Phase operator: Collapse and revival of the phase

C. Trefzger, M2 – ICFP

27th September 2012

1 Naive approach for large N

1. **Phase operator.** Sometimes it is useful to pretend to have a phase operator $\hat{\phi}$, which satisfies
$$[\hat{\phi}, \hat{n}] = i, \tag{1}$$
where $\hat{n}$ is the operator number of particles of a bosonic mode.

   (a) Calculate $a^\dagger a$ and show that the operators defined as follows
   $$a = \sqrt{\hat{n}+1}\, e^{-i\hat{\phi}} \qquad a^\dagger = e^{i\hat{\phi}}\sqrt{\hat{n}+1}, \tag{2}$$
   have the expected commutation relations.

   (b) From Eq. (1), show that if $\hat{\phi}$ is hermitian $\hat{n}$ can have a continuous non-positive spectrum.

2. **Collapse of the phase.** For large n we forget about this problem for the moment and we make use of relation (1). Let's consider the Kerr-type Hamiltonian
$$H = \frac{\hbar\chi}{2}\hat{n}^2, \tag{3}$$
and, as an initial state, we take a Glauber coherent state: $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$, where α is a complex number.

   (a) Calculate the standard deviation of n in such a state. Then using the Heisenberg uncertainty principle calculate the standard deviation of ϕ.

   (b) Show qualitatively that there is a collapse of the phase after a collapse time t_{cl} which you will estimate.

2 Rigorous approach

1. **Phase operator.** One of the most used definition of the phase operator of a monomode bosonic field is given by
$$(\widehat{e^{-i\phi}}) = (a^\dagger a + 1)^{-1/2} a. \tag{4}$$

(a) Write $(\widehat{e^{-i\phi}})$ in the Fock state basis $|n\rangle$.

(b) Calculate explicitly the commutator of $(\widehat{e^{-i\phi}})$ with its hermitian conjugate $(\widehat{e^{-i\phi}})^\dagger$ and with the operator number $\hat{n} = a^\dagger a$. Is the operator $(\widehat{e^{-i\phi}})$ unitary?

(c) The operator $(\widehat{e^{-i\phi}})$ has eigenvectors with eigenvalues $e^{-i\phi'}$.

$$(\widehat{e^{-i\phi}})|e^{-i\phi'}\rangle = e^{-i\phi'}|e^{-i\phi'}\rangle. \tag{5}$$

Express these eigenvectors on the Fock basis by choosing the normalization $\langle 0|e^{-i\phi'}\rangle = 1$.

(d) Demonstrate the closure relation:

$$\int_0^{2\pi} d\phi\, |e^{-i\phi}\rangle\langle e^{-i\phi}| = 2\pi. \tag{6}$$

As a consequence, show that the state of the system can be described by the "wavefunction" $\psi(\phi) = \langle e^{-i\phi}|\psi\rangle$. What is $\psi(\phi)$ for a Fock state $|n\rangle$?

(e) For a Glauber coherent state $|\alpha\rangle$, show that

$$\psi_{\alpha e^{i\phi_0}}(\phi) = \psi_\alpha(\phi + \phi_0)\,. \tag{7}$$

For α real and $|\alpha| \gg 1$ find a Gaussian approximation for the coefficients of $|\alpha\rangle$ on the different Fock states. Estimate, in the continuous limit, the ϕ-width of $|\psi_\alpha(\phi)|^2$.

2. **Collapse and revival of the phase**.

(a) consider the Hamiltonian:

$$\mathcal{H} = \frac{\hbar\chi}{2}\,\hat{n}^2. \tag{8}$$

Write the Schrödinger equation for the time evolution of $\psi(\phi)$. Determine the eigenstates and the eigenenergies.

(b) Determine qualitatively the time evolution of $\psi(\phi, t)$ starting from a coherent state $|\alpha\rangle$ with $|\alpha| \gg 1$. What is the characteristic time $t_{\rm cl}$ for the reduction of the initial wave packet (phase *collapse*) ? Determine the instants $t_{\rm res}$ where the wave packet becomes identical to the initial one within a translation (phase *revival*).

(c) With the help of the equation for $\psi(\phi, t)$ show that at $t = t_{\rm res}/2$ the system is in a state of Shrödinger cat type.

Given :

$$e^{-i\pi n^2/2} = \frac{1}{\sqrt{2}}\left[e^{-i\pi/4} + e^{i\pi(n+1/4)}\right]\,. \tag{9}$$