Bose-Einstein statistics and beam splitter

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Consider an ideal beam splitter (BS) as shown in Fig. 1.

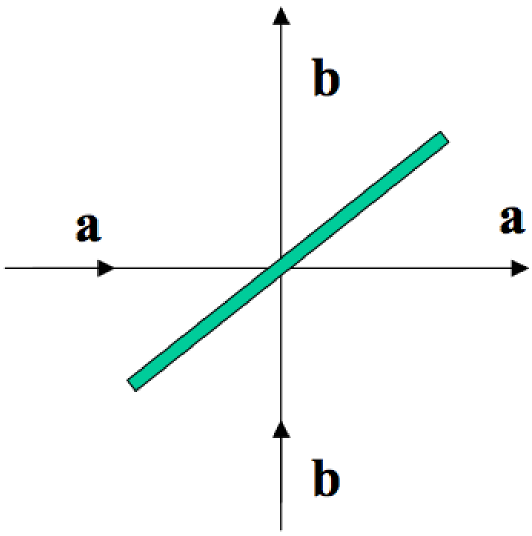

Figure 1: An ideal beam splitter (BS).

A single-particle wave packet arriving on the BS in the mode a generates two single-particle wave packets

- a wave packet which is "transmitted" in the same mode a with a probability amplitude t, t being real
- a wave packet which is "reflected" in the mode b with a probability amplitude r also real.

Similarly, a single-particle wave packet arriving on the BS in the mode b is transmitted in the same mode with a probability amplitude t, and reflected in the mode a with a probability amplitude $-r$.

This corresponds to an evolution operator $U(1)$ which in the $|a\rangle, |b\rangle$ base has the following representation:

$$U(1) = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \tag{1}$$

1) Explain the origin of the $-$ sign and why the condition $t^2 + r^2 = 1$ must apply.

2) We send two bosonic particles on the BS, one arriving from the channel a and the other arriving from the channel b. Assuming the particles do not interact, generalize the operator $U(1)$ of §1 to the case of two particles [matrix U(1,2)]. Calculate the probability amplitude to have at the exit of the BS respectively:

- the two bosons in the channel a
- the two bosons in the channel b
- one boson in each channel a and b.

Verify that for a perfect BS (transmission coefficient equal to reflection coefficient,$t = r$), the two bosons exit always from the same channel.

What would the result be for polarized fermionic particles?

3) We would like to generalize the matrix U to the case of any number of incident bosons by working in second quantization. We define a and b the operators annihilating one particle in the state $|a\rangle$ and $|b\rangle$ respectively.

We recall the definition of the creation operator of a particle in the state $|u\rangle$ acting on any bosonic state $|\Psi\rangle$:

$$c^{\dagger}_{|u\rangle}|\Psi\rangle = \sqrt{\hat{N}} S\left(|u\rangle \otimes |\Psi\rangle\right), \qquad (2)$$

where S is the operator that symmetrizes and $\hat{N}$ the operator of the total number of particles.

a) Show that

$$U c^{\dagger}_{|u\rangle} U^{\dagger} = c^{\dagger}_{U|u\rangle} \qquad (3)$$

b) Deduce the transformations $Ua^{\dagger}\, U^{\dagger}$ and $Ub^{\dagger}\, U^{\dagger}$ of the operators $a^{\dagger}$ et $b^{\dagger}$.

c) Recalculate the previous results for two bosons.

4) We send $2N$ bosons on the BS, N from the channel a and N from the channel b. For a perfect BS calculate the probability that all bosons exit from the same channel. How does this probability vary in the limit $N \to +\infty$? Compare with the case of distinguishable particles.

We recall the Stirling's approximation :

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

when $n \to +\infty$.