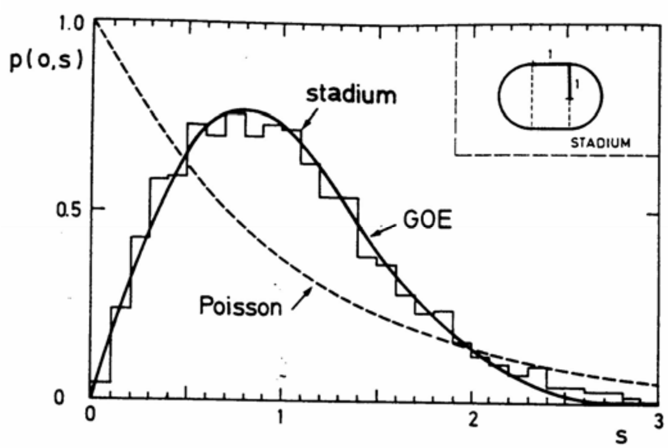




Chaotic classical analog

Wigner- Dyson spectral statistics

No quantum numbers except energy

Q: What does it mean Quantum Chaos ?

Two possible definitions

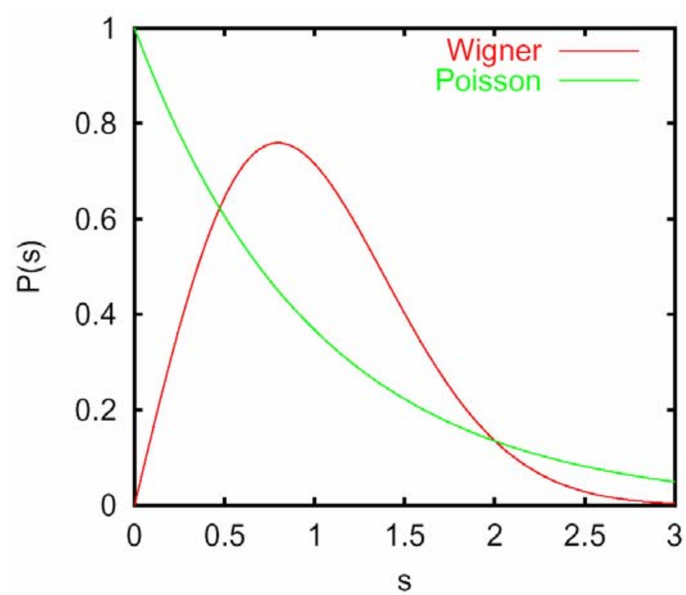
Chaotic classical analog

Wigner - Dyson-like spectrum

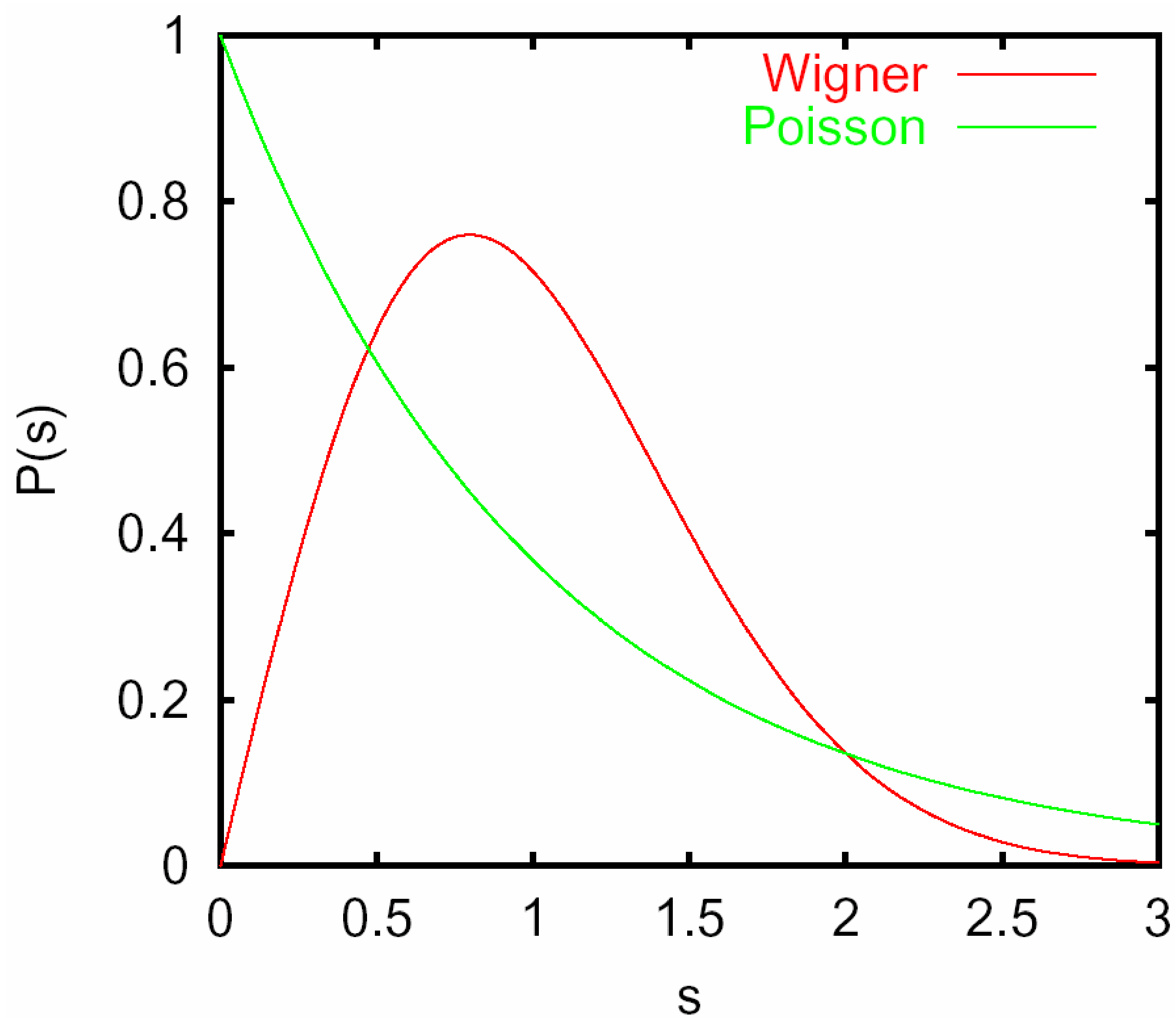
Classical Quantum

Integrable ? ⟺ **Poisson**

Chaotic ? ⟺ **Wigner-Dyson**

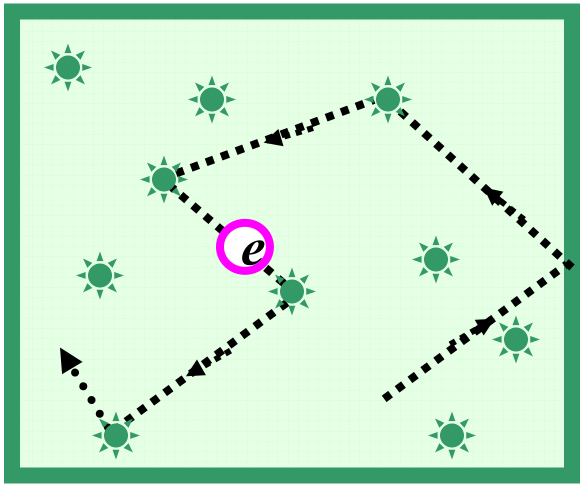


Poisson to Wigner-Dyson crossover



Poisson to Wigner-Dyson crossover

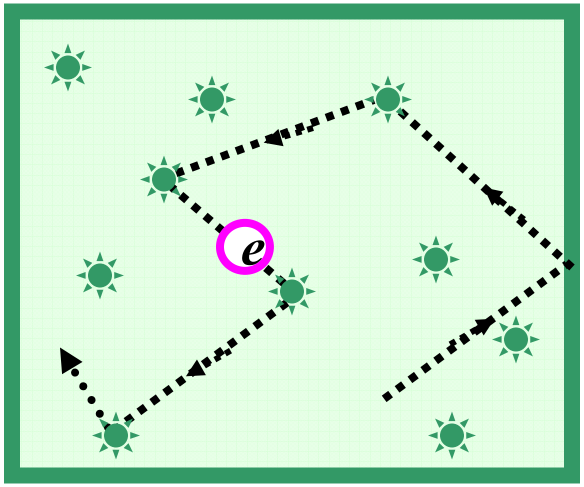
Important example: quantum particle subject to a random potential – disordered conductor



Scattering centers, e.g., impurities

Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor



Scattering centers, e.g., impurities

- As well as in the case of Random Matrices (RM) there is a luxury of ensemble averaging.
- The problem is much richer than RM theory
- There is still a lot of universality.

