# Superfluidity versus Bose-Einstein condensation: Continuation

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22nd November 2012

## 1 The response of the system

We suppose that the system before the action of the time-dependent perturbation $W(t)$ that tries to move all particles with a constant velocity $\mathbf{v} = v\,\mathbf{e}_x$ along the $x$-direction (see previous lecture), was at thermodynamic equilibrium at the temperature $T$ in the canonical ensemble, therefore with the density operator

$$\sigma_0 = \frac{1}{Z_0} e^{-\beta H_0} \quad \text{where} \quad Z_0 = \mathrm{Tr}\, e^{-\beta H_0}, \tag{1}$$

with $H_0$ given in the previous lecture and $\beta = 1/(k_B T)$ as usual. Furthermore, we suppose that the system in the presence of the perturbation $W$ and after the unitary transformation $U(t)$ (see previous lecture) also reaches thermodynamic equilibrium at sufficiently large times and with the same temperature $T$,

$$\tilde{\sigma_0} = \frac{1}{Z} e^{-\beta \tilde{H}} \quad \text{where} \quad Z = \mathrm{Tr}\, e^{-\beta \tilde{H}}. \tag{2}$$

a) Justify the fact that the total momentum operator of the gas $P_x$ has zero average in the non-perturbed thermal state $\sigma_0$.

b) The normal fraction of the gas $f_n$, that is the complementary to one of the superfluid fraction of the gas $f_s$, $f_s + f_n = 1$, is defined as follows :

$$f_n = \lim_{v \to 0} \lim_{\mathcal{W} \to 0} \frac{\langle P_x \rangle}{Nmv}, \tag{3}$$

where $\langle P_x \rangle$ is the expectation value taken in the thermalized state in the presence of the perturbation :

$$\langle P_x \rangle = \mathrm{Tr}[P_x \tilde{\sigma}]. \tag{4}$$

Justify this definition from physical arguments. For this we will consider the extreme case of an ordinary fluid (what is the value of $\langle P_x \rangle$ ?), and the extreme case of a fluid that is totally superfluid (what is then the value of $\langle P_x \rangle$ ?).

c) After taking the limit where $\mathcal{W}$ tends to zero, develop $\tilde{\sigma}$ up to to first order in $v$ (first order included) and justify all the passages. Show that

$$f_n = \frac{\langle P_x^2 \rangle_0}{N m k_B T}, \tag{5}$$

where the expectation value is taken here in the non-perturbed thermal state :

$$\langle P_x^2 \rangle_0 = \mathrm{Tr}[P_x^2 \sigma_0]. \tag{6}$$

d) Using result (6) and (5) calculate explicitly the normal fraction when the temperature of the gas is small compared to its chemical potential, $k_B T \ll \mu$. What is the value of $f_n$ in the limit where $T \to 0$ ? We give the following integral :

$$\int_0^{+\infty} dx \frac{x^4 e^x}{(e^x - 1)^2} = \frac{4\pi^4}{15}. \tag{7}$$