Energy minimization in the BCS state and excitation spectrum (II)

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8th November 2012

1 BCS Hamiltonian and excitation spectrum

To describe fermionic superfluidity one often uses the so-called BCS Hamiltonian, which is a quadratic Hamiltonian in the field operator. In order to find it we decompose

$$\hat{\psi}_\uparrow \hat{\psi}_\downarrow = \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle + \left(\hat{\psi}_\uparrow \hat{\psi}_\downarrow - \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle \right), \tag{1}$$

and we neglect the terms which in the interaction energy are quadratic in the "fluctuations" $\left(\hat{\psi}_\uparrow \hat{\psi}_\downarrow - \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle \right)$.

a) Determine the BCS Hamiltonian starting from the model Hamiltonian of question 1 a) of the previous lecture, where we will replace μ par $\tilde{\mu}$.

b) In the Heisenberg point of view the equations for the time-evolution of the field operators are liner:

$$i\hbar \frac{d}{dt} \begin{pmatrix} \hat{\psi}_\uparrow(x) \\ \\ \hat{\psi}_\downarrow(x) \\ \\ \hat{\psi}_\uparrow^\dagger(x) \\ \\ \hat{\psi}_\downarrow^\dagger(x) \end{pmatrix} = \mathcal{L} \begin{pmatrix} \hat{\psi}_\uparrow(x) \\ \\ \hat{\psi}_\downarrow(x) \\ \\ \hat{\psi}_\uparrow^\dagger(x) \\ \\ \hat{\psi}_\downarrow^\dagger(x) \end{pmatrix}. \tag{2}$$

Calculate the operator $\mathcal{L}$. Is it Hermitian? We will use a 2×2 block notation where we introduce the spin base $|\uparrow\rangle, |\downarrow\rangle$.

c) Show that

$$x \to \begin{pmatrix} \tilde{u}_k(x) \\ 0 \\ 0 \\ \tilde{v}_k(x) \end{pmatrix} = \begin{pmatrix} U_k \\ 0 \\ 0 \\ V_k \end{pmatrix} \frac{e^{ikx}}{\sqrt{L}} \tag{3}$$

with U_k and V_k reals, are eigenvectors of $\mathcal{L}$ and calculate the corresponding eigenvalues λ_k. Show that the constants U_k and V_k are related to the u_k et v_k of the BCS ansatz. In particular we get: $U_k = u_k = 1/\sqrt{1+\Gamma_k^2}$ and $V_k = -v_k = -\Gamma_k/\sqrt{1+\Gamma_k^2}$.

Show that

$$x \rightarrow \begin{pmatrix} 0 \\ \tilde{u}_k(x) \\ -\tilde{v}_k(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ U_k \\ -V_k \\ 0 \end{pmatrix} \frac{e^{ikx}}{\sqrt{L}} \tag{4}$$

are also eigenvectors of $\mathcal{L}$.

d) Show that we can write

$$\hat{\psi}_\uparrow(x) = \sum_k \tilde{u}_k(x) b_{k\uparrow} - \tilde{v}_k^*(x) b_{k\downarrow}^\dagger \tag{5}$$

$$\hat{\psi}_\downarrow(x) = \sum_k \tilde{u}_k(x) b_{k\downarrow} + \tilde{v}_k^*(x) b_{k\uparrow}^\dagger \,, \tag{6}$$

and that

$$U a_{k\uparrow} U^\dagger = b_{k\uparrow} \,. \tag{7}$$

e) Show that $|\psi_{BCS}\rangle$ is the vacuum of the operators $b_{k\sigma}$.

f) Put the BCS Hamiltonian in the canonical form as a function of the $b_{k\sigma}$ and the $b_{k\sigma}^\dagger$.

g) For $g = 0$ draw the spectrum $k \rightarrow \lambda_k$ and give a physical interpretation in terms of creation of holes and particles.

h) Now we consider weak interactions such that $\Delta \ll \epsilon_F$. Draw λ_k in this case and show that there is a gap of size Δ that appears in the spectrum of excitation.