Anderson localization (1956)

At strong enough disorder all eigenstates are localized in space

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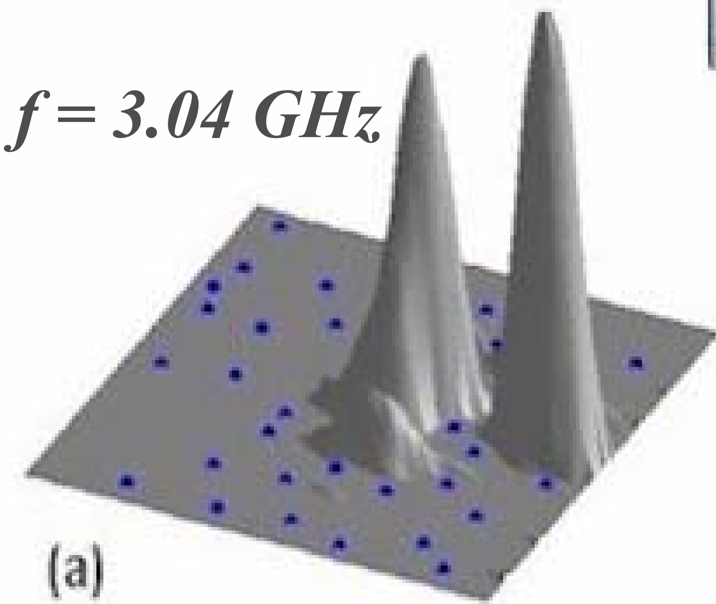
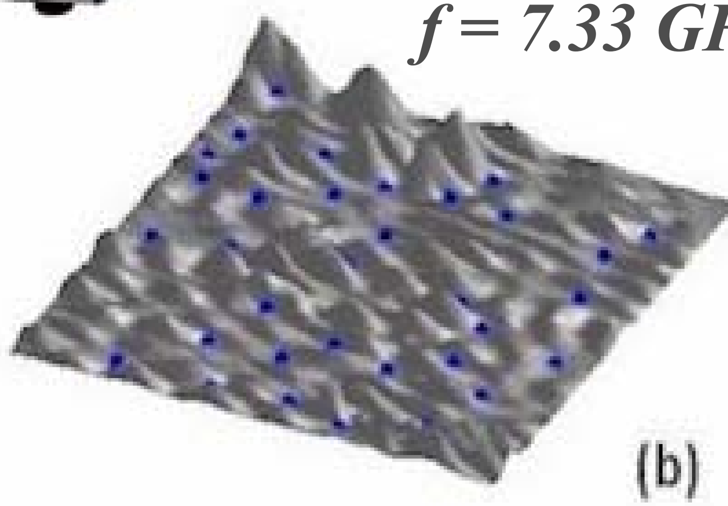
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

***f* = 3.04 GHz**



***f* = 7.33 GHz**

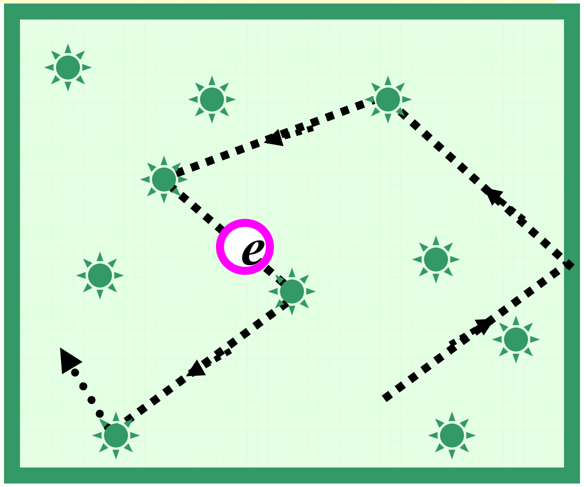


Anderson Insulator

Anderson Metal

Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor



Scattering centers, e.g., impurities

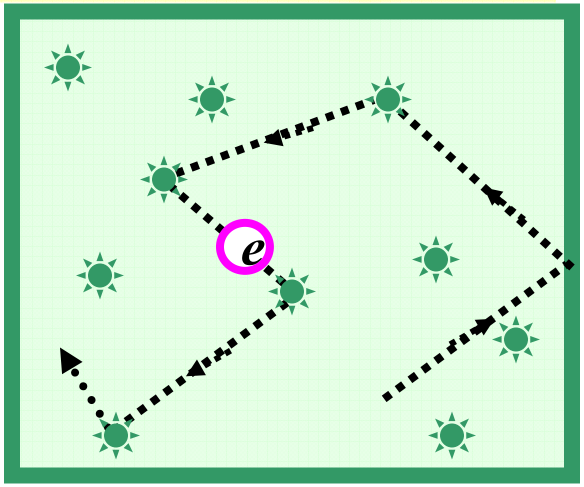
Models of disorder:

Randomly located impurities

$$U(\vec{r}) = \sum_i u(\vec{r} - \vec{r}_i)$$

Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor



Scattering centers, e.g., impurities

Models of disorder:

Randomly located impurities $U(\vec{r}) = \sum_i u(\vec{r} - \vec{r}_i)$

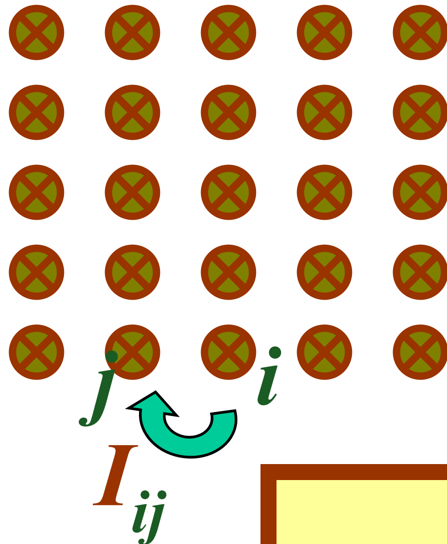
White noise potential $u(\vec{r}) \to \lambda \delta(\vec{r}) \quad \lambda \to 0 \quad c_{im} \to \infty$

Anderson model – tight-binding model with onsite disorder

Lifshits model – tight-binding model with offdiagonal disorder

⋮

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

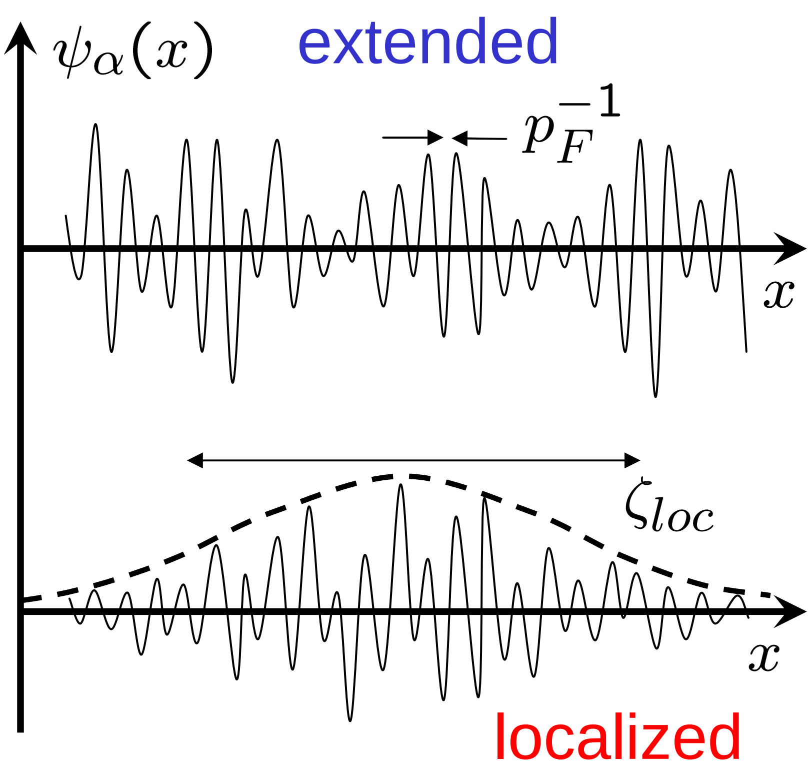
$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$I < I_c$	$I > I_c$
Insulator	***Metal***
All eigenstates are localized	*There appear states extended all over the whole system*
Localization length ξ	

Localization of single-electron wave-functions:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



Localization of single-electron wave-functions:

