Lifetime of a Bogoliubov excitation

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We consider a spatially homogeneous gas at thermodynamic equilibrium in the regime of an almost pure condensate. The particles of the gas are spinless and have an interaction of negligible range and scattering length strictly positive. The Bogoliubov Hamiltonian predicts the existence of eigenmodes, the Bogoliubov modes, but neglects the interaction between these modes. Here we would like to take into account the interaction between the modes and show that it leads to a finite lifetime for the Bogoliubov excitations. We will calculate the lifetime of the Bogoliubov excitation of wavevector $\mathbf{q}$ in the thermodynamic limit using the Fermi's golden rule.

1 Interaction Hamiltonian

We recall the model Hamiltonian written in second quantization for the Bose gas interacting through a Dirac δ potential :

$$H = \int_{L^3} d^3r \left[-\frac{\hbar^2}{2m} \hat{\psi}^\dagger(\mathbf{r}) \Delta \hat{\psi}(\mathbf{r}) + \frac{g}{2} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}) \right]. \tag{1}$$

The field operator is given by $\hat{\psi}(\mathbf{r})$, m is the mass of the particle and $g > 0$ is the coupling constant. The spatial integral is taken on a cubic box of size $[0, L]^3$ and we impose periodic boundary conditions.

1. Write the condensate wavefunction $\phi(\mathbf{r})$ correctly normalized.
2. We perform the decomposition of the field operator as follows :

$$\hat{\psi}(\mathbf{r}) = \phi(\mathbf{r}) a_0 + \delta\hat{\psi}(\mathbf{r}) \tag{2}$$

   where a_0 annihilates a particle in the condensate mode. Explain why in the Hamiltonian H one must look for cubic terms in $\delta\hat{\psi}(\mathbf{r})$ if we want to calculate the first correction to the Bogoliubov Hamiltonian.
3. Show that the kinetic energy terms T in the Hamiltonian H involve the operator $\delta\hat{\psi}(\mathbf{r})$ but not a_0. Deduce that T cannot make appear cubic terms in $\delta\hat{\psi}(\mathbf{r})$.
4. We insert the decomposition Eq.(2) into the interaction part of the Hamiltonian. Write the terms of order three.

5. We make use now of the modulus–phase representation :

$$\hat{a}_0 = e^{i\hat{\theta}}\sqrt{\hat{n}_0} \quad , \quad [\hat{n}_0, \hat{\theta}] = i \tag{3}$$

where $\hat{n}_0$ is the operator number of particles in the condensate. We eliminate $\hat{n}_0$ using the relation

$$\hat{n}_0 = \hat{N} - \int d^3r\, \delta\hat{\psi}^\dagger(\mathbf{r})\delta\hat{\psi}(\mathbf{r}) \tag{4}$$

and we eliminate the phase operator of the condensate mode, supposed to be hermitian, by introducing the field

$$\hat{\Lambda}(\mathbf{r}) = e^{-i\hat{\theta}}\delta\hat{\psi}(\mathbf{r})\,. \tag{5}$$

Show that after elimination of the condensate mode, the terms of the previous question give

$$H_{\text{cube}} = g\sqrt{\rho}\int d^3r\, \hat{\Lambda}^\dagger(\mathbf{r})\left(\hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}^\dagger(\mathbf{r})\right)\hat{\Lambda}(\mathbf{r})\,, \tag{6}$$

where $\rho = N/V$ is the density. H_{cube} is then the first correction to the Bogoliubov Hamiltonian.

2 Fermi's golden rule

At zero temperature the main source of decay of a Bogoliubov excitation comes from elementary Beliaev processes, shown in Fig.1, where the excitation $\mathbf{q}$ disappears by giving two other excitations $\mathbf{k}_1$ and $\mathbf{k}_2$.

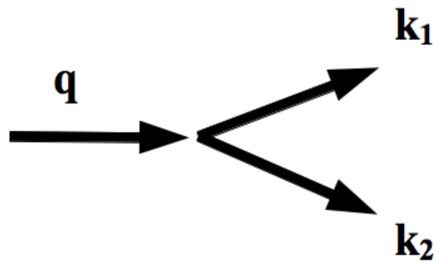

FIG. 1 – Beliaev processes.

We recall Fermi's golden rule :

$$\Gamma^{\text{elem}}_{i\to f} = \frac{2\pi}{\hbar}\,|\langle f|V|i\rangle|^2\,\delta(E_f - E_i)\,. \tag{7}$$

$\Gamma^{\text{elem}}_{i\to f}$ is the elementary transition rate of the initial state towards the final state that belongs to a continuum. The total rate of depart from the initial state is obtained by integrating the elementary transition rate over all the final states.

1. Write in second quantization the initial state $|i\rangle$ and the final state $|f\rangle$ of a Beliaev process.

2. Apply Fermi's golden rule and express the total rate of depart from the initial state Γ_q as a sum which is a function of the squared matrix element $|\langle f|H_{\rm cube}|i\rangle|^2$.
3. Using Wick's theorem, calculate the matrix element $|\langle f|H_{\rm cube}|i\rangle|^2$ and show that in the thermodynamic limit one finds :

$$\Gamma_q = \frac{g^2\rho}{(2\pi)^2\hbar}\int d^3k\, 2\,|\mathcal{B}^q_{k,k'}|^2\,\delta(\epsilon_k+\epsilon_{k'}-\epsilon_q) \tag{8}$$

where $k' = |\mathbf{k}-\mathbf{q}|$ and the coefficient $\mathcal{B}^q_{k,k'}$ is given by

$$\mathcal{B}^q_{k,k'} = U_qU_kU_{k'} + (U_q+V_q)(V_kU_{k'}+U_kV_{k'}) + V_qV_kV_{k'}\,. \tag{9}$$

in terms of the usual Bogoliubov functions U_k, V_k. We recall the decomposition of the field $\hat{\Lambda}$:

$$\hat{\Lambda}(\mathbf{r}) = \sum_{\mathbf{k}\neq\mathbf{0}} \hat{b}_{\mathbf{k}}U_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} + \hat{b}^\dagger_{\mathbf{k}}V_k\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}\,. \tag{10}$$

3 Limit where $\epsilon_q \ll \mu$.

We can perform the angular integration in (8), which gives the following known result

$$\Gamma_q = \frac{g}{2\pi\hbar}\frac{2m\mu}{\hbar^2}\int_0^q dk\,\left(\mathcal{B}^q_{k,k'}\right)^2\frac{k}{q}\frac{\epsilon_q-\epsilon_k}{\left[\mu^2+(\epsilon_q-\epsilon_k)^2\right]^{1/2}} \tag{11}$$

where ϵ_k are the eigenenergies of the Bogoliubov mode with k' such that :

$$\mu + \frac{\hbar^2k'^2}{2m} = \left[\mu^2+(\epsilon_q-\epsilon_k)^2\right]^{1/2}\,. \tag{12}$$

In addition, in the limit where $\epsilon_q \ll \mu$ one obtains

$$\left(\mathcal{B}^q_{k,k'}\right)^2 \simeq \left(\frac{3}{4\sqrt{2}}\right)^2\left(\frac{\hbar}{mc}\right)^3 qkk'\,. \tag{13}$$

1. Calculate ϵ_q, ϵ_k and k' as a function of q, k and the sound velocity $c_s = \sqrt{\mu/m}$ in the limit $\epsilon_q \ll \mu$.
2. Calculate Γ_q and find the result of Beliaev, Sov. Phys. JETP **7**, 299 (1953) :

$$\Gamma_q \simeq \frac{3}{320}\frac{\hbar}{\pi m\rho}q^5\,. \tag{14}$$