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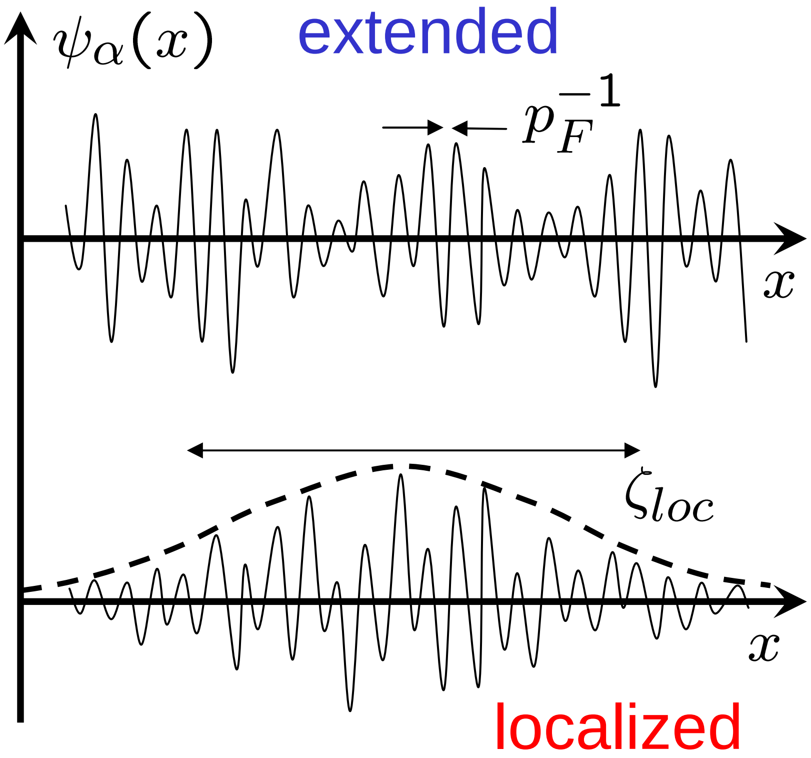


d=1; All states are *localized*

d=2; All states are *localized*

d>2; ***Anderson transition***

Anderson Transition

$I < I_c$

Insulator

All eigenstates are localized

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

Poisson spectral statistics

$I > I_c$

Metal

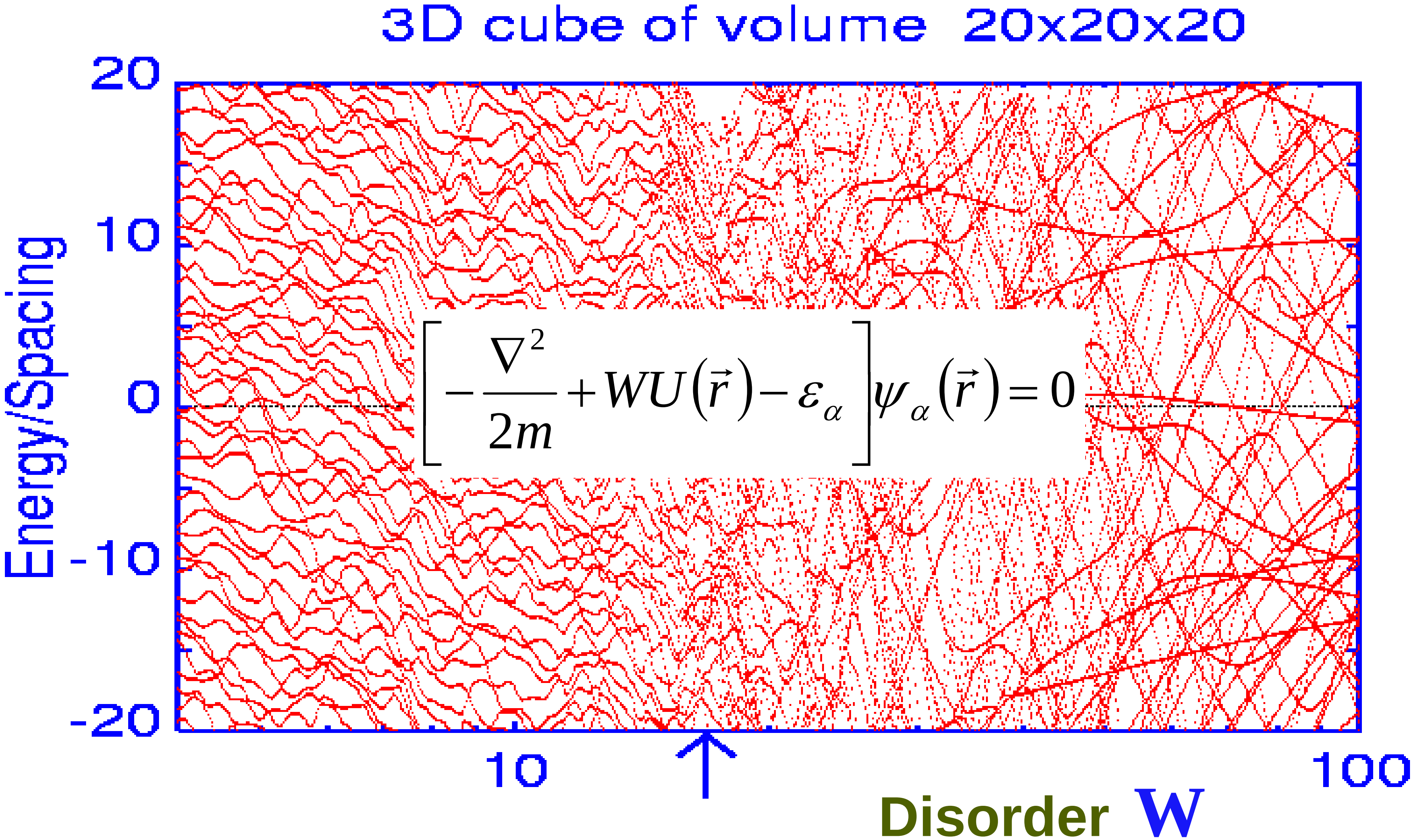
There appear states extended all over the whole system

Any two extended eigenstates repel each other

Wigner – Dyson spectral statistics

Zharekeschev & Kramer.

Exact diagonalization of the Anderson model



Q: What does it mean Quantum Chaos ?

Two possible definitions

Chaotic classical analog

?

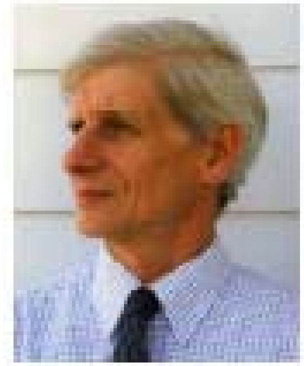
Wigner - Dyson-like spectrum

Are the two definitions equivalent?

Maybe not because of the localization!

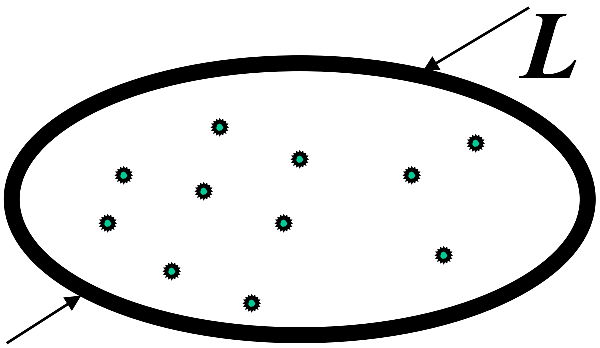
Quantum particle in a random potential (*Thouless, 1972*)

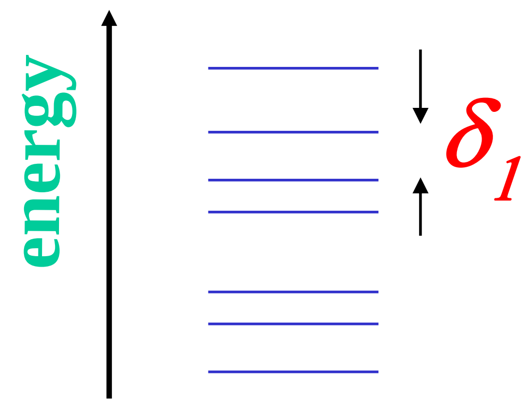
Energy scales



1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T *has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)*

$$g = E_T / \delta_1$$

dimensionless *Thouless* *conductance*

$$g = Gh/e^2$$

Thouless Conductance and One-particle Spectral Statistics

0 — **1** — ***g***

Localized states
Insulator

Poisson spectral statistics

Extended states
Metal

Wigner-Dyson spectral statistics

$N \times N$
Random Matrices

Quantum Dots with Thouless conductance ***g***

$N \to \infty$

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

$g \to \infty$

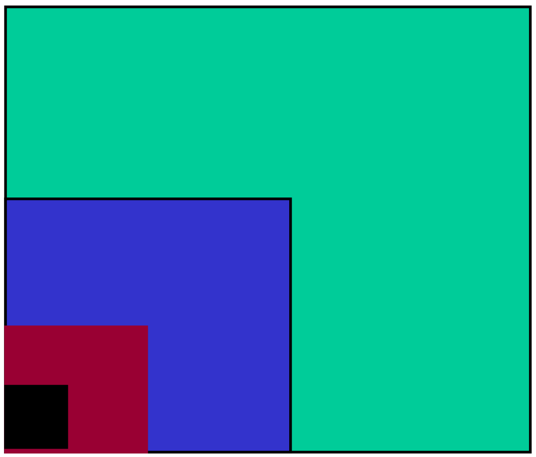
Scaling theory of Localization

(Abrahams, Anderson, Licciardello and Ramakrishnan 1979)

$g = E_T / \delta_1$

Dimensionless Thouless conductance

$g = Gh/e^2$



$$L = 2L = 4L = 8L \;....$$

without quantum corrections

$$E_T \propto L^{-2} \qquad \delta_1 \propto L^{-d}$$

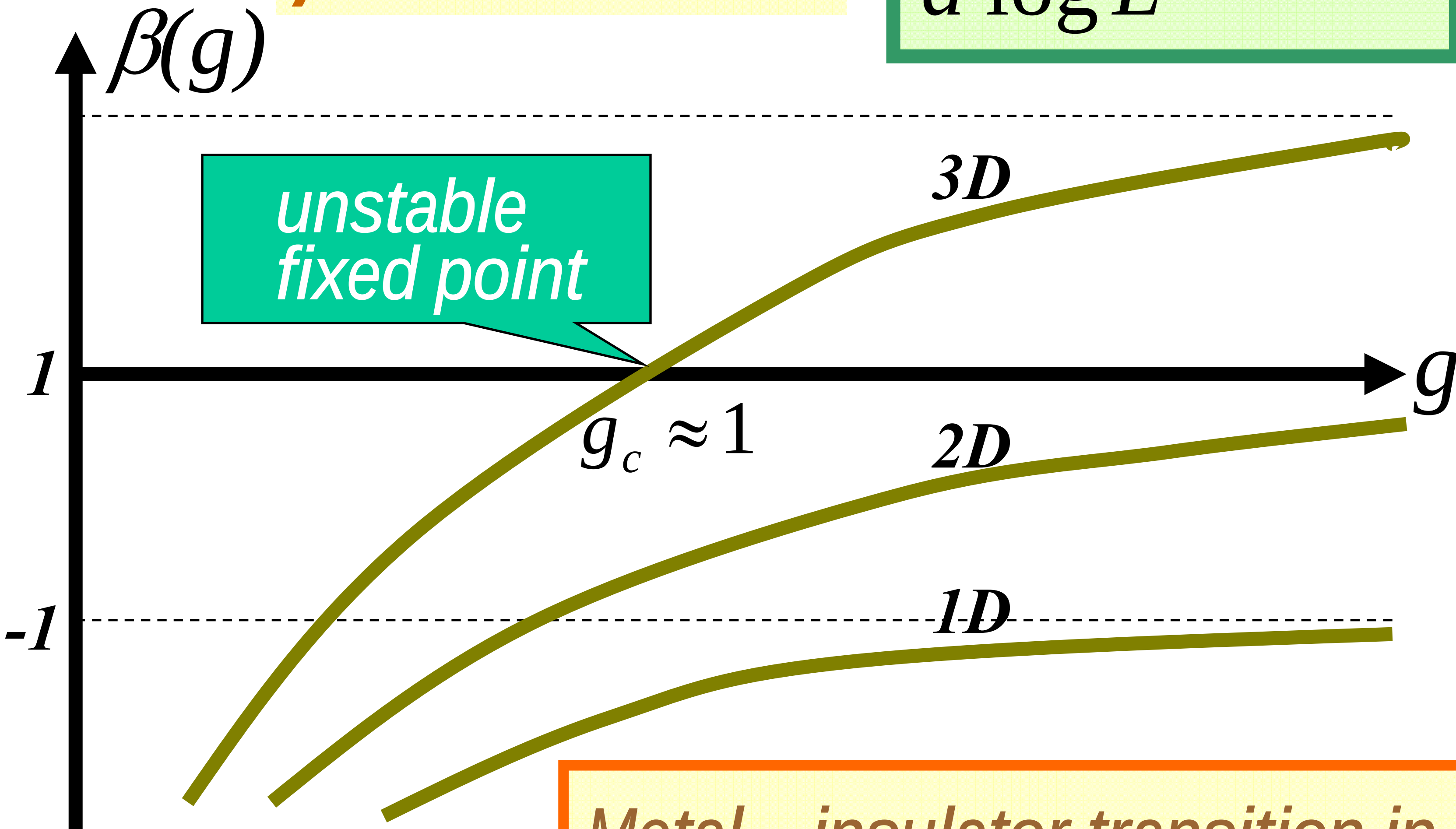
$$\frac{E_T}{\delta_1} \longrightarrow \frac{E_T}{\delta_1} \longrightarrow \frac{E_T}{\delta_1} \longrightarrow \frac{E_T}{\delta_1}$$

$$g \longrightarrow g \longrightarrow g \longrightarrow g$$

$$\boxed{\frac{d(\log g)}{d(\log L)} = \beta(g)}$$

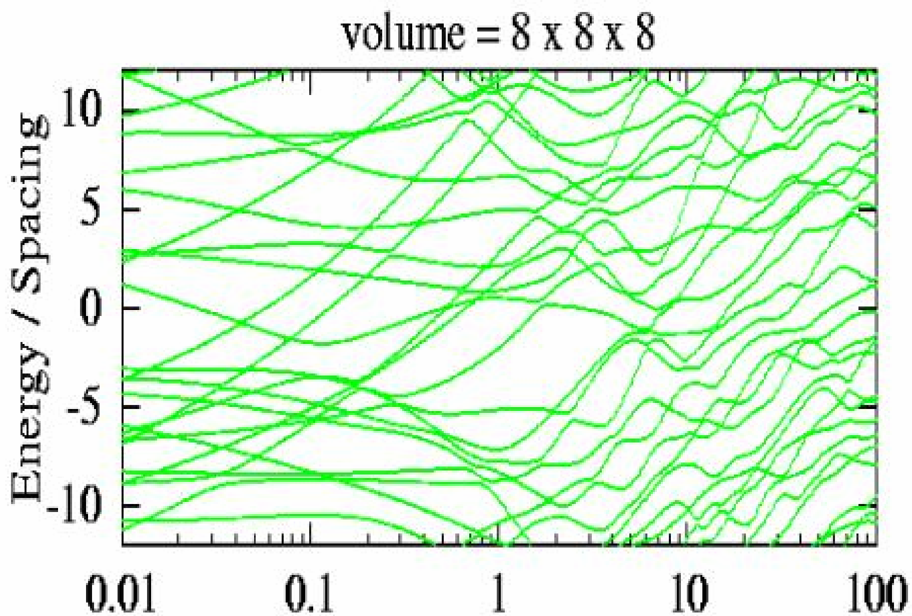
β - function

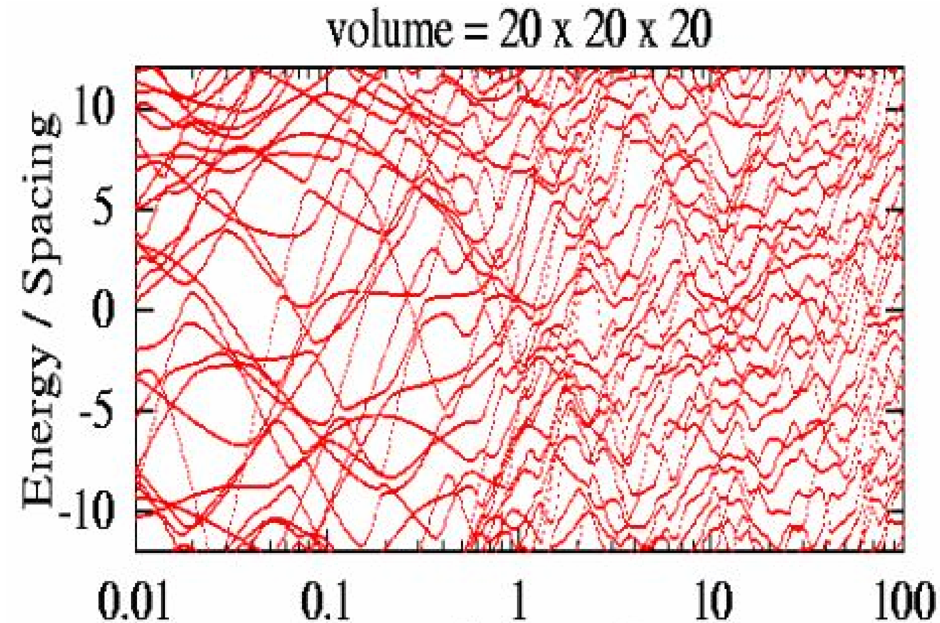
$$\frac{d \log g}{d \log L} = \beta(g)$$



Metal – insulator transition in ***3D***

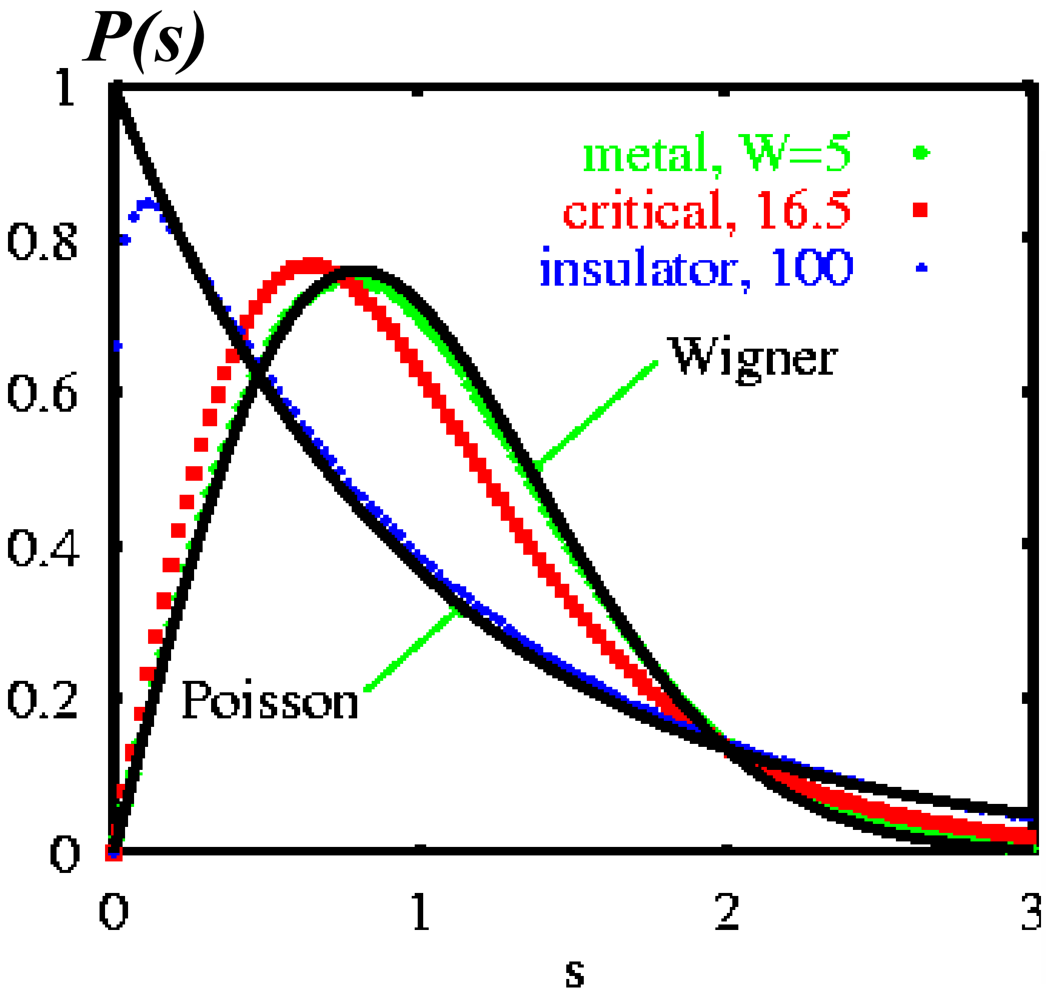
All states are localized for ***d=1,2***

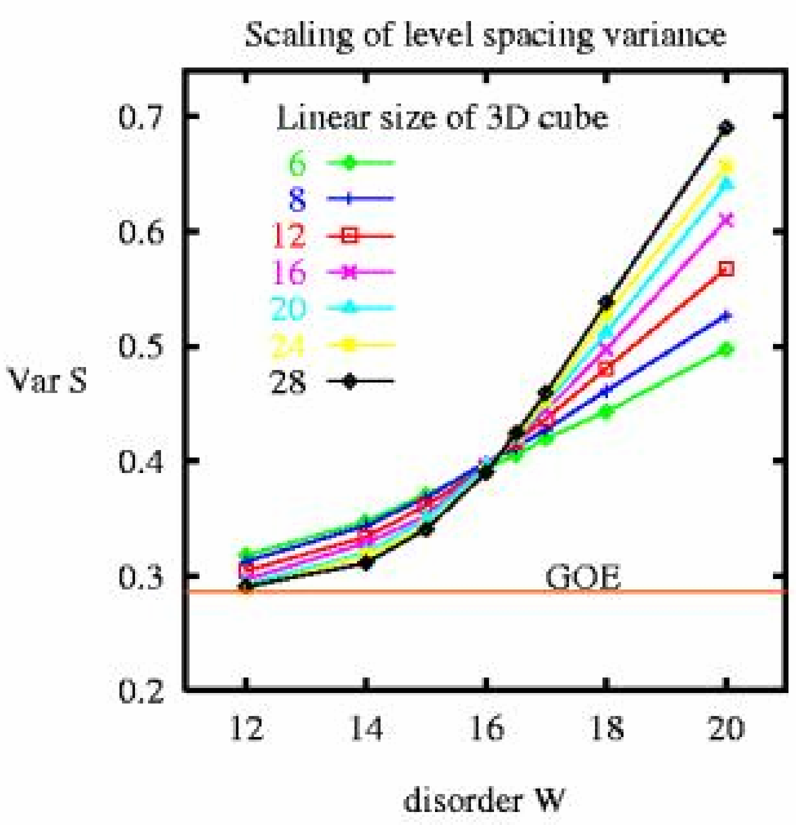




Conductance g

Anderson transition in terms of pure level statistics



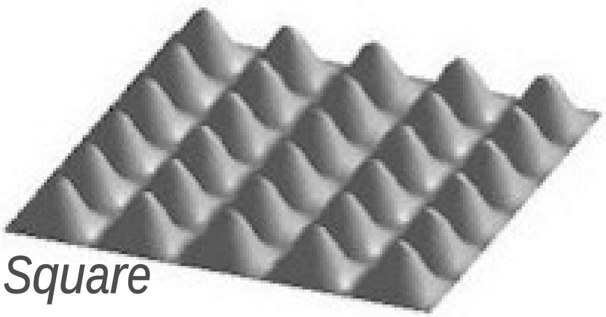


VOLUME 85, NUMBER 11 PHYSICAL REVIEW LETTERS 11 SEPTEMBER 2000

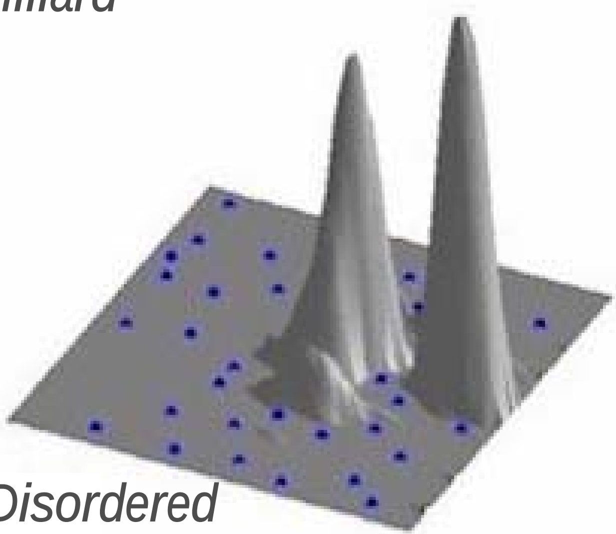
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

Integrable

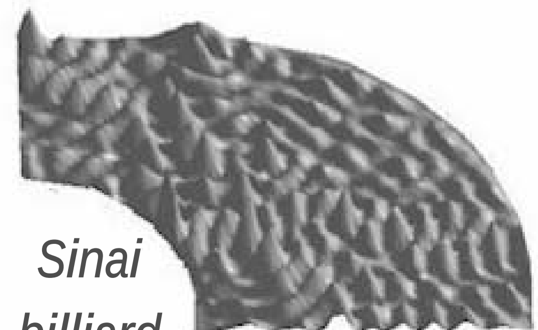


Square billiard

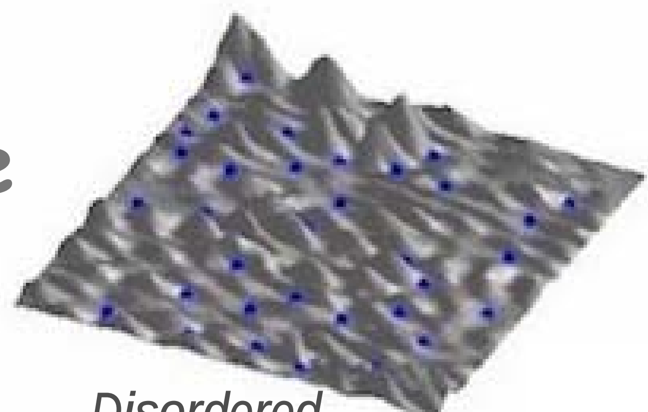


Disordered
localized

Chaotic



Sinai billiard



Disordered
extended

All chaotic systems resemble each other.

All integrable systems are integrable in their own way

Disordered Systems:

$E_T > \delta_1; \quad g > 1$

Anderson ***metal;*** ***Wigner-Dyson*** *spectral statistics*

$E_T < \delta_1; \quad g < 1$

Anderson ***insulator;*** ***Poisson*** *spectral statistics*

Q: *Is it a generic scenario for the Wigner-Dyson to Poisson crossover* **?**

Speculations

*Consider an **integrable** system. Each state is characterized by a **set of quantum numbers.***

*It can be viewed as a point in the **space of quantum numbers**. The whole set of the states forms a **lattice** in this space.*

*A **perturbation** that violates the integrability provides matrix elements of the **hopping** between different sites (**Anderson model !?**)*

Q: *Does Anderson localization provide a generic scenario for the Wigner-Dyson to Poisson crossover* **?**

Consider an integrable system. Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?)

Weak enough hopping - Localization - Poisson
Strong hopping - transition to Wigner-Dyson

The very definition of the localization is **not invariant** – one should specify in which space the eigenstates are localized.

Level statistics **is invariant**:

Poissonian statistics $\exists$ basis where the eigenfunctions are localized

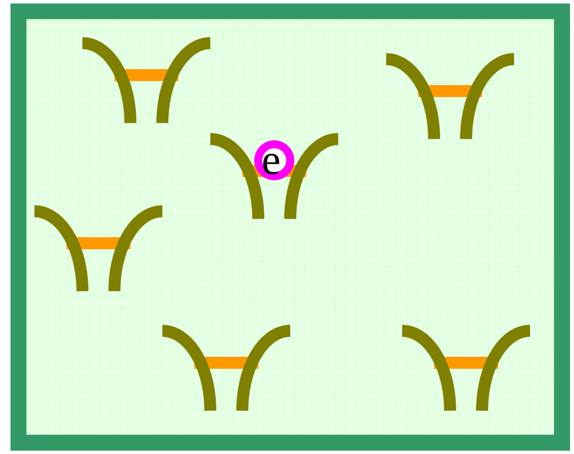
Wigner -Dyson statistics $\forall$ basis the eigenfunctions are extended

Example 1

Doped semiconductor

Low concentration of donors ⇒ Electrons are localized on donors ⇨ Poisson

Higher donor concentration ⇒ Electronic states are extended ⇨ Wigner-Dyson

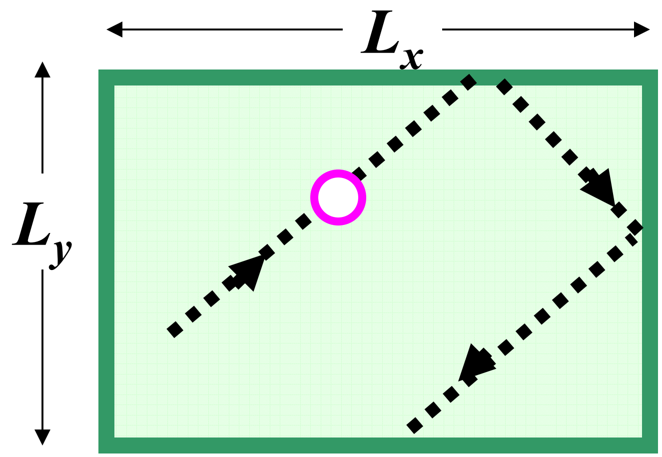


Example 2

Rectangular billiard

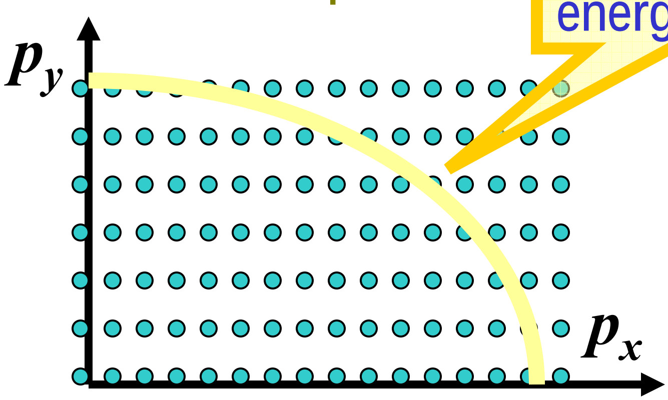
Two integrals of motion

$$p_x = \frac{\pi n}{L_x}; \quad p_y = \frac{\pi m}{L_x}$$



Lattice in the momentum space

Line (surface) of constant energy



Ideal billiard – localization in the momentum space ⇨ Poisson

Deformation or smooth random potential – delocalization in the momentum space ⇨ Wigner-Dyson

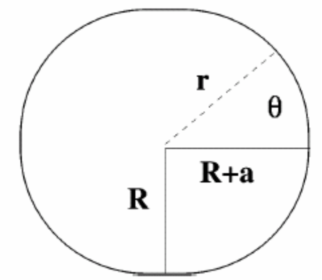
VOLUME 77, NUMBER 23 PHYSICAL REVIEW LETTERS 2 DECEMBER 1996

Localization and diffusion in the angular momentum space

Diffusion and Localization in Chaotic Billiards

Fausto Borgonovi,[1,3,4] Giulio Casati,[2,3,5] and Baowen Li[6,7]
[1] *Dipartimento di Matematica, Università Cattolica, via Trieste 17, 25121 Brescia, Italy*
[2] *Università di Milano, sede di Como, Via Lucini 3, Como, Italy*
[3] *Istituto Nazionale di Fisica della Materia, Unità di Milano, via Celoria 16, 22100, Milano, Italy*
[4] *Instituto Nazionale di Fisica Nucleare, Sezione di Pavia, Pavia, Italy*
[5] *Instituto Nazionale di Fisica Nucleare, Sezione di Milano, Milano, Italy*
[6] *Department of Physics and Centre for Nonlinear and Complex Systems, Hong Kong Baptist University, Hong Kong*
[a7] *Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, 2000 Maribor, Slovenia*
(Received 29 July 1996)

$$\varepsilon \equiv \frac{a}{R}$$



$\varepsilon > 0$ Chaotic stadium

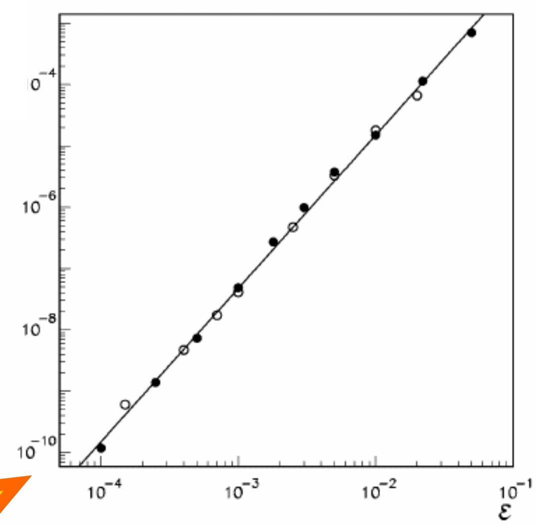
$\varepsilon \to 0$ Integrable circular billiard

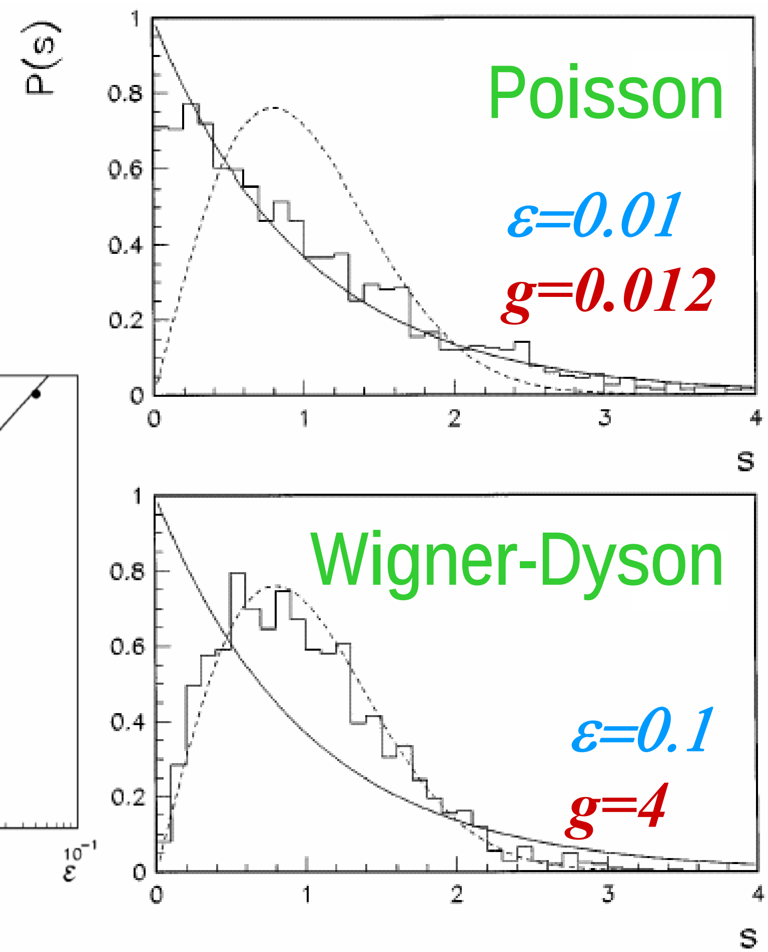
Angular momentum is the integral of motion

$\hbar = 0; \quad \varepsilon << 1$

Diffusion in the angular momentum space

$$D \propto \varepsilon^{5/2}$$





D.Poilblanc, T.Ziman, J.Bellisard, F.Mila & G.Montambaux
Europhysics Letters, v.22, p.537, 1993

1D Hubbard Model on a periodic chain

$$H = t\sum_{i,\sigma}\left(c^{+}_{i,\sigma}c_{i+1,\sigma} + c^{+}_{i+1,\sigma}c_{i,\sigma}\right) + U\sum_{i,\sigma} n_{i,\sigma}n_{i,-\sigma} + V\sum_{i,\sigma,\sigma'} n_{i,\sigma}n_{i+1,\sigma'}$$

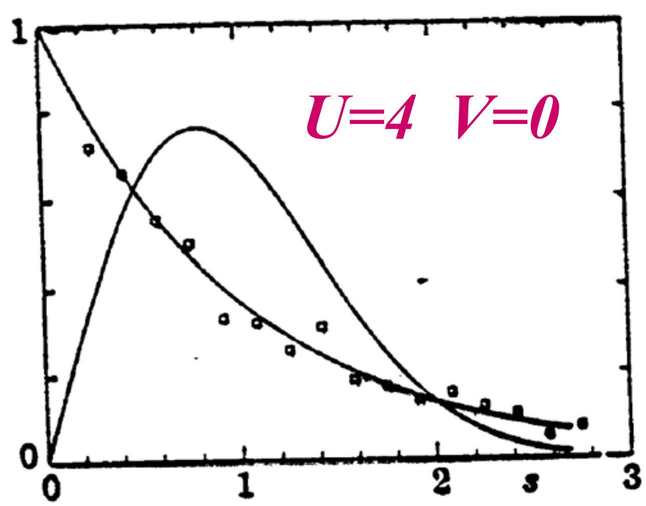
Onsite interaction

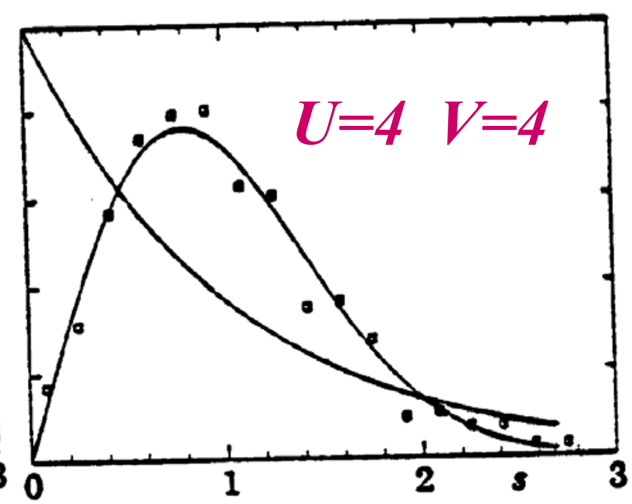
n. neighbors interaction

$V = 0$ Hubbard model **integrable**

$V \neq 0$ extended Hubbard model **nonintegrable**

12 sites
3 particles
Zero total spin
Total momentum $\pi/6$





Finite number N of electrons:

$$\hat{H}\Psi_\alpha = E_\alpha \Psi_\alpha$$

No interactions between electrons →
Shrodinger eqn in d dimensions

Integrable system – each energy is conserved

Poissonian many-body spectrum

In the presence of the interactions
between electrons →
Shrodinger eqn in dN dimensions

Finite number N of electrons:

$$\hat{H}\Psi_\alpha = E_\alpha \Psi_\alpha$$

No interactions between electrons →
Shrodinger eqn in d dimensions

Integrable system – each energy is conserved

Poissonian many-body spectrum

In the presence of the interactions between electrons →
Shrodinger eqn in dN dimensions

Q: Can interaction between the particles drive this system into chaos and make it ergodic ?

Random Matrics statistics of nuclear spectra

II. With interactions

Fermi Liquid and Disorder
Zero Dimensional Fermi Liquid

Fermi Liquid

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

Fermi Liquid

What does it mean?

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

$\Rightarrow$ ***Fermi Liquid***

It means that

1. *Excitations are similar to the excitations in a Fermi-gas:*

   a) the same quantum numbers – momentum, spin ½ , charge e

   b) decay rate is small as compared with the excitation energy

2. *Substantial renormalizations. For example, in a Fermi gas*

$$\partial n/\partial\mu, \quad \gamma = c/T, \quad \chi/g\mu_B$$

are all equal to the one-particle density of states.
These quantities are different in a Fermi liquid

Signatures of the Fermi - Liquid state ?!

1. *Resistivity is proportional to* T^2 *:*

L.D. Landau & I.Ya. Pomeranchuk ***"To the properties of metals at very low temperatures";*** **Zh.Exp.Teor.Fiz., 1936, v.10, p.649**

…The increase of the resistance caused by the interaction between the electrons is proportional to T^2 and at low temperatures exceeds the usual resistance, which is proportional to T^5.

… the sum of the moments of the interaction electrons can change by an integer number of the periods of the reciprocal lattice. Therefore the momentum increase caused by the electric field can be destroyed by the interaction between the electrons, not only by the thermal oscillations of the lattice.

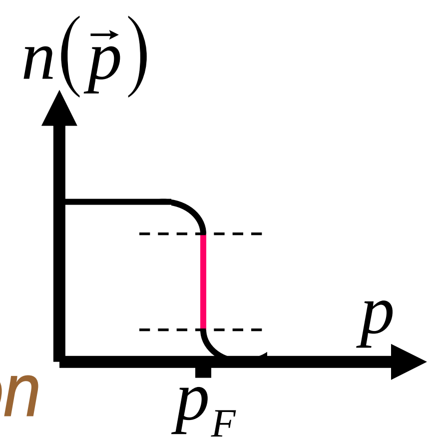
Signatures of the Fermi - Liquid state ?!

1. Resistivity is proportional to T^2 :

L.D. Landau & I.Ya. Pomeranchuk *“To the properties of metals at very low temperatures”*; Zh.Exp.Teor.Fiz., 1936, v.10, p.649

Umklapp electron – electron scattering dominates the charge transport (?!)

2. Jump in the momentum distribution function at $T=0$.



2a. Pole in the one-particle Green function

$$G(\varepsilon, \vec p) = \frac{Z}{i\varepsilon_n - \xi(\vec p)}$$

Fermi liquid = $0<Z<1$ (?!)

Landau Fermi - Liquid theory

Momentum	$\vec{p}$
Momentum distribution	$n(\vec{p})$
Total energy	$E\{n(\vec{p})\}$
Quasiparticle energy	$\xi(\vec{p}) \equiv \delta E / \delta n(\vec{p})$
Landau f-function	$f(\vec{p},\vec{p}\,') \equiv \delta\xi(\vec{p}) / \delta n(\vec{p}\,')$

Q: Can **Fermi - liquid** survive without the **momenta**?

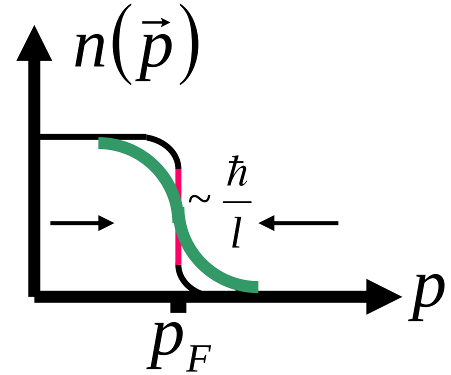
Does it make sense to speak about the **Fermi - liquid** state in the presence of a **quenched disorder**?

Q: *Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder* ?

1. *Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, l. The step in the momentum distribution function is broadened by this uncertainty*

Q: Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder ?

1. Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, l. The step in the momentum distribution function is broadened by this uncertainty



2. Neither resistivity nor its temperature dependence is determined by the umklapp processes and thus does not behave as T^2

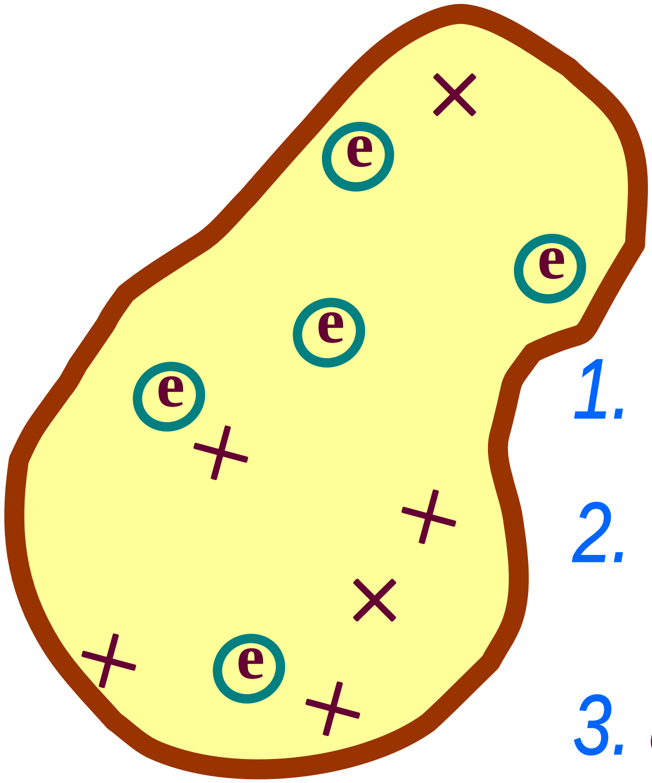
3. Sometimes (e.g., for random quenched magnetic field) the disorder averaged one-particle Green function even without interactions does not have a pole as a function of the energy, ε. The residue , Z, makes no sense.

Nevertheless even in the presence of the disorder

I. Excitations are similar to the excitations in a disordered Fermi-gas.

II. Small decay rate

III. Substantial renormalizations



Quantum Dot

1. *Disorder (×impurities)*
2. *Complex geometry*

} *chaotic one-particle motion*

3. ***e-e*** *interactions*

Realizations:

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes
-
-

Finite System $\Longrightarrow$ Thouless energy E_T

$$\boxed{\varepsilon << E_T \xRightarrow{def} 0D}$$

At the same time, we want the typical energies, ε, to exceed the mean level spacing, δ_1:

$$\delta_1 << \varepsilon << E_T$$

$$\boxed{g \equiv \frac{E_T}{\delta_1} >> 1}$$

Two-Body Interactions

$|\alpha,\sigma>$ *Set of one particle states.* σ *and* α *label correspondingly spin and orbit.*

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha a^+_{\alpha,\sigma} a_{\alpha,\sigma} \qquad \hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^+_{\alpha,\sigma} a^+_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

ε_α *-one-particle orbital energies*

$M_{\alpha\beta\gamma\delta}$ *-interaction matrix elements*

Nuclear Physics	ε_α	*are taken from the shell model*
	$M_{\alpha\beta\gamma\delta}$	*are assumed to be random*
Quantum Dots	ε_α	*RANDOM; Wigner-Dyson statistics*
	$M_{\alpha\beta\gamma\delta}$	*? ? ? ? ? ? ? ?*

Thouless Conductance and One-particle Quantum Mechanics

0 ——————— **1** ——————— ***g*** →

Localized states	***Extended states***
Insulator	***Metal***
Poisson spectral statistics	**Wigner-Dyson spectral statistics**

$N \times N$ Random Matrices

Quantum Dots with dimensionless conductance g

$N \to \infty$

$g \to \infty$

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

Matrix Elements

$$\hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^{+}_{\alpha,\sigma} a^{+}_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

Matrix Elements $M_{\alpha\beta\gamma\delta}$

Diagonal - $\alpha,\beta,\gamma,\delta$ *are equal pairwise*

$\alpha=\gamma$ *and* $\beta=\delta$ *or* $\alpha=\delta$ *and* $\beta=\gamma$ *or* $\alpha=\beta$ *and* $\gamma=\delta$

Offdiagonal - *otherwise*

It turns out that

in the limit $g \to \infty$

- *Diagonal matrix elements are much bigger than the offdiagonal ones*

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- *Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging*

Toy model: *Short range e-e interactions*

$$U(\vec{r}) = \frac{\lambda}{\nu}\delta(\vec{r})$$

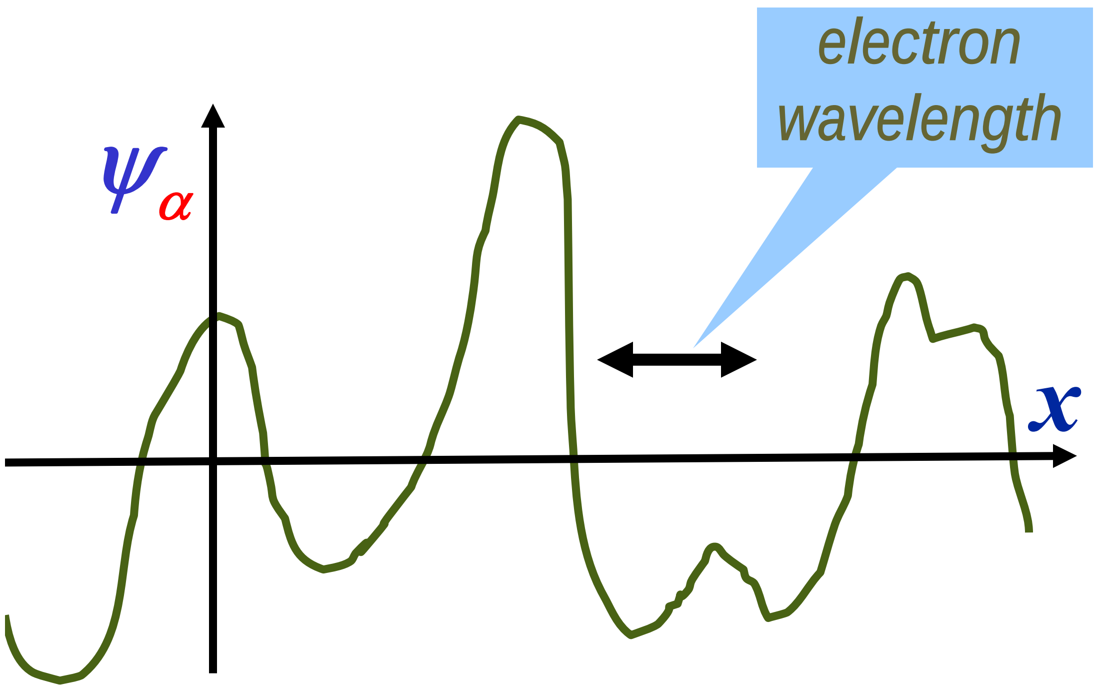
λ *is dimensionless coupling constant*

ν *is the electron density of states*

$$\boxed{M_{\alpha\beta\gamma\delta} = \frac{\lambda}{\nu}\int d\vec{r}\,\psi^*_\alpha(\vec{r})\psi^*_\beta(\vec{r})\psi_\gamma(\vec{r})\psi_\delta(\vec{r})}$$

$\psi_\alpha(\vec{r})$

one-particle eigenfunctions



$\Psi_\alpha(x)$ *is a random function that rapidly oscillates*

$$|\psi_\alpha(x)|^2 \geq 0$$

$$\psi_\alpha(x)^2 \geq 0$$

as long as ***T****-invariance is preserved*

In the limit

$$g \to \infty$$

- Diagonal matrix elements are much bigger than the offdiagonal ones

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging

$$M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu} \int d\vec{r} \left|\psi_\alpha(\vec{r})\right|^2 \left|\psi_\beta(\vec{r})\right|^2$$

$$\left|\psi_\alpha(\vec{r})\right|^2 \Rightarrow \frac{1}{\text{volume}}$$

$$\boxed{M_{\alpha\beta\alpha\beta} = \lambda\delta_1}$$

More general: *finite range interaction potential* $U(\vec{r})$

$$\boxed{M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu} \int \left|\psi_\alpha(\vec{r}_1)\right|^2 \left|\psi_\beta(\vec{r}_2)\right|^2 U(\vec{r}_1 - \vec{r}_2) d\vec{r}_1 d\vec{r}_2}$$

The same conclusion

Universal (Random Matrix) limit - Random Matrix symmetry of the correlation functions:

All correlation functions are invariant under arbitrary orthogonal transformation:

$$\tilde{\psi}_{\mu}(\vec{r}) = \sum_{\nu} \int d\vec{r}_1 O_{\mu}^{\nu}(\vec{r},\vec{r}_1) \psi_{\nu}(\vec{r}_1)$$

$$\int d\vec{r}_1 O_{\mu}^{\nu}(\vec{r},\vec{r}_1) O_{\nu}^{\eta}(\vec{r}_1,\vec{r}\,') = \delta_{\mu\eta}\delta(\vec{r}-\vec{r}\,')$$

There are *only* *three operators, which are quadratic in the fermion operators* a^+, a , *and invariant under RM transformations:*

$$\hat{n} = \sum_{\alpha,\sigma} a^+_{\alpha,\sigma} a_{\alpha,\sigma}$$

total number of particles

$$\hat{S} = \sum_{\alpha,\sigma_1,\sigma_2} a^+_{\alpha,\sigma_1} \vec{\sigma}_{\sigma_1,\sigma_2} a_{\alpha,\sigma_2}$$

total spin

$$\hat{K}^+ = \sum_{\alpha} a^+_{\alpha,\uparrow} a^+_{\alpha,\downarrow}$$

????

Charge conservation (gauge invariance) - no $\hat{K}$ or $\hat{K}^{+}$ only $\hat{K}\,\hat{K}^{+}$

Invariance under rotations in spin space - no $\hat{S}$ only $\hat{S}^{2}$

Therefore, in a very general case

$$\boxed{\hat{H}_{int} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{K}^{+}\hat{K}.}$$

Only three coupling constants describe **all** of the effects of e-e interactions

In a very general case only three coupling constants describe all effects of electron-electron interactions:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \hat{H}_{int}$$

$$\hat{H}_{int} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{K}^{+} \hat{K}.$$

I.L. Kurland, I.L.Aleiner & B.A., 2000

See also

P.W.Brouwer, Y.Oreg & B.I.Halperin, 1999

H.Baranger & L.I.Glazman, 1999

H-Y Kee, I.L.Aleiner & B.A., 1998

In a very general case only three coupling constants describe all effects of electron-electron interactions:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \hat{H}_{int}$$

$$\hat{H}_{int} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{K}^{+} \hat{K}.$$

For a short range interaction with a coupling constant λ

$$E_c = \frac{\lambda\delta_1}{2} \qquad J = -2\lambda\delta_1 \qquad \lambda_{BCS} = \lambda\delta_1(2-\beta)$$

where δ_1 is the one-particle mean level spacing

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha}$$

$$\hat{H}_{int} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{K}^+ \hat{K}.$$

Only one-particle part of the Hamiltonian, $\hat{H}_0$ ***, contains randomness***

!

$$\boxed{\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}}$$

$$\boxed{\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha n_\alpha}$$

$$\boxed{\hat{H}_{int} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{K}^+\hat{K}.}$$

E_c determines the charging energy (Coulomb blockade)

J describes the spin exchange interaction

λ_{BCS} determines effect of superconducting-like pairing

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_\alpha \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{int} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{K}^+\hat{K}.$$

I. *Excitations are similar to the excitations in a disordered Fermi-gas.*

II. *Small decay rate*

III. *Substantial renormalizations*

Isn't it a Fermi liquid ?

Fermi liquid behavior follows from the fact that different wave functions are almost uncorrelated